Lecture 15

X. Supersymmetry and Supergravity

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On many occasions during the discussion of superstring theory we have obtained results that are consistent with the presence of space-time supersymmetry. These include the counting of the number of bosonic versus fermionic states at all mass levels, as well as the study of the spectrum at zero mass. In this section, we shall exhibit supersymmetry on the spectrum and on the scattering amplitudes in an explicit fashion. We begin by completing the study of the RNS formulation of string theory, and use the Ramond (fermion) vertex operator to produce a conserved supercharge that generates space-time supersymmetry. Next, we introduce and study the Green-Schwarz formulation of superstring theory, in which space-time supersymmetry is manifestly realized. Finally, we exhibit the low energy field theory approximations to the various superstring theories we have introduced, and identify them with known supergravity theories, in which we have local or gauged supersymmetry. We show how the Green-Schwarz formulation of superstrings may be used to couple strings to supergravity in a covariant way.

§10.1. Global space-time supersymmetry in the RNS Formulation

In the previous chapter, we have provided a complete prescription for the vertex operators and for the calculation of transition amplitudes in the RNS formulation of superstrings. The RNS formulation does not exhibit space-time supersymmetry in a manifest fashion however: the construction of bosonic and fermionic vertex operators is asymmetrical.

Nonetheless, it is possible to construct a conserved current $j_\alpha(z)$ on the worldsheet, whose charge Q_α, defined by

$$Q_\alpha = \oint \frac{dz}{2\pi}\, j_\alpha(z) \tag{10.1}$$

obeys the supersymmetry algebra of 10-dimensional space-time:

$$\begin{aligned}&[P_\mu, P_\nu] = [P_\mu, Q_\alpha] = 0\\ &\{Q_\alpha, Q_\beta\} = 2\,\Gamma^\mu_{\alpha\gamma}(C^{-1})^\gamma{}_\beta\,P_\mu = 2\,\gamma^\mu_{\alpha\beta}\,P_\mu\end{aligned} \tag{10.2}$$

Here P_μ is the translation operator, and we use the Dirac algebra defined in Lecture 9. For $N = 2$ supersymmetric theories such as Type II A, B, there will be an independent supercharge Q' arising from the $\bar z$ sector of the theory, whose discussion is analogous to that of Q. The current $j_\alpha(z)$ is related to the Ramond vertex operator. To see this, we notice that Q_α should map states of given M^2 into states of the same M^2, and should transform as a spinor. The R vertex operator, at zero space-time momentum, precisely has these properties. Thus, we identify $j_\alpha(z)$ with

$$j_\alpha(z) = e^{-\phi/2}(z)S_\alpha(z) \tag{10.3}$$

of ghost number $-1/2$ or

$$j_\alpha(z) = e^{\phi/2}(\gamma_\mu)_{\alpha\beta}S'^\beta(z)\partial_z x^\mu \tag{10.4}$$

of ghost number $1/2$. Both are holomorphic and, with the help of

$$\begin{aligned}e^{-\phi/2}(z)e^{\phi/2}(w) &\sim \frac{1}{(z-w)^{-1/4}}\\ S_\alpha(z)S'^\beta &\sim \frac{\delta^\beta_\alpha}{(z-w)^{5/4}}\end{aligned} \tag{10.5}$$

we find

$$j_\alpha(z)j_\beta(w) \sim (\gamma_\mu)_{\alpha\beta}\partial_z x^\mu \frac{1}{z-w} \tag{10.6}$$

and recover the superalgebra of Q_α and P_μ upon integration.

Under the action of Q_α, vertex operators in the NS and R sectors are mapped into one another. Considering left moving vertex operators for massless states, for example,

$$\begin{aligned}W_{\rm NS}(z,\varepsilon,k) &= \varepsilon_\mu\partial_z x^\mu e^{ik\cdot x}\\ W_{\rm R}(z,u,k) &= e^{-\phi/2}S_\alpha u^\alpha e^{ik\cdot x}\end{aligned} \tag{10.7}$$

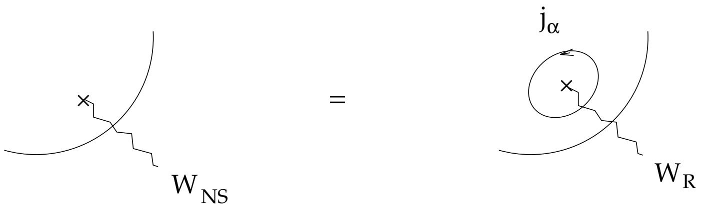
where we have considered $W_{\rm NS}$ in the picture with zero ghost number and $W_{\rm R}$ in the picture with ghost number $-1/2$. We have

$$\begin{aligned}[\zeta^\alpha Q_\alpha, W_{\rm NS}(z,\varepsilon,k)] &= W_{\rm R}(z,u,k)\\ [\zeta^\alpha Q_\alpha, W_{\rm R}(z,v,k)] &= W_{\rm NS}(z,\eta,k)\end{aligned} \tag{10.8}$$

with

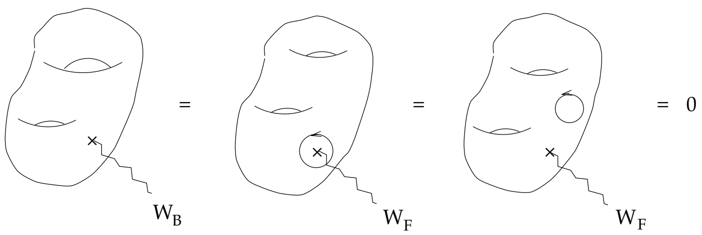
$$
(10.9) \qquad \begin{aligned} u^\alpha &= k^\mu \epsilon^\nu (\gamma_{\mu\nu})^\alpha{}_\beta \zeta^\beta \\ \eta^\mu &= v^\alpha (\gamma^\mu)_{\alpha\beta} \zeta^\beta \end{aligned}
$$

Pictorially, we may represent this action by



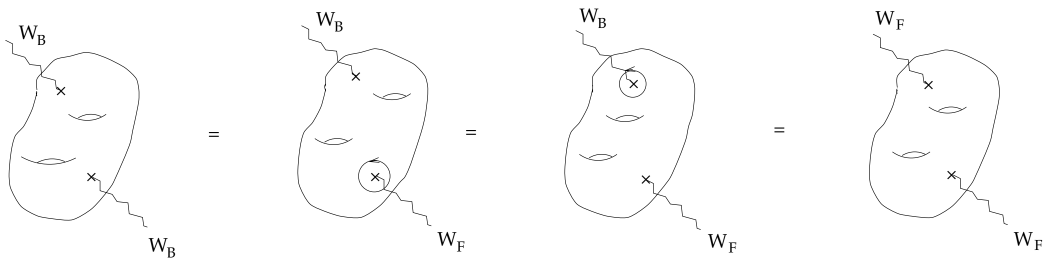
and similarly for $W_{\rm R}$ insertions.

From the action of supersymmetry, we obtain several important relations between transition amplitudes. For the one-point function, we have



by conservation of the current $j_\alpha(z)$. The only vertex operator that enters is the dilaton at zero momentum, and the above argument now shows directly that this amplitude must vanish, as we have indeed established to tree and one-loop level, by explicit calculation.

For the two point function, we obtain



By chiral symmetry, the W_F two point function must vanish for massless fermions, and hence the W_B two point function must also vanish, as we have indeed established to tree and one-loop level, by explicit calculation.

It is hoped that the use of the supersymmetry currents will ultimately allow one to show that superstring theory in the RNS formulation is finite ot all orders, but subtleties with the superghost system have prevented us so far from carrying out this program in full.

§10.2. The Green-Schwarz formulation

The string degrees of freedom in the Green-Schwarz (GS) formulation of superstring theory (in D space-time dimensions) are

(1) the bosonic coordinate $x^\mu, \quad \mu = 0, 1, \ldots, D-1$

(2) the fermionic coordinate $\theta^{\alpha I}$

The x^μ coordinates are as before. The $\theta^{\alpha I}$ coordinates are spinors under $\mathrm{Spin}(1, D-1)$, labeled by the spin index α, with $I = 1, 2, \ldots, N$ describing the degree of extension of the supersymmetry algebra of the Poincaré group. $\theta^{\alpha I}$ are Grassmann valued worldsheet scalars. For simplicity, we shall assume that $\mathrm{Spin}(1, D-1)$ admits Majorana spinors, and we shall use Majorana notation. (For $D = 10$, $\theta^{\alpha I}$ are Majorana-Weyl spinors, with $N = 1$ for Type I and heterotic and $N = 2$ for Type II.)

Poincaré supersymmetry acts on the coordinates x^μ and $\theta^{\alpha I}$ as follows (we use the conventions of §9):

$$\left\{ \begin{aligned} \delta\theta^I &= \varepsilon^I \\ \delta\bar{\theta}^I &= \bar{\epsilon}^I \\ \delta x^\mu &= i\bar{\epsilon}^I \Gamma^\mu \theta^I \end{aligned} \right. \tag{10.10}$$

where ε^I is a constant set of Majorana spinors, labeled by $I = 1, \ldots, N$.

We seek a worldsheet action in terms of x^μ and θ^I for the various superstring theories, that possesses manifest space-time supersymmetry as defined above.

As a warm-up, we obtain the action for a massless point particle, propagating on a worldline L, parametrized by τ (and $\dot{} = \frac{d}{d\tau}$). To do so, we need a generalization of $\dot{x}^\mu$ that has good supersymmetry transformation properties. The following combination is easily seen to be *invariant* under supersymmetry

$$\Pi^\mu \equiv \dot{x}^\mu - i\bar{\theta}^I \Gamma^\mu \dot{\theta}^I \tag{10.11}$$

and the following worldline action is susy invariant as well as $Diff(L)$-invariant:

$$S = \int_L d\tau \, e^{-1}(\tau)(\dot{x}^\mu - i\bar{\theta}^I \Gamma^\mu \dot{\theta}^I)^2 \tag{10.12}$$

Here, e is the worldline metric. The variational equations are

$$\Pi^2 = 0, \quad \dot{\Pi}^\mu = 0, \quad \Gamma_\mu \Pi^\mu \dot{\theta}^I = 0 \tag{10.13}$$

Interpreting Π^μ as particle momentum, we find that the particle propagates with constant momentum, that it is massless, and that because $(\Gamma \cdot \Pi)^2 = 0$, half of the θ^J's are unconstrained. This last property is due to the fact that S exhibits a further symmetry, called *κ-symmetry*:

$$\begin{cases} \delta\theta^I i\Gamma \cdot \Pi \kappa^I \\ \delta x^\mu = i\bar{\theta}^I \Gamma^\mu \delta\theta^I \\ \delta e = 4e\dot{\bar{\theta}}^I \kappa^I \end{cases} \tag{10.14}$$

The presence of κ-symmetry guarantees that only half of the θ's are physical degrees of freedom of the particle.

We now generalize the above construction to obtain a worldsheet action with space-time supersymmetry. There is an obvious generalization of momentum, which is space-time supersymmetric:

$$\Pi_m{}^\mu \equiv \partial_m x^\mu - i\bar{\theta}^I \Gamma^\mu \partial_m \theta^I \tag{10.15}$$

Thus there is an obvious worldsheet action (with worldsheet metric g, which does not transform under supersymmetry)

$$S_1[x, \theta] = \frac{1}{8\pi} \int_\Sigma d\mu_g g^{mn} \Pi_m{}^\mu \Pi_n{}^\nu \eta_{\mu\nu} \tag{10.16}$$

invariant under $Diff(\Sigma)$, $Weyl(\Sigma)$ and space-time supersymmetry. However, the κ-symmetry of the point particle does not extend to S_1. This is a problem, because in $D = 10$, $N = 1$ superstrings, e.g. we will now have 8 physical degrees of freedom of x^μ, but 16 for θ. The κ-symmetry is required to have the correct on-shell particle content of the string spectrum.

When $N = 1, 2$ or $N = 1$, the action S_1 may be modified to $S = S_1 + S_2$ in such a way that S is $Diff(\Sigma)$, $Weyl(\Sigma)$, space-time supersymmetric, but also invariant under κ-supersymmetry. The additional part S_2 is found to be

$$S_2[x, \theta] = \frac{1}{4\pi} \int_\Sigma \{ -i\, dx^\mu \wedge (\bar\theta^1 \Gamma_\mu d\theta^1 - \bar\theta^2 \Gamma_\mu d\theta^2) \\ \bar\theta^1 \Gamma_\mu d\theta^1 \wedge \bar\theta^2 \Gamma^\mu d\bar\theta^2 \} \tag{10.17}$$

Actually, the space-time supersymmetry of S_2 is not automatic. Restricting to $N = 1$ only for simplicity, the ε-transform of S_2 yields

$$\delta S_2 = \frac{1}{4\pi} \int_\Sigma (\bar\epsilon \Gamma^\mu d\theta \wedge \bar\theta \Gamma_\mu d\theta - i\, dx^\mu \wedge \bar\epsilon \Gamma_\mu d\theta) \tag{10.18}$$

The second term in δS_2 is a total derivative and leads to no further requirements. The vanishing of the first term in δS_2 requires that a certain Γ-matrix identity be satisfied:

$$\Gamma^\mu \psi_1 (\bar\psi_2 \Gamma_\mu \psi_3) + \Gamma^\mu \psi_2 (\bar\psi_3 \Gamma_\mu \psi_1) + \Gamma^\mu \psi_3 (\bar\psi_1 \Gamma_\mu \psi_2) = 0 \tag{10.19}$$

This identity can be realized in the following situations:

(10.20)

(i)	$D = 10$,	θ	Majorana-Weyl	8 θ's, 8 x's
(ii)	$D = 6$,	θ	Weyl	4 θ's, 4 x's
(iii)	$D = 4$,	θ	Majorana	2 θ's, 2 x's
(iv)	$D = 3$,	θ	Majorana	1 θ, 1 x .

(Note, for $D = 6$, certain formulas must be adapted, since the corresponding spinors θ are not Majorana.)

The κ-symmetry may be expressed in terms of a worldsheet vector κ^J_m, which is also a space-time spinor:

$$\begin{cases} \delta\theta^I = 2i\, \Gamma \cdot \Pi_m \kappa^{Im} \\ \delta x^\mu = \bar\theta^I \Gamma^\mu \delta\theta^I \\ \delta g_{mn} = \qquad \ldots \end{cases} \tag{10.21}$$

The worldsheet vector κ^I_m produces a 1-form $\kappa^I \equiv \kappa^I_m d\xi^m$ with the following self-duality properties

$$\begin{cases} {}^*\kappa^1 = -i\, \kappa^1 \\ {}^*\kappa^2 = i\, \kappa^2 \end{cases} \tag{10.22}$$

The κ-symmetry closes onto a combination involving a κ-symmetry, a $Diff(\Sigma)$, $Weyl(\Sigma)$ transformation, and an additional bosonic symmetry that has no dynamical consequences.

On a worldsheet with flat metric g, the field equations for g, x and θ are

$$
\begin{aligned}
&\bullet \quad \Pi_z{}^2 = \Pi_{\bar z}{}^2 = 0 \\
(10.23) \qquad &\bullet \quad \partial_z(\partial_{\bar z} x^\mu - 2i\bar\theta^1\Gamma^\mu\partial_{\bar z}\theta^1) + \partial_{\bar z}(\partial_z x^\mu - 2i\bar\theta^2\Gamma^\mu\partial_z\theta^2) = 0 \\
&\bullet \quad \Gamma_\mu\Pi_z^\mu\partial_{\bar z}\theta^1 = \Gamma_\mu\Pi_{\bar z}^\mu\partial_z\theta^2 = 0 \ .
\end{aligned}
$$

These equations of motion are highly non-linear. As a result, quantization of the GS string in covariant form is fraught with difficulties, arising in particular from the fact that θ's do not have a standard kinetic term. No satisfactory approach exists to date.

§10.3. Lightcone gauge quantization of the GS Formulation

In the lightcone gauge, the non-linearities of the GS field equations disappear, and quantization may be carried out in complete parallel with the lightcone quantization of the RNS string (Problem Set #9). As usual, we impose

$$(10.24) \qquad x^+ = q^+ + p^+(z+\bar z)$$

and assume that $p^+ \neq 0$. (If $p^+ = 0$, then by positivity of energy, we must have $p = 0$, and we only have the vacuum state.) Hence

$$(10.25) \qquad \Pi_z^+ = \Pi_{\bar z}^+ = p^+$$

On-shell, we have $\Pi_z^2 = \Pi_{\bar z}^2 = 0$, so at a given point on the worldsheet, we may rotate Π so that $\Pi^- = \Pi^i = 0$, with only p^+ remaining as the only non-vanishing component. At that point, the κ-transformation on θ^I becomes

$$(10.26) \qquad \delta\theta^I = 2i\,\Gamma^- p^+(\kappa^{Iz} + \kappa^{I\bar z}) \ .$$

Thus, by suitable κ-transformation, all θ^I's in the image of Γ^- may be rotated away. Thus, by κ-symmetry, we may impose the additional gauge choice

$$(10.27) \qquad \Gamma^+\theta^I = 0$$

This choice implies that the equations for $\partial_{\bar z}\theta^1$ and $\partial_z\theta^2$ simplify; to see this, multiply each to the left by Γ^+, and use $\{\Gamma^+,\Gamma^-\}=2$:

$$\partial_{\bar z}\theta^1 = \partial_z\theta^2 = 0 \tag{10.28}$$

As a result, the equation for x^μ linearizes as well

$$\partial_z\partial_{\bar z}x^\mu = 0\ . \tag{10.29}$$

The only remaining non-linearity is the mass-shell constraint, given by

$$\begin{aligned}\Pi_z^2 &= 2p^+\Pi_z^- + \Pi_z^i\Pi_z^i = 0\\ \Pi_{\bar z}^2 &= 2p^+\Pi_{\bar z}^- + \Pi_{\bar z}^i\Pi_{\bar z}^i = 0\ .\end{aligned} \tag{10.30}$$

As always in the lightcone gauge, the Lorentz group is reduced (in $D=10$) to SO(8), under which x^i transforms as a vector 8_v, while θ^I (or more precisely, its eight dimensional restriction S^I) transforms as a spinor of SO(8): 8_s or 8_c. These representations are permuted into one another under *triality* of SO(8).

The lightcone quantization of the GS string is equivalent to the lightcone quantization of the RNS superstring. Let us exhibit how this works on a cylinder, where the RNS field ψ_+^μ has periodic boundary conditions, suitable to a space-time spinor state. In complete parallel with the covariant construction of the Ramond vertex operator, we may construct $S^{\alpha 1}(z)$ from $\psi_+^i(z)$ (and $S^{\alpha 2}(\bar z)$ out of $\psi_-^i(\bar z)$). One begins by bosonizing the Cartan generators of SO(8) in the RNS representation:

$$\psi_+^{2i-1}(z)\psi_+^{2i}(z) = -i\partial_z\phi^i \qquad i=1,2,3,4 \tag{10.31}$$

Recall that the fields ψ_+^i then have an exponential form in terms of ϕ^i, with $\pm$ weights

$$\frac{1}{\sqrt{2}}(\psi_+^{2i-1}\pm i\psi_+^{2i})(z) = e^{\pm i\phi^i} \tag{10.32}$$

and the spin fields are also exponentials with $\pm\frac12$ weights:

$$\begin{aligned}S_\alpha^1(z) &= e^{i\alpha\cdot\phi(z)}\\ \alpha &= (\pm\tfrac12,\ \pm\tfrac12,\ \pm\tfrac12,\ \pm\tfrac12)\end{aligned} \tag{10.33}$$

The conformal weight of $S^1_\alpha(z)$ is now $\frac{1}{2}\,\alpha^2 = \frac{1}{2}$! This means that the lightcone gauge has made $S^1_\alpha(z)$ into a worldsheet spinor.

§10.4. Flat superspace GS formulation

A more intrinsic formulation of the GS worldsheet action is obtained by expressing it in superspace We concentrate on the $D=10$, $N=2$ case, for which superspace is of dimension $(10|32)$, with 32 Majorana spinor components. Coordinates are $X^M=(x^m,\theta^{\mu I})$, where $m=0,1,\ldots,9$ and $\mu=1,\ldots,16$ while $I=1,2$ labels the two irreducible Majorana-Weyl (MW) multiplets.

Flat superspace is characterized by a frame $E^A = dX^M E_M{}^A$ where $A=(a,\alpha I)$ are frame indices, with $a=0,1,\ldots,9$ labeling the defining representation of $\mathrm{SO}(1,9)$, α labeling the MW spinor of $\mathrm{Spin}(1,9)$ and $I=1,2$. The $\mathrm{Spin}(1,9)$ connection vanishes in flat superspace, and the superderivatives are

$$\mathcal{D}_A = E_A{}^M \partial_M \qquad\qquad \partial_M \equiv \frac{\partial}{\partial X^M} \tag{10.34}$$

and they satisfy

$$\{\mathcal{D}_{\alpha I}, \mathcal{D}_{\beta J}\} = 2(\Gamma^a)_{\alpha\gamma}(C^{-1})^\gamma{}_\beta \mathcal{D}_a \delta_{IJ} \tag{10.35}$$

with all others commuting. The combination Π^r_m, encountered previously, has a very natural interpretation as a pull-back of the superspace frame E^A to the worldsheet. Indeed

$$\begin{aligned} X^*(E^a) &\equiv E_p{}^a d\xi^p = (\partial_p x^a - i\bar\theta^I \Gamma^a \partial_p \theta^I) d\xi^p \\ X^*(E^{\alpha I}) &\equiv E_p{}^{\alpha I} d\xi^p = \partial_p \theta^{\alpha I} d\xi^p \end{aligned} \tag{10.36}$$

so that $\Pi_p{}^a = E_p{}^a$. This identification readily allows us to write the first part of the GS superstring action in superspace:

$$S_1[X] = \frac{1}{8\pi}\int_\Sigma d\mu_g g^{pq} E_p{}^a E_q{}^b \eta_{ab} \tag{10.37}$$

The second part of the GS action may be recast in the form of a Wess-Zumino-Witten action. To show that this is indeed the case, we consider the following 3-form

$$H \equiv E^a \wedge (\bar E^1 \Gamma_a \wedge E^1 - \bar E^2 \Gamma_a \wedge E^2) \tag{10.38}$$

where we have used the notation $\bar{E}^I \equiv (E^I)^T\Gamma^0$ and suppressed spinor index contraction. To construct a Wess-Zumino-Witten action, we require as usual that the form H be closed $dH = 0$. This condition implies

$$
\begin{aligned}
0 = dH &= dE^a \wedge (\bar{E}^1\Gamma_a \wedge E^1 - \bar{E}^2\Gamma_a \wedge E^2) \\
&= -i\bar{\theta}^I\Gamma^a d\theta^I(\bar{E}^1\Gamma_a \wedge E^1 - \bar{E}^2\Gamma_a \wedge E^2) \\
&= -i(\bar{E}^1\Gamma^a E^1 + \bar{E}^2\Gamma^a E^2)(\bar{E}^1\Gamma_a \wedge E^1 - \bar{E}^2\Gamma_a E^2) \\
&= -i(\bar{E}^1\Gamma^a E^1 \bar{E}^1\Gamma_a E^1 - (1 \to 2)) \\
&= -(\Gamma^0\Gamma^a)_{\alpha\beta}(\Gamma^0\Gamma_a)_{\gamma\delta}E^{1\alpha}E^{1\beta}E^{1\delta} - (1 \to 2)
\end{aligned}
\tag{10.39}
$$

The last line vanishes since the wedge product of the E^1's is symmetry in $(\alpha\beta\gamma\delta)$ in view of the identity (10.19) we used to show that the S_2 action is supersymmetric.

It is easy to show that $H = db$, where b is the 2-dimensional integrand of S_2. Indeed,

$$
\begin{aligned}
H &= (dx^a - i\bar{\theta}^j\Gamma^a d\theta^j) \wedge (d\bar{\theta}^1\Gamma_a d\theta^1 - d\bar{\theta}^2\Gamma_a d\theta^2) \\
&= d(dx^a(\bar{\theta}^1\Gamma_a d\theta^1 - \bar{\theta}^2\Gamma_a d\theta^2)) \\
&\quad - i(\bar{\theta}^1\Gamma^a\theta^1 d\bar{\theta}^1\Gamma_a d\theta^1 - \bar{\theta}^2\Gamma^a d\theta^2 d\bar{\theta}^2\Gamma_a d\theta^2) \\
&\quad - i\bar{\theta}^2\Gamma^a d\theta^2 d\bar{\theta}^1\Gamma_a d\theta^1 + \bar{\theta}^1\Gamma^a d\theta^1 d\bar{\theta}^2\Gamma_a d\theta^2
\end{aligned}
\tag{10.40}
$$

Now, the second line again vanishes by the Γ-identity (10.19), and the last term is a total differential:

$$
H = d\{dx^a(\theta^1\Gamma_a d\theta^1 - \bar{\theta}^2\Gamma_a d\theta^2) + i\bar{\theta}^1\Gamma^a d\theta^1\bar{\theta}^2\Gamma_a d\theta^2\}
\tag{10.41}
$$

Identifying with the term in S_2, we find

$$
S_2[x] = \frac{-i}{4\pi}\int_B X^*(H) \ .
\tag{10.42}
$$

Finally, it is very instructive to exhibit the action of the κ-symmetry in the superspace formulation. It turns out that the κ-symmetry is most easily expressed in terms of a worldsheet scalar (space-time spinor)

$$
\kappa \equiv i\, E_p^a\, \Gamma_a(\kappa^{p1} + \kappa^{p2})
\tag{10.43}
$$

In terms of this parameter, one has

$$
\begin{aligned}
\delta X^M E_M{}^a &\equiv \delta X^a = 0 \\
\delta X^M E_M{}^1 &= (1+\Gamma)\kappa \\
\delta X^M E_M{}^2 &= (1-\Gamma)\kappa
\end{aligned}
\tag{10.44}
$$

where the operator Γ is defined by

$$
\Gamma \equiv \frac{1}{2\sqrt{h}}\epsilon^{pq}E_p{}^a E_q{}^b \epsilon_{ab}
\tag{10.45}
$$

and h is such that $\Gamma^2 = I$.

§10.5. Supergravity and low energy superstrings

We have shown in the first part of this chapter that the RNS and GS formulations of superstring theory exhibit 10-dimensional (global) Poincaré supersymmetry, in flat Minkowski space-time $M = \mathbb{R}^{10}$.

Actually, superstring theory also exhibits $Diff(M)$ invariance when formulated on a general manifold M. For the bosonic string, we had shown this in §VI by coupling bosonic string theory to background fields, including the metric $G_{\mu\nu}$, the anti-symmetry tensor field $B_{\mu\nu}$ and the dilaton Φ, in a $Diff(M)$ covariant fashion. For the superstring, we may again couple superstring theory to $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ in a $Diff(M)$ covariant way. The relevant ($N=1$ on the worldsheet) supersymmetric non-linear sigma model for the type II superstrings is given by

$$
\begin{aligned}
S[X;G,B,\Phi] = &\frac{1}{4\pi}\int_\Sigma d\mu_E \mathcal{D}_- X^\mu \mathcal{D}_+ X^\nu (G_{\mu\nu}(X) + iB_{\mu\nu}(X)) \\
&+ \frac{1}{4\pi}\int_\sigma d\mu_E R_{+-}\Phi(X)
\end{aligned}
\tag{10.46}
$$

This worldsheet action defines the RNS superstring in an arbitrary space-time manifold M, with the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. (For the $N=1$ theories, this sigma model must be modified: for heterotic theories, one replaces $\mathcal{D}_-$ by $\partial_{\bar z}$ and R_{+-} by $R_{+\bar z}$, while for Type I theories, the $B_{\mu\nu}$ coupling will not occur here, but will result from the R-R sector.)

Of course, it must be realized that for any of the superstrings, the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ do *not* correspond to the full set of massless states discussed in §VII and VIII. First of all, there are additional space-time fermions: the gravitino and the dilatino. But, there are also additional boson fields: in Type II theories, these are the R-R gauge fields; in heterotic and Type I theories these are the Yang–Mills bosons. Following the discussion provided for the study of bosonic strings, on general manifolds, with general background fields, we should include in the non-linear sigma model all space-time background fields corresponding to massless string states. The inclusion of fermions or R-R bose fields is complicated by the fact that it requires the use of the Ramond vertex operator. Nonetheless, let us imagine that such a formulation were used. (We shall exhibit it explicitly later, with the use of the GS formulation.)

Now, since the superstring theory is covariant under $Diff(M)$ as well as under global Poincaré supersymmetry, it is fully expected that it will in fact be invariant under *local supersymmetry transformations* (also called supergravity transformations). This is expected because the composition of $Diff(M)$ and Poincaré supersymmetry transformations does *not* close onto these, but does close onto local supersymmetry. (We have already examined in §9 how this happens in $D = 2$ within the context of super-Riemann surfaces.) Field theories that are invariant under local supersymmetry transformations are so-called *supergravities.*

In the low energy approximation, superstring theory can be approximated by the spectrum and dynamics of its massless states only, summarized in terms of a *local field theory.* The massless sector of superstring theory will inherit invariance under local supersymmetry, and will then be a supergravity.

The dynamics of the NS-NS sector of Type II A, B superstrings, in the massless sector, will yield the part of the corresponding supergravity involving only the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ (with the fermion fields and R-R fields set to zero). This dynamics may be deduced directly from the above non-linear sigma model, and results from demanding that its quantum dynamics be superconformal invariant. It turns out that the conditions of superconformal invariance of $S[X; G, B, \Phi]$ to leading order (i.e. two derivatives or fewer on fermion fields) are precisely the same as the condition found for the bosonic string, except that $D = 10$ now. The field equations for G, B and Φ could be deduced from an action for those fields on the space-time manifold M:

$$I_1(G, B, \Phi) = -\frac{1}{2K^2} \int_M d\mu_G \left\{ R_G - 4D_\mu \Phi D^\mu \Phi + \frac{1}{12} H^2 \right\} e^{-2\Phi} \tag{10.47}$$

Here, we have expressed the metric $G_{\mu\nu}$ in the *string* convention, so that an extra overall factor of the dilaton coupling appears.

For the heterotic strings, the fields G, B and Φ result from the NS sector of the left-movers coupled to the space-time part of the right movers, and the resulting action is still given by I_1.

§10.6. Type II A, $D = 10$, $N = 2$ and $D = 11$, $N = 1$ supergravities

We begin by discussing the low energy field theory (at most 2 derivatives on bose fields, at most 1 on fermion fields) associated with Type II Asuperstring theory, i.e. Type II A, $N = 2$, $D = 10$ supergravity. The existence of this theory was predicted by Nahm, based on representation theory of supersymmetry algebras and was constructed by Cremmer, Julia and Scherk, by dimensional reduction from the $D = 11$, $N = 1$ supergravity. The Type II Afield content is readily identified from §VII, 4:

• $G_{\mu\nu}$	NS-NS	space-time metric (equivalently frame $e_m{}^a$)
• $B_{\mu\nu}$		anti-symmetric NS-NS tensor
• Φ		dilaton
• $A^{(1)}_\mu$	R-R	graviphoton
• $A^{(3)}_{\mu\nu\rho}$		rank 3 anti-symmetric tensor
• $\chi_\mu{}^\alpha$	R-NS and NS-R	32 component Majorana gravitino
• λ_α		32 component Majorana dilatino
		(both contain both chiralities.)

The action I_1 corresponds to the dynamics in which $A^{(1)} = A^{(3)} = \chi = \lambda = 0$, and we now seek the generalization of I_1 that also describes the remaining fields.

It turns out that the easiest way to do this is to reorganize the field contents in such a way that it manifestly results from dimensional reduction of an $D = 11$ supergravity theory with $N = 1$ supersymmetry. The precise statement is that the low energy limit of the $D = 11$ supergravity, compactified on a circle with small radius, is precisely the Type II Asupergravity. This re-organization of field variables is advantageous because it will turn out that the $N = 1$, $D = 11$ supergravity is described by a relatively simple Lagrangian in terms of relatively few fields.

We begin by showing that the field contents may be organized according to multiplets of Spin$(1, 10)$. We use space-time indices $\mu, \nu, \ldots = 0, 1, \ldots, 9$ and $\bar{\mu}, \bar{\nu} = 0, 1, \ldots, 9, 10$. We denote the compactified coordinate by x^{10}. Upon dimensional reduction on a circle, the only parts that survive to the low energy approximation of the $D = 10$ theory are the fields that are independent of x^{10}. Indeed, x^{10}-dependence renders the fields massive, with mass on the order of $1/R$.

A metric tensor in $D = 11$ decomposes as follows:

$$G_{\bar{\mu}\bar{\nu}}(x^\mu, x^{10}) \longrightarrow G_{\bar{\mu}\bar{\nu}}(x^\mu) \tag{10.48}$$

Under the action of SO$(1, 10)$, this multiplet is irreducible, but under SO$(1, 9)$, we find $\square\!\square \to \square\!\square \oplus \square \oplus \cdot$, so that the objects of Type II Atheory that naturally fit into the $D = 11$ metric multiplet are

$$(G_{\mu\nu}, A^{(1)}_\mu, \Phi) \to G_{\bar{\mu}\bar{\nu}} \tag{10.49}$$

Proceeding analogously for an anti-symmetric rank 3 tensor

$$A_{\bar{\mu}\bar{\nu}\bar{\rho}}(x^\mu, x^{10}) \to A_{\bar{\mu}\bar{\nu}\bar{\rho}}(x^\mu) \tag{10.50}$$

and under SO$(1, 10)$, this tensor is again irreducible, but decomposes under SO$(1, 9)$ as

$$\begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array} \to \begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array} \oplus \begin{array}{c}\square\\[-4pt]\square\end{array},$$

so that

$$(A^{(3)}_{\mu\nu\rho}, B_{\mu\nu}) \to A_{\bar{\mu}\bar{\nu}\bar{\rho}} \tag{10.51}$$

Thus, all the NS-NS and R-R bosons fit into two fields: $G_{\bar\mu\bar\nu}$ and $A_{\bar\mu\bar\nu\bar\rho}$.

The spinors also fit together. The smallest dimensional representation of a spinor of Spin$(1,10)$ is a Majorana spinor of dimension 32. Under Spin$(1,9)$, it will reduce into 2 Majorana-Weyl spinors of opposite chirality, and this is precisely what we need for Type II A, where the gravitino as well as the dilatino are non-chiral Majorana spinors. Furthermore, the dilatino may be appended to ${\chi_\mu}^\alpha$ as its 11-th component:

$$({\chi_\mu}^\alpha, \lambda_\alpha) \to \chi^\alpha_{\bar\mu}\ . \tag{10.52}$$

§**10.6.1** $N = 1$**,** $D = 11$ **supergravity**

Based upon general arguments by Nahm, the largest possible dimension D in which supersymmetric multiplets can exist with $spin \leq 2$ only is $D = 11$, with a single local supersymmetry. Its field contents are precisely the ones that we have described above:

- $G_{\bar\mu\bar\nu}$ (equiv. ${e_{\bar\mu}}^{\bar a}$) graviton: 44 states
- $A_{\bar\mu\bar\nu\bar\rho}$ $U(1)$gauge field: 84 states
- $\chi^\alpha_{\bar\mu}$ gravitino: 128 states.

The dimensional reduction is carried out by setting x^{10} dependence to zero, and by letting $G_{\bar\mu\bar\nu}$ decompose as

$$G_{\mu\nu}\ ; \qquad G_{\mu 10} = G_{10\mu} = A^{(1)}_\mu \qquad G_{10\,10} = e^{-2\Phi} \tag{10.53}$$

which yields the action I_1 when $A^{(1)}_\mu = 0$, and $A^{(3)}_{\mu\nu\rho} = \chi^\alpha_\mu = \lambda_\alpha = 0$. The normalization of the field strength $H = dB$ of B in I_1 provides a normalization of the field strength $F = dA$ in the $D = 11$ supergravity theory. The remaining terms are fixed by local supersymmetry invariance.

The full Lagrangian is obtained by seeking $N = 1$ local supersymmetry and was established by Cremmer, Julia and Scherk:

$$I(G, A, \chi) = -\frac{1}{2\kappa^2}\int_{\bar M} d\mu_G \mathcal{L} - \frac{\sqrt{2}\,\kappa}{3^4 56}\int_{\bar M} A \wedge F \wedge F \tag{10.54}$$

where $\mathcal{L}$ is given by

$$\mathcal{L} = R_G + \frac{1}{12}\kappa^2 F^2 + \kappa^2 \bar{\chi}_{\bar{\mu}} \Gamma^{\bar{\mu}\bar{\nu}\bar{\rho}} D_{\bar{\nu}} \chi_{\bar{\rho}} \\ + \frac{\sqrt{2}\,\kappa^3}{384} \left(\bar{\chi}_{\bar{\mu}} \Gamma^{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}\bar{\tau}\bar{\omega}} \chi_{\bar{\omega}} + 12 \bar{\chi}^{\bar{\nu}} \Gamma^{\bar{\rho}\bar{\nu}\bar{\sigma}} \chi^{\bar{\tau}} \right) (F + \widehat{F})_{\bar{\nu}\bar{\rho}\bar{\sigma}\bar{\tau}} \tag{10.55}$$

In this action, we have used the standard notation

$$\Gamma^{\mu_1 \cdots \mu_\rho} \equiv \Gamma^{[\mu_1} \Gamma^{\mu_2} \ldots \Gamma^{\mu_\rho]}$$

where the square brackets stand for complete anti-symmetrization of the indices. $F = dA$ was defined previously, and

$$\widehat{F} = F + \chi\text{-terms}$$

such that the supersymmetry variation of $\widehat{F}$ does not contain derivatives of η — the susy parameter. $D_{\bar{\nu}}$ is the covariant derivative with respect to $\frac{1}{2}(\omega_{\bar{M}} + \widehat{\omega}_{\bar{M}})$ where

$$\widehat{\omega}_{\bar{\mu}\bar{a}\bar{b}} = \omega_{\bar{\mu}\bar{a}\bar{b}} + \frac{1}{8} \bar{\chi}^{\bar{\nu}} \Gamma_{\bar{\nu}\bar{\mu}\bar{a}\bar{b}\bar{\rho}} \chi^{\bar{\rho}} \tag{10.58}$$

and $\omega_{\bar{\mu}}$ is the spin connection, which is determined by its equation of motion. For completeness, we record the supergravity transformations:

$$\begin{aligned} \delta e_{\bar{\mu}}{}^{\bar{a}} &= \frac{\kappa}{2} \bar{\eta} \Gamma^{\bar{a}} \chi_{\bar{\mu}} \\ \delta A_{\bar{\mu}\bar{\nu}\bar{\rho}} &= -\frac{\sqrt{2}}{8} \bar{\eta} \Gamma_{[\bar{\mu}\bar{\nu}} \chi_{\bar{\rho}]} \\ \delta \chi_{\bar{\mu}} &= \frac{1}{\kappa} D'_{\bar{\mu}} \eta + \frac{\sqrt{2}}{288} \left(\Gamma_{\bar{\mu}}{}^{\bar{\nu}\bar{\rho}\bar{\sigma}\bar{\tau}} - 8 \delta_{\bar{\mu}}{}^{\bar{\nu}} \Gamma^{\bar{\rho}\bar{\sigma}\bar{\tau}} \right) \widehat{F}_{\bar{\nu}\bar{\rho}\bar{\sigma}\bar{\tau}} \end{aligned} \tag{10.59}$$

The covariant derivative D' is here defined with the connections $\widehat{\omega}$.

§10.6.2 Type II Asupergravity

The Type II Asupergravity Lagrangian may now be deduced from $D = 11$ supergravity by the compactification and dimensional reduction discussed previously. The reduction of the metric $G_{\bar{\mu}\bar{\nu}}$ was already given; the reduction of the $A_{\bar{\mu}\bar{\nu}\bar{\kappa}}$ field proceeds by

$$\begin{aligned} A_{\mu\nu\kappa} &= \kappa^{\frac{1}{4}} A^{(3)}_{\mu\nu\kappa} \\ A_{\mu\nu 10} &= \frac{1}{\kappa} B_{\mu\nu} \end{aligned} \tag{10.60}$$

The bosonic part of the Type II Asupergravity is then

$$
\begin{aligned}
I_{\mathrm{II\ A}}&(G, B, \Phi; A^{(1)}, A^{(3)}) \\
&= -\frac{1}{2\kappa^2}\int_M d\mu_G \Big\{ e^{-2\Phi}\Big(R_G - 4D_\mu\Phi D^\mu\Phi + \frac{1}{12}\,H^2\Big) \\
&\qquad\qquad + \sqrt{\kappa}\,G_A^2 + \sqrt{\kappa}\,\frac{1}{12}\,F^2\Big\} \\
&\quad - \frac{\sqrt{\kappa}}{288\kappa^2}\int_M B\wedge F\wedge F
\end{aligned}
\tag{10.61}
$$

here $G_A = dA^{(1)}$, $H = dB$, $F = dA^{(3)}$. Notice that the R-R part of the action does not occur with the dilaton rescaling factor $e^{-2\Phi}$; thus, these contributions are in some sense of order 1 in the string loop expansion.

§10.7. Type II B, $D = 10$, $N = 2$ supergravity

The Type II Bfield contents are as follows (see Lecture 7).

- $G_{\mu\nu}$(equiv. ${e_\mu}^a$) graviton.
- $B_{\mu\nu} + i\,A^{(2)}_{\mu\nu}$ complex rank 2 antisymmetric tensor.
- $\Phi + i\,A^{(0)}$ complex dilaton-axion.
- $A^{(4)}_{\mu\nu K\lambda}$ real, self-dual $F = dA^{(4)}$.

- χ^α_μ (chirality+) complex Majorana-Weyl spinor gravitino.
- λ_α (chirality−) complex Majorana-Weyl spinor dilatino.

Clearly, the field contents do not arise from a simple dimensional reduction, since the spinor content is chiral here. In fact, since the theory contains a self-dual field, F, no off-shell Lorentz-covariant formulation is known for this theory.

The field equations may be written covariantly though, either in components or in superspace. The gauge group is $\mathrm{Spin}(1,9)\times U(1)_R$ and the complex dilaton-axion field $\Phi + i\,A^{(0)}$ lives in $\mathrm{SU}(1,1)/U(1)_R$.

§10.8. Type I – Heterotic, $D = 10$, $N = 1$ supergravities

Both for the Type I and for the heterotic theories, we have only a single space-time supersymmetry: $N = 1$. However, both theories now also have an $N = 1$ super-Yang–Mills multiplet, which we shall have to include.

The field contents of the $N = 1$ *supergravity multiplet* are obtained (say for the heterotic string) by considering the left-moving part of Type II, tensored with the right-moving 10-dimensional part of the bosonic string. The corresponding fields are

•	$G_{\mu\nu}$	space-time metric
•	$B_{\mu\nu}$	real rank 2 anti-symmetric tensor
•	Φ	real dilaton
•	${\chi_\mu}^\alpha$	1 Majorana-Weyl (real) gravitino (+ chirality)
•	λ_α	1 Majorana-Weyl (real) dilatino (− chirality)

the Lagrangian for this theory exists (since there are no fields with self-dual field strength) and may be obtained from the Type II Atheory by setting

$$A^{(1)}_\mu = A^{(3)}_{\mu\nu\rho} = 0 \qquad \chi^\alpha_\mu\big|_- = \lambda_\alpha\big|_+ = 0 \tag{10.62}$$

where the $\big|_\pm$ indicates the chirality. This truncation is consistent with the field equations of Type II A, since both $A^{(1)}$ and $A^{(3)}$ occur quadratically in the bosonic Lagrangian, and since the fermionic Lagrangian does not mix between chiralities. We limit ourselves to examining only the bosonic action of the supergravity multiplet, and notice immediately that it must be just I_1. The fermionic part may be deduced from the $D = 11$ supergravity Lagrangian by the above dimensional reductions and restrictions.

The field contents of the $N = 1$ *super Yang–Mills multiplet* are obtained (again for the heterotic string) by considering the left-moving part of the Type II string, tensored with the right-moving internal part of the bosonic string. The corresponding fields are

•	A	gauge field of $E_8 \times E_8$ or $\mathrm{Spin}(32)/Z_2 = G$ (adjoint rep.)
•	ψ	gaugino of $E_8 \times E_8$ or $\mathrm{Spin}(32)/Z_2 = G$ (adjoint rep.)

The dynamics are governed by the action (in flat M, with $G = \eta$)

$$I^0_{YM} = \int_M d\mu_G \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2} \bar\psi^a \Gamma^\mu (D_\mu \psi)^a \right) \tag{10.63}$$

which is *global* $N = 1$ Poincaré supersymmetry invariant. Here, we use the notation ($a = 1, \dots, \dim G$) and

$$A = A^a_\mu \, t^a \, dx^\mu \qquad F = dA + gA \wedge A = \frac{1}{2} F^a_{\mu\nu} \, t^a \, dx^\mu \, dx^\nu \ . \tag{10.64}$$

It remains to construct an action with $N=1$ local supersymmetry, that combines both the supergravity multiplet and the super-Yang–Mills multiplet. First, the action I^0_{YM} is adapted as follows.

$$I_{YM} = \frac{1}{\kappa}\int_M d\mu_G e^{-2\Phi}\left\{-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}\bar{\psi}^a\Gamma^\mu D_\mu\psi^a\right\} \tag{10.65}$$

For $G=U(1)$ gauge theory, $I_1 + I_{YM} +$ the fermionic terms of the supergravity multiplet yield an $N=1$ susy action. For non-Abelian G, this is not so, and the form of the B-field strength H in I_1 must be modified as follows:

$$\widetilde{H} = dB - \frac{\kappa}{\sqrt{2}}\omega_3(A) \tag{10.66}$$

Here, $\omega_3(A)$ is the so-called Chern-Simons form of degree 3:

$$\omega_3(A) \equiv \mathrm{tr}\left(A\wedge dA + \frac{2}{3}gA\wedge A\wedge A\right) , \tag{10.67}$$

with the properties that

$$\begin{aligned} &d\omega_3(A) = \mathrm{tr}\, F\wedge F \\ &\delta_\Lambda\omega_3(A) = \mathrm{tr}(d\Lambda\wedge dA) \qquad \text{under } \delta_\Lambda A = d\Lambda + [A,\Lambda] . \end{aligned} \tag{10.68}$$

To render $\widetilde{H}$ gauge invariant under non-Abelian transformation in G, B must transform as

$$\delta_\Lambda B = \frac{\kappa}{\sqrt{2}}\mathrm{tr}\Lambda dA$$

The supergravity + super Yang–Mills action is now (for bosons)

$$I = -\frac{1}{2K^2}\int_M d\mu_G\left\{e^{-2\Phi}\left(R_G + 4D_\mu\Phi D^\mu\Phi - \frac{1}{12}\widetilde{H}^2\right) + e^{-2\Phi}\left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\right)\right\} \tag{10.69}$$

where the first part coincides with I_1, except for the fact that H has been replaced with its G-invariant version $\widetilde{H}$.

§10.9. Superspace formulation of supergravities in $D=11$ and $D=10$

In §10.6, §10.7 and §10.8, we have discussed supergravities in $D=11$ and $D=10$ in the component formulation. While supersymmetry invariance transformations of these actions are known, there is the problem that the composition of two supergravity transformations closes onto $Diff(M)$ only upon the use of the equations of motion. Also, the supergravity invariance is not manifest.

The superspace approach provides a formulation in which $Diff(M)$ and local supersymmetry transformations appear on the same footing and are united into the group of diffeomorphisms of superspace. However, the superspace formulation in $D = 11$ and $D = 10$ supergravity — and super Yang-Mills-theory — is known only for the equations of motion. No superspace action is known that describes the off-shell configurations as well.

To exhibit local supersymmetric covariant coupling of superstrings to supergravity, it is most convenient to make use of the superspace formulation of supergravity, as we shall show in the next section. For this issue, the fact that only the on-shell formulation is available causes no problem. In fact, already superstring theory itself, and as we showed previously, Type II Bsupergravity, do not admit an off-shell formulation; it is thus perhaps not so surprising that superspace supergravity does not either.

$N = 1,\ D = 11$ *supergravity in superspace*

From many points of view, the $D = 11$ theory is the simplest theory, and we shall begin by giving its formulation. The superspace data are as follows: local supercoordinates $X^M = (x^m, \theta^\mu)$, with $m = 0, 1, \ldots, 10$ and $\mu = 1, \ldots, 32$ parametrize a real supermanifold M of dimension $(11|32)$. The fields are most conveniently expressed as

(10.70)

- $E^A = dX^M E_M{}^A$
- $\Omega_A{}^B = dX^M \Omega_{MA}{}^B$
- $X = \frac{1}{3!} E^C \wedge E^B \wedge E^A X_{ABC}$

Here, A, B, C sum over frame indices: (a, α), (b, β), (c, γ) $a, b, c = 0, 1, \ldots 10$; $\alpha, \beta, \gamma = 1, \ldots, 32$, and transform under the defining representation and the Majorana spinor representations of Spin$(1, 10)$, respectively. The field E^A is a frame, belonging to $\Omega^1(M)$; the field $\Omega_A{}^B$ is a Spin$(1, 10)$ connection, which admits the following decomposition under

Spin(1, 10):

$$\begin{aligned} &\Omega_{ab} = -\Omega_{ba} \qquad\qquad \Omega_{\alpha b} = \Omega_{\alpha\beta} = 0 \\ &\Omega_{\alpha\beta} = \frac{1}{4}(\Gamma^{ab})_{\alpha\beta}\Omega_{ab} \end{aligned} \tag{10.71}$$

Finally X is a 3 form in superspace, and a Spin(1, 10) invariant. It contains the rank 3 field $A^{(3)}_{\bar\mu\bar\nu\bar\rho}$.

Curvature, torsion and X field strength are defined by

$$\begin{array}{llr} T^A & \equiv dE^A + E^B{\wedge}\Omega_B{}^A & \left(\equiv \frac{1}{2}E^c{\wedge}E^B T_{BC}{}^A\right) \\ R_A{}^B & \equiv d\Omega_A{}^B + \Omega_A{}^C{\wedge}\Omega_C{}^B & \left(\equiv \frac{1}{2}E^D{\wedge}E^c R_{CD,A}{}^B\right) \\ H & \equiv dX & \left(\equiv \frac{1}{4!}E^D{\wedge}E^C{\wedge}E^B{\wedge}E^A{\wedge}H_{ABCD}\right) \ . \end{array} \tag{10.72}$$

These fields satisfy Bianchi identities

$$\begin{array}{ll} \text{(a)} & DT^A = E^B{\wedge}R_B{}^A \\ \text{(b)} & DR_A{}^B = 0 \\ \text{(c)} & dH = 0. \end{array} \tag{10.73}$$

written in differential form notation. It is most convenient to write these Bianchi identities out in terms of the Spin(1, 10) tensors $T_{AB}{}^C$, $R_{AB,C}{}^D$ and H_{ABCD}; (parentheses $(ABC\dots)$ stand for cyclic, graded summation of indices)

$$\begin{array}{ll} \text{(a)} & \sum\limits_{(ABC)}(R_{ABC}{}^D - D_A T_{BC}{}^D - T_{AB}{}^E T_{EC}{}^D) = 0 \\ \text{(b)} & \sum\limits_{(ABC)}(D_A R_{BCD}{}^E + T_{AB}{}^F R_{FCD}{}^E) = 0 \\ \text{(c)} & \sum\limits_{(ABCDE)}(D_A H_{BCDE} + T_{AB}{}^F H_{FCDE}) \\ & \qquad - \sum\limits_{(ADBEC)}(-)^{BC+BD+CE} T_{AD}{}^F H_{FBEC} = 0 \end{array} \tag{10.74}$$

This geometry is too general, and contains many more fields than are necessary for the $D = 11$ supergravity theory that we constructed in components. We must impose suitable "constraint equations". (We already encountered this problem in $D = 2$, $N = 1$ supergravity, where certain constraints were also to be imposed, see §9.1.) The choice of constraints is dictated by the structure of the component fields that one expects to remain after the constraints have been imposed, by dimensional analysis, and of course by Lorentz invariance. We shall limit ourselves here to quoting the results from Brink and Howe (1980):

$$\begin{aligned} T_{\alpha\beta}{}^{\gamma} = T_{\alpha b}{}^{c} = T_{ab}{}^{c} = 0;\ T_{\alpha\beta}{}^{c} = -i(\Gamma^{c})_{\alpha\beta} \\ H_{\alpha\beta\gamma\delta} = H_{\alpha\beta\gamma c} = H_{\alpha bcd} = 0;\ H_{\alpha\beta cd} = i(\Gamma_{cd})_{\alpha\beta} \end{aligned} \tag{10.75}$$

It can be shown that all the torsion constraints can be solved in terms of a single superfield H_{abcd}. We shall not list out the various components here, but limit ourselves to providing the two field equations that result. We introduce

$$R_{ab} = \eta^{cd} R_{acbd} \qquad R = \eta^{ab} R_{ab} \tag{10.76}$$

and have the equations of motion

$$\begin{aligned} R_{ab} = \frac{1}{2}\eta_{ab}R \quad &= -\tfrac{1}{12}\, H_{acde}H_{b}{}^{cde} + \tfrac{1}{96}\,\eta_{ab}H_{cdef}H^{cdef} \\ D^{a}H_{abcd} \quad &= -\tfrac{1}{1728}\epsilon_{bcde_1\ldots e_4 f_1\ldots f_4} H J^{e_1\ldots e_4} H^{f_1\ldots f_4} \end{aligned} \tag{10.77}$$

Retaining only the $\theta = 0$ part of these equations, we readily observe that they are the bosonic field equations that would also have been obtained from the component formulation.

We conclude by remarking that the $D = 10$, $N = 2$ Type II Asupergravity may be obtained again by dimensional reduction. The resulting formulation is rather involved. There also exist a superspace formulation for Type II Bsupergravity.

$N = 1$, $D = 10$ *supergravity in superspace*

The superspace data are as follows: local supercoordinates $X^{M} = (x^{m}, \theta^{\mu})$, with $m = 0, 1, \ldots, 9$ and $\mu = 1, \ldots 16$, parametrize a real supermanifold of dimension $(10|16)$. The fundamental fields are

$$\begin{aligned} \bullet \qquad E^{A} &= dX^{M}E_{M}{}^{A} \\ \bullet \qquad \Omega_{A}{}^{B} &= dX^{M}\Omega_{MA}{}^{B} \end{aligned} \tag{10.78}$$

where A,B run over frame indices: (a, α), (b, β), $a, b = 0, 1, \ldots, 9$ and $\alpha, \beta = 1, \ldots, 16$. The spinors here transform under a single Majorana-Weyl representation of Spin$(1, 9)$. Notice that there is **no** further anti-symmetric tensor as there was in $D = 11$. The $B_{\mu\nu}$ field that occurs in $D = 10$, $N = 1$ supergravity is contained in the fields E^A and $\Omega_A{}^B$. The field $\Omega_A{}^B$ is again a spin connection, but now for Spin$(1,9)$, acting on the direct sum of the defining representation and the Majorana-Weyl representation:

$$\begin{aligned} &\Omega_{ab} = -\Omega_{ba} \qquad\qquad \Omega_{\alpha b} = \Omega_{a\beta} = 0 \\ &\Omega_{\alpha\beta} = \tfrac{1}{4}(\Gamma^{ab})_{\alpha\beta}\Omega_{ab} \end{aligned} \tag{10.79}$$

Torsion and curvature are defined by

$$\begin{aligned} T^A &\equiv dE^A + E^B \wedge \Omega_B{}^A \qquad && \left(\equiv \tfrac{1}{2}\, E^C \wedge E^B T_{BC}{}^A\right) \\ R_A{}^B &\equiv d\Omega_A{}^B + \Omega_A{}^C \wedge \Omega_C{}^B && \left(\equiv \tfrac{1}{2}\, E^D \wedge E^C R_{CDA}{}^B\right) \ . \end{aligned} \tag{10.80}$$

As in the case of $D = 11$, $N = 1$, these quantities satisfy Bianchi identities

$$DT^A = E^B \wedge R_B{}^A \qquad\qquad DR_A{}^B = 0 \ . \tag{10.81}$$

(A component decomposition for these equations is given in (10.74).)

Again, this geometry is too general, and torsion constraints should be imposed. The choice of constraints is not unique, and depends upon the choice of connection, among others. The constraints here were first derived by Nilsson, and reformulated by Witten:

$$\begin{aligned} &T_{\alpha\beta}{}^\gamma = T_{\alpha b}{}^c = 0 \qquad\qquad T_{\alpha\beta}{}^c = 2\Gamma^c_{\alpha\beta} \\ &T_{\alpha\beta}{}^\gamma = (\Gamma_a)_{\beta\delta}\phi^{\delta\gamma} \quad \text{for some } \phi^{\delta\gamma} \end{aligned} \tag{10.82}$$

One may show that the resulting fields, after elimination of all auxiliary fields, are precisely those of the $N = 1$ supergravity multiplet in $D = 10$, with the same equations of motion as those that we obtained from the component formulation.

§10.10. Local supersymmetric coupling of superstrings to supergravity

It is now very easy to produce a GS worldsheet action, formulated in superspace, that couples in a manifestly locally supersymmetric way to the superspace fields of supergravity in $D = 10$. For simplicity, we consider the case with $N = 1$ supersymmetry only.

Recall that the flat superspace action was

$$S[X] = \frac{1}{8\pi}\int_\Sigma d\mu_g g^{pq} E_p{}^a E_a{}^b \eta_{ab} - \frac{i}{4\pi}\int_B X^*(H)$$

where ${E_p}^a d\xi^p \equiv X^*(E^a)$ is the pullback to X of E^a, and where $\partial B = \Sigma$, and H was a closed 3-form on M. The form of H was dictated by the condition that we had to have κ-symmetry in order to obtain the correct spectrum.

In general curved superspace backgrounds, we may again propose $S[X]$ as a suitable action, provided the superspace geometry admits a closed 3-form H, and upon inserting a factor of a scalar superfield k:

$$S[X] = \frac{1}{8\pi} \int_\Sigma d\mu_g g^{pq} {E_p}^a {E_q}^b \eta_{ab} e^{4k} - \frac{i}{4\pi} \int_B X^*(H)$$

where ${E_p}^a d\xi^p = X^*(E^a)$, just as in the flat superspace case. The superfield k cannot be arbitrary, of course.

It can be shown that when the torsion constraints for $N = 1$, $D = 10$ supergravity are satisfied, then there exists a natural closed 3-form H, such that $S[X]$ is invariant under κ-symmetry. In other words, if we assume (locally) that $H = dB$, then the requirement of κ-invariance demands that certain torsion constraints are satisfied. Up to equivalences, these constraints are precisely the torsion constraints.