— DRAFT — IASSNS-HEP-97/72

Lectures 2, 3, 4

II. Spectrum of Free Bosonic Strings

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We begin by constructing the Hilbert space of physical states and the spectrum of a single free bosonic closed oriented string, whose worldsheet has the topology of a cylinder (or equivalently an annulus) and of a single free open oriented string, whose worldsheet has the topology of half a cylinder (or equivalently half an annulus).

A free string should not couple to any other strings, or to any non-trivial background fields in the space-time manifold M. Thus we take $M = \mathbb{R}^D$, with constant $G_{\mu\nu}$ and $B_{\mu\nu} = \Psi = T = 0$ to start. We wish to examine the properties of physical strings in physical space-time, so it is essential that the space-time metric be Minkowskian, with $G_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(- + + \cdots +)$ and $\mu, \nu = 0, 1, \ldots, D-1$. In quantum field theory, the problem of constructing the Hilbert space of states and the spectrum of a single particle amounts to finding the 1-particle unitary representations of the Poincaré group — a problem for which the Minkowski nature of space-time is again essential.

A single string has an infinite number of excitation states, each of which corresponds to a representation of the Poincaré group. In this lecture, we wish to identify which representations of the Poincaré group occur (i.e. find the spectrum) and determine the conditions under which all these representations are unitary. We will find that conformal invariance plays an essential rôle.

Before we discuss the details of this fundamental analysis of string theory, let us outline the main steps of our strategy.

1) In the first lecture, we postulated a prescription for the transition amplitudes of N external string states — represented by vertex operators V_i, $i = 1, \ldots, N$ — in a general Riemannian space-time M with metric G (as well as possibly other fields $B_{\mu\nu}$, Φ, T)

$$A = \sum_{h=0}^{\infty} \int_{\mathrm{Met}(\Sigma)} Dg \frac{1}{\mathcal{N}(g)} \int_{\mathrm{Diff}(\Sigma, M)} Dx\, V_1 \ldots V_N\, e^{-S[x,g]} \tag{2.1}$$

2) To investigate the spectrum of the string in Minkowski $\mathbb{R}^D_M$ we shall start with the above amplitudes in Euclidean $\mathbb{R}^D_E$ with $G_{\mu\nu} = \delta_{\mu\nu}$, $B_{\mu\nu} = \Phi = T = 0$ and action

$$S[x,g] = \frac{1}{8\pi} \int_\Sigma d\mu_g g^{mn} \partial_m x \cdot \partial_n x \ . \tag{2.2}$$

(For flat Euclidean or Minkowskian metric, we have $x \in \mathbb{R}^D_E$ or $\mathbb{R}^D_M$ respectively and we use the notation $(v, v') = v \cdot v'$.) The worldsheet metric g is always kept Riemannian. The string tension κ introduced in §I has been set to $\kappa = 1/8\pi$. In the physics literature one often finds the string tension expressed in terms of α' (the Regge slope parameter) by $\kappa = \frac{1}{4\pi\alpha'}$; so in our conventions $\alpha' = 2$.

3) At fixed g, the quantum field theory defined by $S[x,g]$ provides an example of a *conformal field theory* (see also the lectures by Gawedzki). All we shall need of the conformal field theory to solve for the free string spectrum is contained in the operator product expansions.

4) Once we have derived the OPE's for Euclidean $\mathbb{R}^D_E$, we shall continue the time component of the map x, let $x^0 \to ix^0$ and obtain well-defined OPE's for the operators in Minkowskian $\mathbb{R}^D_M$. In particular, we obtain two Virasoro algebras $\mathrm{Vir}_L \otimes \mathrm{Vir}_R$ for the closed string, and a single Virasoro algebra Vir_L for the open string.

5) From the Minkowskian OPE's and associated algebra, we construct Fock spaces $\mathcal{F}_{\mathrm{open}}$ and $\mathcal{F}_{\mathrm{closed}}$ for open and closed strings respectively along the same lines that Gawedzki constructed a Hilbert space. Due to the Minkowski signature of the space-time metric, however, the Fock space we obtain does *not* carry a positive definite inner product. $\mathrm{Vir}_L \otimes \mathrm{Vir}_R$ acts on $\mathcal{F}_{\mathrm{closed}}$ and Vir_L acts on $\mathcal{F}_{\mathrm{open}}$.

6) The remaining integration over g — which is *not* present in conformal field theory, but characteristic to string theory — effectively selects out of $\mathcal{F}$ (either $\mathcal{F}_{\mathrm{closed}}$ or $\mathcal{F}_{\mathrm{open}}$) a subspace $\mathcal{F}^+$ on which we have a positive definite inner product, under certain conditions which we shall determine. In particular, this is how the critical dimension $D = 26$ arises.

(The occurrence of negative norm states in a standard Fock space construction is not unique to string theory. It will take place for gauge theories and for gravity as well, or more generally in any theory with particles of spin ≥ 1.)

A) **Basics of Conformal Field Theory**

• An oriented surface Σ, with a (fixed) Riemannian metric g automatically carries a complex structure, and we choose local complex coordinates z and $\bar z$ in which g is conformally flat, and given by

$$g = g_{z\bar z}(dz \otimes d\bar z + d\bar z \otimes dz) = 2g_{z\bar z}|dz|^2 \tag{2.3}$$

This metric is left invariant under analytic or *conformal* diffeomorphisms $z \to z' = f(z)$, with f complex analytic, combined with Weyl transformations on g. A quantum theory invariant under this transformation is a conformal field theory (abbreviated CFT).

• Fundamental operators in CFT are conformal tensors or *primary fields* ϕ of conformal weight $(h, \bar h)$ satisfying the transformation law

$$\phi = \phi(z, \bar z)(dz)^h (d\bar z)^{\bar h} = \phi'(z', \bar z')(dz')^h (d\bar z)^{\bar h} \ . \tag{2.4}$$

For integer weights h and $\bar h$, these are operator-valued sections of $K^h \otimes \bar K^{\bar h}$, where K is the holomorphic cotangent (or canonical) bundle of Σ. In general, h and $\bar h$ don't have to be integers, but their difference is the spin of the field ϕ and must satisfy $h - \bar h = \text{spin} \in \frac{1}{2}\mathbb{Z}$.

• In the conformal field theories we shall deal with in these lectures, the matrix elements of products of primary fields (and other fields) may be expressed in terms of functional integrals:

$$\langle \phi_1(\xi_1) \dots \phi_N(\xi_N) \rangle_g = \int Dx \ \phi_1(\xi_1) \dots \phi_N(\xi_N) e^{-S[x,g]} \tag{2.5}$$

where it is understood that ϕ_i can be expressed in terms of the canonical fields x. Notice that here we do not normalize correlation functions by dividing out by the partition function $\langle 1 \rangle_g$.

• The stress tensor is defined as the variation of the action S with respect to the inverse metric $g^{-1} = g^{mn} \partial_m \otimes \partial_n$, where the metric g is viewed as a background source field.

$$\delta_g S = -\frac{1}{4\pi} \int_\Sigma d\mu_g T_{mn} \delta g^{mn} \tag{2.6}$$

Assuming that the variation g has support away from the operator positions $\xi_1, \ldots, \xi_N$, the variation of correlation functions is given by

$$\delta_g \langle \phi_1(\xi_1) \ldots \phi_N(\xi_N) \rangle = \frac{1}{4\pi} \int_\Sigma d\mu_g \delta g^{mn} \langle T_{mn} \phi_1(\xi_1) \ldots \phi_N(\xi_N) \rangle \tag{2.7}$$

The latter equation may be used as a definition of T_{mn}, even in conformal field theories where no Lagrangian formulation is available.

• Applying Noether's theorem to conformal symmetry yields the following *conserved currents*:

$$j_z = v^z T_{zz} \qquad\qquad j_{\bar z} = v^{\bar z} T_{\bar z \bar z} \tag{2.8}$$

where v^z (resp. $v^{\bar z}$) is any (locally) holomorphic (resp. anti-holomorphic) vector field. Conformal symmetry requires $\partial_{\bar z} j_z = 0$, $\partial_z j_{\bar z} = 0$, so that we must also have

$$\partial_{\bar z} T_{zz} = 0 \qquad\qquad \partial_z T_{\bar z \bar z} = 0 \tag{2.9}$$

Thus, $T(z) = T_{zz}$ and $\widetilde{T}(\bar z) = T_{\bar z \bar z}$ are complex analytic and anti-analytic respectively. In correlation functions, however, a product of $T(z)$ and another field $\phi(w)$ may produce singularities, as $z \to w$, so the above analyticity condition is to be understood as valid when T is away from other operators.

• It was shown in Gawedzki's lectures how to extract these OPE singularities from conformal Ward identities for locally flat metrics g; for the case of general metric g, the derivation is the subject of Problem Set #2. One finds

$$\begin{aligned} T(z)\phi(w,\bar w) &\sim \left(\frac{h}{(z-w)^2} + \frac{1}{z-w} \frac{\partial}{\partial w} \right) \phi(w, \bar w) \\ T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T_{ww} + \frac{1}{z-w} \partial_w T_{ww} \end{aligned} \tag{2.10}$$

where c is the *central charge* of the CFT and h is the holomorphic conformal weight of ϕ. (Analogous relations hold for $\widetilde{T}$.) As always, these relations are to be understood as valid when inserted into any correlation function. The $\sim$ sign used in (2.10) stands for the fact that we have retained only the singular part on the r.h.s. When $c=0$, $T(z)$ is a conformal tensor of weight $(2,0)$, but for $c \neq 0$, T transforms as a projective connection instead, with

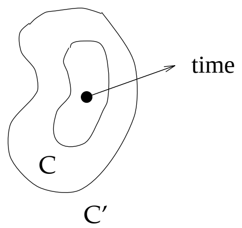
$$T'(z')(dz')^2 = T(z)(dz)^2 - \frac{c}{12}\{z';z\}(dz)^2 \tag{2.11}$$

where the Schwarzian derivative $\{\ ;\ \}$ is defined by

$$\{f;z\} \equiv f'''/f' - (3/2)(f''/f')^2, \qquad f' = \partial f/\partial z.$$

- Analytic currents $j_\alpha = j_{\alpha z}dz$, α labeling different species of currents, give rise to time-independent or conserved charges:

$$Q_\alpha(C) = \frac{1}{2\pi i}\oint_C j_\alpha$$

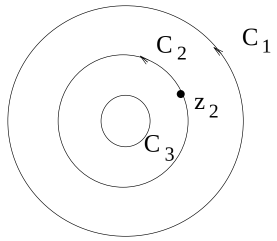


Here $Q_\alpha(C)$ is the charge enclosed in the region bounded by C and equals $Q_\alpha(C')$ provided no sources for Q_α occur between C and C'. The action of charges on fields ϕ may be evaluated using only the residues in the OPE of j_α with ϕ:

$$Q_\alpha(C)\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_C j_\alpha(z)\phi(w,\bar{w}) = \operatorname*{Res}_{z\to w} j_\alpha(z)\phi(w,\bar{w}) \tag{2.12}$$

provided no other sources for j_α lie within C. The commutator between two charges is recovered as follows:

$$\begin{aligned} &Q_\alpha(C_1)Q_\beta(C_2) - Q_\alpha(C_3)Q_\beta(C_2) \\ &\quad = [Q_\alpha, Q_\beta](C_2) \\ &\quad = \frac{1}{2\pi i}\oint_{C_2} \operatorname*{Res}_{z\to z_2} j_\alpha(z)j_\beta(z_2) \end{aligned} \tag{2.13}$$



- The conserved charges associated with conformal transformations are obtained from vector fields $v^z_{(m)} = z^{m+1}$ and $v^{\bar{z}}_{(m)} = \bar{z}^{m+1}$:

$$\begin{aligned} L_m &\equiv \oint \frac{dz}{2\pi i} z^{m+1} T(z) \\ \tilde{L}_m &\equiv \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{m+1} \widetilde{T}(\bar{z}). \end{aligned} \tag{2.14}$$

From the OPE of two T's (and two $\widetilde{T}$'s), we recover two Virasoro algebras

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\
[\tilde{L}_m, \tilde{L}_n] &= (m-n)\tilde{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}.
\end{aligned}
\tag{2.15}
$$

From the OPE of T (and $\widetilde{T}$) with a primary field ϕ of conformal weight $(h, \bar{h})$, we obtain

$$
\begin{aligned}
[L_m, \phi(w,\bar{w})] &= w^{m+1}\partial_w \phi(w,\bar{w}) + h(m+1)w^m \phi(w,\bar{w}) \\
[\tilde{L}_m, \phi(w,\bar{w})] &= \bar{w}^{m+1}\partial_{\bar{w}} \phi(w,\bar{w}) + \bar{h}(m+1)\bar{w}^m \phi(w,\bar{w})
\end{aligned}
\tag{2.16}
$$

• An important example of a CFT is the free scalar field $x \colon \Sigma \to \mathbb{R}$ (or $\mathbb{R}/\lambda\mathbb{Z}$ as discussed in Witten's lectures), with action

$$
S[x] = \frac{1}{4\pi}\int_\Sigma d^2 z\, \partial_z x \partial_{\bar{z}} x
\tag{2.17}
$$

The OPE is derived from the Green function for x:

$$
x(z,\bar{z})x(w,\bar{w}) \sim -\ln|z-w|^2
\tag{2.18}
$$

It is customary to define normal ordering by

$$
: x(z,\bar{z})x(w,\bar{w}) :\equiv x(z,\bar{z})x(w,\bar{w}) + \ln|z-w|^2
\tag{2.19}
$$

The normal ordered product admits a smooth limit as $w \to z$. The renormalized (also called the quantum) stress tensor may now be defined by

$$
T(z) = -\frac{1}{2} : (\partial_z x)^2 :\equiv -\frac{1}{2}\lim_{w\to z}\left(\partial_z x \partial_w x + \frac{1}{(z-w)^2}\right) .
\tag{2.20}
$$

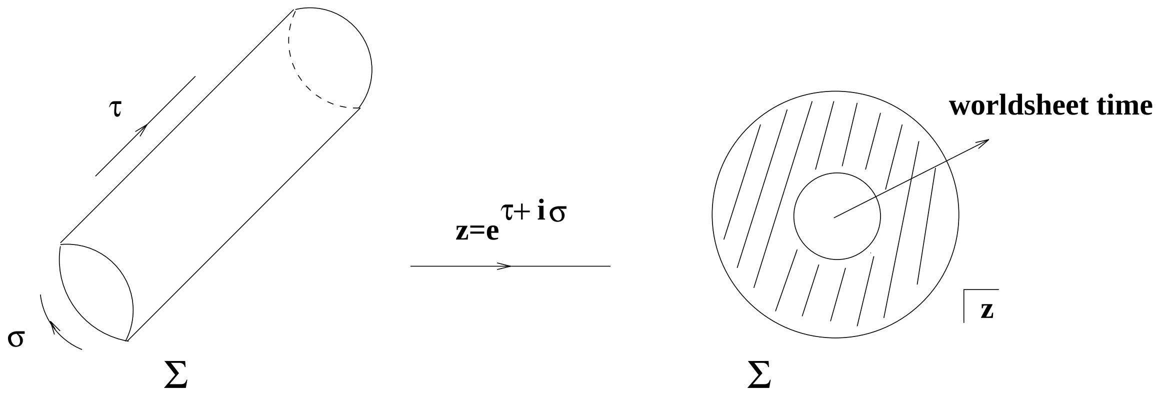
Elementary primary fields in this CFT are listed below, with their corresponding conformal weight.

$$
\begin{array}{ll}
\partial_z x & (1,0) \\
\partial_{\bar{z}} x & (0,1) \\
: \exp ikx : & \left(\frac{1}{2}k^2, \frac{1}{2}k^2\right)
\end{array}
\tag{2.21}
$$

The central charge is $c=1$. Notice that while the derivative of x and the exponential are primary fields, x itself is not a well-defined quantum field (cf. Witten's lecture).

B) **The free closed bosonic string conformal field theory**

The worldsheet topology for the free closed bosonic string is that of the cylinder or equivalently the annulus.



The metric g determines *global* complex coordinates z, $\bar{z}$ in which $g = 2g_{z\bar{z}}|dz|^2$.

Recall from 2) that our starting point is the flat Euclidean $\mathbb{R}^D_E$ worldsheet action

$$S = \frac{1}{4\pi}\int_\Sigma d^2z\, \partial_z x \cdot \partial_{\bar{z}} x \tag{2.22}$$

which for fixed metric g defines a CFT of D scalar fields x^μ, which are the coordinates of the map x: $\Sigma \to \mathbb{R}^D_E$. The quantum stress tensor is

$$T(z) = -\frac{1}{2} : \partial_z x \cdot \partial_z x : \tag{2.23}$$

and the central charge is $c = D$.

Following point 3), we obtain the OPE's and associated algebraic structure. To do this, we take advantage of the fact that $\partial_z x(z)$ and $T(z)$ are holomorphic, and thus admit a Laurent expansion inside the annulus:

$$\begin{aligned} \partial_z x(z) &= -i\sum_{m\in\mathbb{Z}} x_m z^{-m-1} & \qquad T(z) &= \sum_{m\in\mathbb{Z}} L_m z^{-m-2} \\ \partial_{\bar{z}} x(\bar{z}) &= -i\sum_{m\in\mathbb{Z}} \tilde{x}_m \bar{z}^{-m-1} & \qquad \widetilde{T}(\bar{z}) &= \sum_{m\in\mathbb{Z}} \tilde{L}_m \bar{z}^{-m-2} \end{aligned} \tag{2.24}$$

From the definition of $T \equiv -\frac{1}{2}\!: \partial_z x \cdot \partial_z x\!:$, we have

$$\begin{aligned} L_m &= \frac{1}{2}\sum_n x_{m-n}\cdot x_n \qquad\qquad m \neq 0 \\ L_0 &= \frac{1}{2}x_0^2 + \sum_{n=1}^{\infty} x_{-n}\cdot x_n \end{aligned} \tag{2.25}$$

From the OPE of $\partial_z x$ with itself and with $T(z)$, we get

$$[x^\mu_m, x^\nu_n] = m\delta_{m+n,0}\delta^{\mu\nu} \qquad [L_m, x_n] = -nx_{m+n} \ , \tag{2.26a}$$

also

$$[\tilde{x}^\mu_m, \tilde{x}^\nu_n] = m\delta_{m+n,0}\delta^{\mu\nu} \qquad [\tilde{L}_m, \tilde{x}_n] = -n\tilde{x}_{m+n} \ , \tag{2.26b}$$

while tilded and untilded operators mutually commute. L_m and $\tilde{L}_m$ satisfy Virasoro algebras with central charge $c = D$.

Following point 4), we find that the above OPE's and their algebra contents can safely be continued to $x^0 \to ix^0$ where x^0 is the time component of the map $x\colon \Sigma \to \mathbb{R}^D_M$. It is clear that this means

$$\left.\begin{array}{l} x^0_m \to ix^0_m \\ \tilde{x}^0_m \to i\tilde{x}^0_m \end{array}\right\} \qquad \text{all } m \in \mathbb{Z}. \tag{2.27}$$

The stress tensor is now

$$T(z) = -\frac{1}{2}\colon \partial_z x^\mu \partial_z x^\nu \colon \eta_{\mu\nu} \equiv -\frac{1}{2}\colon \partial_z x \cdot \partial_z x \colon \tag{2.28}$$

and the algebraic relations are

$$\begin{aligned} [x^\mu_m, x^\nu_n] &= m\delta_{m+n,0}\eta^{\mu\nu} \quad [L_m, x^\mu_n] = -nx^\mu_{m+n} \\ L_m &= \frac{1}{2}\sum_n \colon x^\mu_{m-n} x^\nu_n \colon \eta_{\mu\nu} \end{aligned} \tag{2.29}$$

and analogously for $\tilde{x}$ and $\tilde{L}$. The central charge is unchanged, and we still have $\mathrm{Vir}_L \otimes \mathrm{Vir}_R$.

The field x itself is not a conformal tensor, but may be obtained by integrating $\partial_z x$ and $\partial_{\bar{z}} x$:

$$x(z, \bar{z}) = x_L(z) + x_R(\bar{z}) \tag{2.30}$$

with

$$\begin{aligned} x_L(z) &= q_L - ix_0 \ln z + i \sum_{m\neq 0} x_m \frac{z^{-m}}{m} \\ x_R(\bar{z}) &= q_R - i\tilde{x}_0 \ln \bar{z} + i \sum_{m\neq 0} \tilde{x}_m \frac{\bar{z}^{-m}}{m}. \end{aligned} \tag{2.31}$$

Since $x : \Sigma \to \mathbb{R}^D_M$, x must be a single-valued map as a function of z. This will require that $x_0 = \tilde{x}_0 = p$, and p is to be identified with the *momentum* operator of the string, while $q = q_L + q_R$ is the overall position operator of the string.

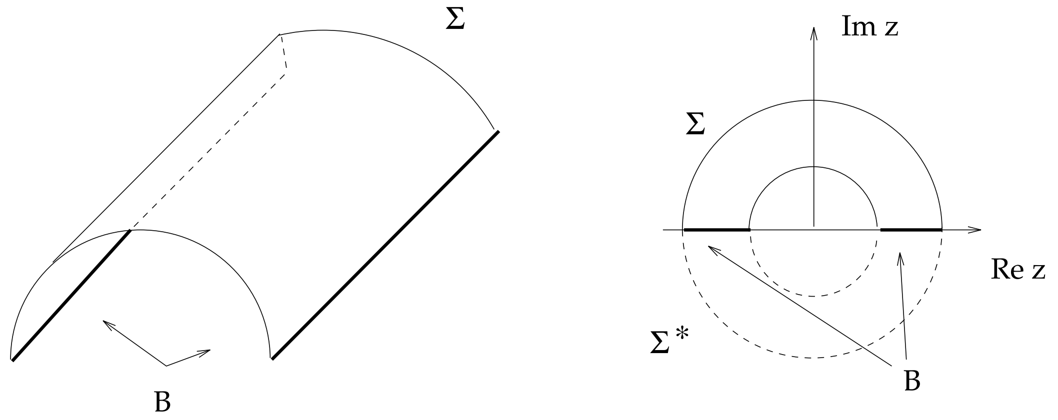
It will turn out to be useful to record the explicit form of the operators L_0 and $\tilde{L}_0$, expressed in terms of p.

$$
\begin{aligned}
L_0 &= \frac{1}{2}p^2 + \sum_{n=1}^{\infty} x_{-n} \cdot x_n \\
\tilde{L}_0 &= \frac{1}{2}p^2 + \sum_{n=1}^{\infty} \tilde{x}_{-n} \cdot \tilde{x}_n
\end{aligned}
\tag{2.32}
$$

For non-simply connected space-time manifolds M, the map x need not be single-valued, since the string can wrap around non-trivial 1-cycles in M. As a result, x_0 need not be equal to $\tilde{x}_0$, but the difference $x_0 - \tilde{x}_0$ belongs to a lattice, dual to $H^1(M, \mathbb{Z})$.

C) **The free open bosonic string conformal field theory**

The worldsheet of a free open string has the topology of a strip or a half annulus.



Fundamental open strings should have freely moving boundary points. Thus Neumann conditions $\partial_{\mathrm{Im}\, z} x = 0$ at the boundary B are appropriate. (Dirichlet b.c. would fix the end points of the string on some submanifold $\mathcal{D}$ of M. Such submanifolds are D-branes, or depending upon their space dimension p: p-branes. In certain string theories, D-branes become dynamical (extended) objects, perhaps to be discussed later on.)

It is standard to double Σ by reflecting Σ onto the lower half plane into Σ^* and then considering the union $\Sigma' = \Sigma \cup \Sigma^*$ which is the full annulus. Functions on Σ with Neumann boundary conditions on B are then even functions under $z \to \bar{z}$. Thus, the mode expansion of the open string is obtained from this restriction applied to $x(z,\bar{z})$

$$x(z,\bar{z}) = x_L(z) + x_L(\bar{z}) \tag{2.33}$$

and only involves the oscillators x_m but there are no oscillators $\tilde{x}_m$. Similarly, we only have Virasoro generators L_n but there are no $\tilde{L}_n$. As a result, the energy-momentum tensor satisfies

$$T(z) = \widetilde{T}(\bar{z}) = 0 \qquad \text{on} \qquad \mathrm{Im}(z) = 0. \tag{2.34}$$

This condition guarantees conservation of T across B, and results from the fact that Diff(Σ) leaves B invariant. We have the mode expansion

$$x(z,\bar{z}) = x + 2ip\ln|z|^2 - i\sum_{m\neq 0} x_m \left(\frac{z^{-m}}{m} + \frac{\bar{z}^{-m}}{m}\right) \tag{2.35}$$

Notice that $x_0 = 2p$, in contrast with the closed string. It is useful to also record the Virasoro generators

$$\begin{aligned} L_0 &= 2p^2 + \sum_{n=1}^{\infty} x_{-n}\cdot x_n \\ L_m &= \sum_{n\in\mathbb{Z}} x_{m-n}\cdot x_n \qquad\qquad m \neq 0 \end{aligned} \tag{2.36}$$

Thus the open string only has a single Virasoro algebra Vir_L.

D) **Fock Space, Negative Norm States**

We now follow point 5) and construct the Fock space of the conformal field theory of x, continued to Minkowski space-time. (This parallels Gawedzki's construction).

To do so, we must choose a polarization and split the operators $\{x_m^\mu\}$ and $\{\tilde{x}_m^\mu\}$ into a group of raising, lowering and central generators. This choice must be consistent with Poincaré invariance. (Also, here we must use a property inherited from the Minkowski nature of the worldsheet, namely that x_m and $\tilde{x}_m$ oscillators are independent, except at $m=0$. We shall not expand upon this issue now, but come back to this part later on.)

The correct choice is $(x_0^\mu)^\dagger = x_0^\mu$, while x_m^μ and $\tilde{x}_m^\mu$ are raising (resp. lowering) for $m < 0$ (resp. $m > 0$). Also

$$(x_m^\mu)^\dagger = x_{-m}^\mu \qquad\qquad (\tilde{x}_m^\mu)^\dagger = \tilde{x}_{-m}^\mu \tag{2.37}$$

and this implies

$$L_m^\dagger = L_{-m} \qquad\qquad \tilde{L}_m^\dagger = \tilde{L}_{-m}. \tag{2.38}$$

The Fock space $\mathfrak{F}_k$ is constructed from x_m^μ oscillators only. The ground state $|\, 0, k\rangle_L$ (also called the highest weight state) is labeled by the momentum k (the eigenvalue of the momentum operator p)

$$\begin{cases} p|\, 0, k\rangle_L = k|\, 0, k\rangle_L \\ x_m^\mu|\, 0, k\rangle_L = 0 \qquad\qquad m > 0. \end{cases} \tag{2.39}$$

We may choose its normalization as follows:

$${}_L\langle 0, k|0, k'\rangle_L = \delta(k - k')\ . \tag{2.40}$$

Arbitrary vectors in $\mathfrak{F}_k$ are obtained by applying x_{-m}^μ's for $m > 0$ to the ground state

$$|\, \varepsilon, k\rangle_L = \varepsilon_{\mu_1 \ldots \mu_n}(k; m_1, \ldots, m_n) x_{-m_1}^{\mu_1}, \ldots x_{-m_n}^{\mu_n}|\, 0, k\rangle_L \tag{2.41}$$

for all possible Lorentz *polarization tensors* $\varepsilon_{\mu_1 \ldots \mu_n}$ $(k; m_1, \ldots, m_n)$; $n \in \mathbb{N}$, with all possible $m_i \in \mathbb{N}$.

- The open string Fock space is a sum over all momenta k of $\mathfrak{F}_k$:

$$\mathfrak{F}_{\text{open}} = \bigoplus_{k \in \mathbb{R}_M^D} \mathfrak{F}_k$$

- The closed string Fock space is a sum over all momenta — which must be the same for left and right movers, in view of the constraint $x_0 = \tilde{x}_0$ — of the tensor product of $\mathfrak{F}_k$ and $\widetilde{\mathfrak{F}}_k$, where $\widetilde{\mathfrak{F}}_k$ is built from $\tilde{x}_m^\mu$:

$$\mathfrak{F}_{\text{closed}} = \bigoplus_{k \in \mathbb{R}_M^D} \mathfrak{F}_k \otimes \widetilde{\mathfrak{F}}_k.$$

• Using the norm on $\mid 0,k\rangle_L$ introduced in (2.40), $\mathcal{F}_k$ naturally inherits a quadratic form. However, $\mathcal{F}_k$ automatically contains *negative norm states*. To see this, consider states $\mid \varepsilon,k\rangle_L = \varepsilon_\mu x^\mu_{-m}\ |0,k\rangle_L$ and with $m>0$ evaluate their norm:

$$ {}_L\langle \varepsilon,k|\varepsilon,k'\rangle_L = \langle 0,k\ |\varepsilon\cdot x^\dagger_{-m}\varepsilon\cdot x_{-m}\ |0,k'\rangle = m\varepsilon^2\delta(k-k') \tag{2.42} $$

For $\varepsilon^2>0$, the norm of the state $\mid \varepsilon,k\rangle_L$ is positive. But, for $\varepsilon^2<0$, which is allowed with Minkowski metric, the norm of $\mid \varepsilon,k\rangle_L$ is negative. Such states are called *ghosts* and their presence violates quantum mechanics: they cannot be present in a physically sensible theory.

E) **Integration over** $\mathrm{Met}(\Sigma)$ — **Virasoro Constraints**

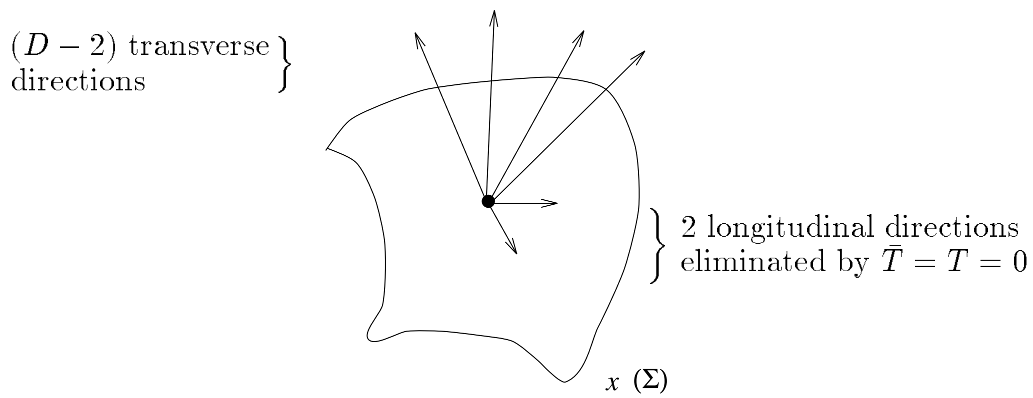
We now deal with point 6): the integration over $\mathrm{Met}(\Sigma)$, and the elimination of ghosts from the spectrum. We shall perform this integration with great care in forthcoming lectures, but we can already here extract its key effects.

The metric g enters in the action S as a non-dynamical field, i.e. without derivatives. The rôle of the metric g here is completely analogous to that of the time-like component A_0 of the gauge field in Yang-Mills theory (see also Faddeev's lecture). The effect of such non-dynamical fields is to supply a constraint on the dynamics of the remaining degrees of freedom, and to reduce their phase space. (You may loosely view the effect in the functional integral as generating a delta function $\int dA_0 e^{iA_0 x}=\delta(x)$.) The constraints also generate a group action on the phase space, and the reduced phase space may be viewed as the quotient by this group action.

Let us analyze the effect of the constraints at the *classical level* first. The variational equations are

$$ \begin{cases} \partial_{\bar z}\partial_z x = 0 & \text{by varying } x \\ T(z)=\bar T(\bar z)=0 & \text{by varying } g \end{cases} \tag{2.43} $$

The second equation is a constraint on x since $T(z)=-\frac{1}{2}\partial_z x\cdot\partial_z x$ (similarly for $\widetilde{T}(\bar z)$). Since $\partial_{\bar z}T(z)=0$, the first equation shows that T will vanish throughout Σ as long as $T=0$ has been imposed on the boundary. The action of the constraints at the classical level is something we have already analyzed, namely the Virasoro algebra but with zero central charge. Geometrically, the constraint reflects the fact that vibrations of $x(\Sigma)$ tangent to the surface $x(\Sigma)$ are eliminated, leaving only transverse direction:



We now analyze the effects of the constraints at the *quantum level*. There are basically three ways of dealing with the constraints:

1) Eliminate the longitudinal degrees of freedom at the classical level by solving for them in terms of the transverse degrees of freedom, and then quantize the transverse degrees of freedom. This procedure goes under the name of *light-cone gauge* quantization. It cannot be achieved in a manifestly Lorentz invariant parametrization. We shall discuss it very briefly later on in this lecture.

2) Ultimately, in lecture VII, we shall reformulate this constraint problem in terms of the powerful BRST quantization method. This is a more complex scheme based on introducing extra, unphysical degrees of freedom, called Faddeev-Popov ghosts, and enforcing BRST (Becchi-Rouet-Stora-Tyutin) invariance. This method is manifestly Lorentz invariant. The physical Hilbert space will in fact arise as a cohomology group of a semi-infinite cohomology complex, as developed by Feigin and later on by Freeman and Olive in physics and Frenkel, Garland and Zuckerman in mathematics.

3) Before launching into BRST, however, we shall analyze the quantum effects of the constraints in a more direct fashion, which will also yield immediate results on the spectrum. In this third method, we impose the constraint as an invariance condition and use it to

select an invariant subspace out of the full Fock space $\mathcal{F}$ (either $\mathcal{F}_{\mathrm{open}}$ for open strings or $\mathcal{F}_{\mathrm{closed}}$ for closed strings). In Yang-Mills theory, the corresponding constraint was just Gauss' law, enforcing gauge invariance on physical states. In string theory, the constraints are the vanishing of the stress tensor, enforcing Diff(Σ) invariance on physical states. Due to the central extension in Vir, the reduced Fock space $\mathcal{F}^+ = \mathcal{F}/\mathrm{Vir}$ will have a positive definite norm only when certain extra conditions are satisfied, which we now determine.

F) **Physical Spectrum, No-Ghost Theorem**

We use the Virasoro constraints to select an invariant subspace out of the full Fock space $\mathcal{F}$. It is tempting to try and impose $L_n|\,\phi\rangle = \tilde{L}_n|\,\phi\rangle = 0$ for all $n \in \mathbb{Z}$. However, unless $c = 0$, these conditions imply that $|\,\phi\rangle = 0$. Instead, we impose invariance under half of Vir. The correct physical state conditions are therefore

$$\begin{array}{llll} |\,\phi\rangle_L \in \mathcal{F}_k^{\mathrm{phys}} & \text{if} & (L_m - a\delta_{m,0})|\,\phi\rangle_L = 0 & m \in \mathbb{N} \\ |\,\phi\rangle_R \in \widetilde{\mathcal{F}}_k^{\mathrm{phys}} & \text{if} & (\tilde{L}_m - a\delta_{m,0})|\,\phi\rangle_R = 0 & m \in \mathbb{N}. \end{array} \tag{2.44}$$

and the full physical Fock spaces are obtained by

$$\mathcal{F}^{\mathrm{phys}}_{\mathrm{open}} = \bigoplus_k \mathcal{F}_k^{\mathrm{phys}} \ ,$$
$$\mathcal{F}^{\mathrm{phys}}_{\mathrm{closed}} = \bigoplus_k \mathcal{F}_k^{\mathrm{phys}} \otimes \widetilde{\mathcal{F}}_k^{\mathrm{phys}} \ .$$

The constant a is undetermined at this point, and is allowed for since L_0 and $\tilde{L}_0$ were naturally defined only up to an additive constant in view of the ordering choice we made in their definition. For closed strings, the value of a is taken the same for L_0 and $\tilde{L}_0$ so that $L_0 - \tilde{L}_0 = \frac{\partial}{\partial\sigma}$ enforces rotation invariance on the annulus. Notice that states $|\,\phi\rangle\,,\,|\,\psi\rangle \in \mathcal{F}^{\mathrm{phys}}$ automatically yield zero matrix elements of Virasoro generators $L_m - a\delta_{m,0}$ for all $m \in \mathbb{Z}$ (similarly for $\tilde{L}_m - a\delta_{m,0}$ in the case of the closed string). Hence, we have now realized the physical state condition $\langle\phi|T|\psi\rangle = \langle\phi|\widetilde{T}|\psi\rangle = 0$!

States of the form $|\,\phi\rangle = L_{-m}|\,\psi\rangle$, for $m > 0$, are called *spurious*, and we define

$$\mathcal{F}_k^{\mathrm{spurious}} = \left\{ |\,\phi\rangle = \bigoplus_{m=1}^{\infty} L_{-m}|\,\chi_m\rangle \right\}$$

It is easy to see that $\mathcal{F}_k^{\text{spurious}}$ is generated by states of the form $L_{-1}|\,\chi_1\rangle \oplus L_{-2}|\,\chi_2\rangle$, using the commutation relations of the Virasoro algebra which imply that $[L_{-1}, L_{-n}]$ is a non-zero multiple of L_{-1-n} for $n \geq 2$. Notice that spurious states are orthogonal to all physical states. States that are both physical and spurious are *null*,

$$\mathcal{F}_k^{\text{spurious}} \cap \mathcal{F}_k^{\text{phys}} \subset \mathcal{F}_k^{\text{null}} ,$$

where $\mathcal{F}_k^{\text{null}}$ stands for the space of null states of momentum k.

The special Virasoro conditions $(L_0 - a)|\,\phi\rangle = (\tilde{L}_0 - a)|\,\phi\rangle = 0$ determine the value of k^2 for a given level of operators, and thus determines the mass of the state. To see this, introduce the number (also called level) operators

$$N \equiv \sum_{n=1}^{\infty} x_{-n} \cdot x_n \qquad\qquad \widetilde{N} \equiv \sum_{n-1}^{\infty} \tilde{x}_{-n} \cdot \tilde{x}_n \ . \tag{2.45}$$

The operators N and $\widetilde{N}$ count the number of operators x_{-n} and $\tilde{x}_{-n}$, $n \geq 1$ with weight n, applied to the ground state $|\,0, k\rangle$. The special Virasoro conditions thus become

$$\begin{array}{llll} \text{closed} & (k^2 + M^2)|\,\phi\rangle = 0 & M^2 = 2N - 2a & \text{and } N = \widetilde{N}|\,\phi\rangle = 0 \\ \text{open} & (k^2 + M^2)|\,\phi\rangle = 0 & M^2 = \frac{1}{2}N - \frac{1}{2}a \ . & \end{array} \tag{2.46}$$

Thus, the mass squared of various string states are spread by $2\times$ integers for closed strings and $\frac{1}{2}\times$ integers for open strings.

The mass squared of a state is one of the Casimir values that label the representation of the Poincaré group under which the state transforms. The remaining data that fully specify the Poincaré representation are those given by the representation of the stabilizer group (i.e. the subgroup that leaves the momentum invariant). This information is contained in the expression of the Lorentz generators of the string.

$$\begin{aligned} J^{\mu\nu} = q^u p^\nu - q^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n}(x^\mu_{-n} x^\nu_n - x^\nu_{-n} x^\mu_n) \\ \left(-i \sum_{n=1}^{\infty} \frac{1}{n}(\tilde{x}^\mu_{-n} \tilde{x}^\nu_n - \tilde{x}^\nu_{-n} \tilde{x}^\mu_{-n}) \quad \text{for closed strings} \right) \end{aligned} \tag{2.47}$$

It follows that the string ground state $|\,0, k\rangle$ transforms as a scalar (i.e. trivial representation under the stabilizer).

We observed in the first lecture that, at long distances compared to the Planck length ℓ_P, the size of strings effectively becomes unobservable. We can then approximate the dynamics of the string by that of point particles, and string theory by quantum field theory. Thus, we shall seek to identify string states with states in quantum field theory. To do so, we require that string and quantum field states transform under the same representations of the Poincaré group, as well as of other symmetry groups that are available. Also their interactions should agree in this long distance limit.

We examine under what conditions the Virasoro constraints eliminate all negative norm states, so that $\mathcal{F}_k^{\rm phys}$ has positive norm. Before stating the general result, let's look at the lowest level cases (for the closed string; for the open string it suffices to let $M^2 \to M^2/4$).

(1) Ground state $|\,0,k\rangle_L$ with mass square $M^2 = -2a$ (actually $|\,0,k\rangle_L \otimes |\,0,k\rangle_R$ for the closed string). The Virasoro conditions are automatically satisfied.

(2) First excited state $|\,\varepsilon,k\rangle_L = \varepsilon\cdot x_{-1}|\,0,k\rangle_L$ with mass square $M^2 = -2a+2$:

$$L_1|\,\varepsilon,k\rangle_L = 0 \Longrightarrow \varepsilon\cdot k = 0 \ .$$

The remaining Virasoro conditions $L_n|\,\varepsilon,k\rangle_L = 0,\ n\geq 2$, are automatically satisfied. There are three possibilities:

(i) $a>1$, so that k^2 is space-like. If we pick $k=(0,k^1,0\dots 0)$ then we are allowed to choose $\varepsilon = (\varepsilon^0,0\dots 0)$, and we see that ghosts are still present.

(ii) $a<1$, so that k^2 time-like. If we pick $k=(k^0,0\dots 0)$ we see that we always have $\varepsilon^2>0$ and there are $D-1$ independent states with positive norm.

(iii) $a=1$, so that k^2 is light-like. If we pick $k=(k^1,k^1,0\dots 0)$, we see that there are $(D-2)$ massless states of positive norm, and one of zero norm.

Case (i) is clearly eliminated: Virasoro constraints do not suffice to eliminate all negative norm states. Case (ii) has massive $M^2>0$ vector states (open string), or massive rank 2 tensor states (closed string). Case (iii) has massless $M^2 = 0$ vector states (open strings), which suggests the presence of massless Yang-Mills states, and has massless $M^2 = 0$ rank 2 tensor states (closed strings) which suggests the presence of gravity (for symmetric tensors).

3) At the next level $|\, \varepsilon, k\rangle_L = \{\varepsilon_{\mu\nu} x^{\mu}_{-1} x^{\nu}_{-1} + \eta_\mu x^{\mu}_{-2}\} |\, 0, k\rangle_L$, with mass squared $M^2 = -2a + 4$ and

$$\begin{matrix} L_1 \ |\, \varepsilon, k\rangle_L = 0 \\ L_2 \ |\, \varepsilon, k\rangle_L = 0 \end{matrix} \qquad \begin{cases} \varepsilon_{\mu\nu} k^\nu + \eta_\mu = 0 \\ \\ \varepsilon^\mu_\mu + 2k_\mu \eta^\mu = 0\,. \end{cases}$$

The norm of the state is

$$_L\langle \varepsilon, k|\varepsilon, k'\rangle_L = 2\mathcal{N}_L\langle 0, k|0, k'\rangle_L$$

with

$$\begin{aligned} \mathcal{N} &= \varepsilon_{\mu\nu}\varepsilon^{\mu\nu} + \eta_\mu \eta^\mu \\ &= (\varepsilon_{00})^2 - 2(\varepsilon_{0i})^2 + (\varepsilon_{ij})^2 - (\eta_0)^2 + (\eta_i)^2. \end{aligned}$$

Choosing a rest frame where $k^0 = M$, $k^i = 0$, the Virasoro conditions become

$$\begin{cases} \varepsilon_{00} M + \eta_0 = 0 \\ \varepsilon_{i0} M + \eta_i = 0 \\ -\varepsilon_{00} + \varepsilon_{ii} + 2M\eta_0 = 0\ . \end{cases}$$

The Virasoro conditions impose no constraints on the traceless part of ε_{ij}, and its contribution to $\mathcal{N}$ is ≥ 0. Hence, we may retain only the trace part: $\varepsilon_{ij} = \varepsilon\delta_{ij}$. Eliminating η_i and η_0 and ε_{00} in favor of ε_{0i} and ε, we have

$$\mathcal{N} = \frac{\varepsilon^2(D-1)}{(2M^2+1)^2}\left\{(2M^2+1)^2 - (M^2-1)(D-1)\right\} + (M^2-2)(\varepsilon_{0i})^2.$$

Assuming $D > 1$, we find the conditions

$$\begin{cases} M^2 \geq 2 \\ \\ (2M^2+1)^2 - (M^2-1)(D-1) \geq 0\ . \end{cases}$$

the first of which is already satisfied since $M^2 = -2a + 4$ and we have seen before that we must have $a \leq 1$. For $M^2 = 2$ (corresponding to massless states at level 1), we find from the second inequality that $D \leq 26$. In fact, for $D = 26$, the second inequality implies $M^2 \leq 2$, so that we must have $M^2 = 2$ for $D = 26$.

For $D \leq 25$, the quadratic form $(2M^2+1)^2-(M^2-1)(D-1)$ is non-negative for all $M^2 \geq 2$, so the state $|\,\varepsilon, k\rangle$ always has non-negative norm.

For $D \geq 27$, the quadratic form is negative for a region in which $M^2 \geq 2$. In fact, by analyzing higher level states it can be shown that for $D \geq 27$ no value of a can be chosen so as to eliminate all ghosts. The complete result can be summarized in the following

No Ghost Theorem (R. Brower, P. Goddard and C. Thorn)**.**

- *For* $D > 26$ **or** $a > 1$

  $\mathfrak{F}^{\text{phys}}$ *contains negative norm states;*

- *For* $D = 26$ **and** $a < 1$

  $\mathfrak{F}^{\text{phys}}$ *contains negative norm states;*

- *For* $D = 26$ **and** $a = 1$

  $\mathfrak{F}^{\text{phys}}$ *contains only non-negative norm states, in particular, two towers of null states and* $\mathfrak{F}^{+} \equiv \mathfrak{F}^{\text{phys}}/\mathfrak{F}^{\text{null}}$ *has positive definite norm;*

$$\mathfrak{F}^{\text{null}} = \left\{ (L_{-2} + \frac{3}{2}L_{-1}^2)|\,\chi_2\rangle \oplus L_{-1}|\,\chi_1\rangle \right\}$$

- *For* $D < 26$ **and** $a \leq 1$

  $\mathfrak{F}^{\text{phys}}$ *but contains only positive norm states for* $a < 1$ (*contains one tower of null states* $L_{-1}|\,\chi_1\rangle$ *for* $a = 1$).

We shall not prove the no ghost theorem here, since later on, using BRST quantization, a simple proof can be given.

The case $D \leq 25$ and $a \leq 1$ is contained in $D = 26$, $a = 1$ by setting $26-D$ space-like operators to zero, and adjusting the $26-D$ positive squared momenta p_i^2, $i = 1, \ldots, 26-D$, so that

$$\frac{1}{2}\sum_i p_i^2 - a + 1 = 0 \ .$$

G) **Spectrum of the critical bosonic string with $D = 26$, $a = 1$**

Based on the masses and Lorentz tensor structure of the lowest lying states of the spectrum, we may identify which particles they naturally correspond to in field theory.

It is easy to see that the spectrum consists entirely of bosons.

Open strings

(1) States corresponding to $|\,0,k\rangle$ have $M^2 = -\frac{1}{2}$, hence they are *tachyons* (particles traveling faster than light), and Lorentz scalars;

(2) States of the form $\varepsilon \cdot x|\,0,k\rangle$ with $M^2 = 0$ are vectors under $\mathrm{SO}(1, D-1)$. There are $(D-2)$ positive norm states and one null state, which constitute a massless gauge boson in $\mathbb{R}^D$. Gauge transformations have the form $\varepsilon_\mu \to \varepsilon_\mu + k_\mu \gamma$ (invariance under these transformations is clear since they amount to shifting the state by a null state $\gamma L_{-1}|\,0,k\rangle = \gamma k \cdot x_{-1}|\,0,k\rangle$);

(3) $M^2 > 0$ states, with $M^2 = \frac{1}{2}n$, $n \in \mathbb{N}$, transform under various tensor representations of $\mathrm{SO}(1, D-1)$.

Closed strings

(1) States corresponding to $|\,0,k\rangle = |\,0,k\rangle_L \otimes |\,0,k\rangle_R$ have $M^2 = -2$, so they are *tachyons*, and Lorentz scalars;

(2) States of the form $\varepsilon_{\mu\nu} x^\mu_{-1} \tilde{x}^\nu_{-1}|\,0,k\rangle$ have $M^2 = 0$. Corresponding to the trace-part of ε there is a Lorentz scalar, the *dilaton*, with positive norm. The symmetric traceless part of ε gives a Lorentz rank 2 tensor, the *graviton*. There are $\frac{D(D+1)}{2} - 1$ such states, $\frac{D(D-1)}{2} - 1$ of them having positive norm. As for the antisymmetric part of ε, it gives a Lorentz rank 2 tensor called the *anti-symmetric tensor* with $\frac{D(D-1)}{2}$ states, out of which $\frac{(D-1)(D-2)}{2}$ have positive norm. These states exhibit gauge invariance under

$$\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \gamma_\mu k_\nu + \gamma'_\nu k_\mu \tag{2.48}$$

(which amounts to shifting by null states);

(3) $M^2 > 0$ states, with $M^2 = -2n$, $n \in \mathbb{N}$, transform under various tensor representations of $\mathrm{SO}(1, D-1)$.

H) Lightcone gauge, density of states

For $D = 26$ and $a = 1$, the null states in $\mathcal{F}^{\mathrm{null}}$ may be ignored completely. We may retain positive norm states only at the cost of giving up manifest Lorentz invariance, $\mathrm{SO}(1,25)$, by retaining $D-2 = 24$ transverse oscillators only. Let M' be a space-like sub-vector space of M, with dim $M' = 24$, and let ε, $\tilde{\varepsilon}$ be in M'; then $\varepsilon \cdot x_{-n}$ and $\widetilde{\varepsilon} \cdot \tilde{x}_{-n}$ for $n > 0$ create only positive norm states. This construction has manifest $\mathrm{SO}(24)$ invariance. It can be checked that the full Lorentz algebra $\mathrm{SO}(1,25)$ closes in this realization. Also, it can be shown in this case that there is a one-to-one correspondence between those states constructed above and the states in $\mathcal{F}^+ = \mathcal{F}^{\mathrm{phys}}/\mathcal{F}^{\mathrm{null}}$ in the covariant approach.

If the same construction were attempted for $D \neq 26$ or $a \neq 1$, one would find that the Lorentz algebra in this realization does not close.

Using the above correspondence, it is very easy to estimate the total number of states at each mass level. Introducing a generating function

$$
\begin{aligned}
G(w) &= \mathrm{Tr}_{\mathcal{F}^+}(w^{2M^2+2}) = \prod_{n=1}^{\infty} (1 - w^n)^{-24} \\
&\equiv \sum_{n=1}^{\infty} d_n w^{2n} \ .
\end{aligned}
\tag{2.49}
$$

We find the number of states asymptotically for large M^2: $d_{M^2} = M^{-25/2} e^{M/4\pi\sqrt{2}}$.

I) Primary fields and Vertex Operators for Physical States

The construction given above for the physical Hilbert space has the disadvantage that the operators x^μ_{-n} with $n > 0$, create states in Fock space that are not in general physical. We now construct *vertex operators*, which have the property that they send physical states into physical states, and spurious states into other spurious states.

Since physical states obey $(L_n - \delta_{n,0})|\,\psi\rangle = (\tilde{L}_n - \delta_{n,0})|\,\psi\rangle = 0$, $n \geq 0$, it is clear that any operator commuting with the entire Virasoro algebra will yield such a vertex operator.

Also, since $[L_n, V] = [\tilde{L}_n, V] = 0$, V automatically sends spurious states into spurious states.

Since L_n generates conformal reparametrizations, V should be conformal invariant. To construct V's, we consider general tensors, then construct invariants. We use the action of L_n on any *primary field*:

$$\begin{aligned}
[L_n, \phi(z,\bar{z})] &= z^{n+1}\partial_z\phi(z,\bar{z}) + h(n+1)z^n\phi(z,\bar{z}) \\
&= \partial_z(z^{n+1}\phi(z,\bar{z})) + (h-1)(n+1)z^n\phi(z,\bar{z}) \\
[\tilde{L}_n, \phi(z,\bar{z})] &= \partial_{\bar{z}}(\bar{z}^{n+1}\phi(z,\bar{z})) + (\bar{h}-1)(n+1)\bar{z}^n\phi(z,\bar{z}) \ .
\end{aligned} \tag{2.50}$$

It is clear that amongst primary fields, the only operator that commutes with all of L_n and $\tilde{L}_n$ is the unit operator $\phi = 1$.

However, integrals of ϕ may be invariant. For any primary field of conformal weight $(h, \bar{h}) = (1,1)$, its commutators with L_n, $\tilde{L}_n$ are total derivatives, and so its integral will be Virasoro invariant. In addition, primary fields ϕ of weight $(1,1)$ form natural volume forms on Σ. Hence the integral of ϕ

$$V = \int \phi(z,\bar{z}) \qquad \text{satisfies} \qquad [L_n, V] = [\tilde{L}_n, V] = 0 \ . \tag{2.51}$$

This is not the only way to construct vertex operators though: if ϕ is a holomorphic primary field (e.g. $\partial_z x^\mu$), then it will commute with $\tilde{L}_n$ provided $\bar{h} = 0$ and its commutator with L_n is a total derivative if $h = 1$. In addition $(1,0)$ and $(0,1)$ forms are naturally integrated along contours. Thus, the operators

$$\oint dz\, \phi_{(1,0)}$$

and

$$\oint d\bar{z}\, \phi_{(0,1)}$$

are also vertex operators provided that $\partial_{\bar{z}}\phi_{(1,0)} = 0$ and $\partial_z\phi_{(0,1)} = 0$. The operators $\phi_{(1,0)}$ and $\phi_{(0,1)}$ are precisely what we defined previously as conformal currents; their charges are conserved. Such operators do not in fact create new physical states, as will become clear by considering examples below.

Vertex operators for the closed bosonic string are now easily reconstructed from the basic field x^μ associated with D copies of the $c = 1$ CFT. We have the following primary fields

$$\begin{array}{lll} \partial_z x^\mu & \text{holomorphic, weight} & (1,0) \\ \partial_{\bar z} x^\mu & \text{anti-holomorphic, weight} & (0,1) \\ :e^{ik\cdot x}: & \text{weight } \left(\frac{k^2}{2}, \frac{k^2}{2}\right) & \end{array}$$

From the holomorphic current $\partial_x x^\mu$ we get the conserved charge of *momentum*:

$$p^\mu = \frac{1}{2\pi} \oint dz\, \partial_z x^\mu \ . \tag{2.52}$$

Physical states in $\mathcal{F}_k^{\text{phys}}$ will just be eigenstates of p^μ, so p^μ does not really create any new states, but simply measures their momentum k.

A single exponential yields a vertex operator for $k^2 = 2$, i.e. when the conformal weight is $(1,1)$:

$$V_T(k) = \int_\Sigma d^2 z \ :e^{ikx}: \tag{2.53}$$

This scalar operator creates a state of mass $M^2 = -2$, which is the tachyon discovered previously. But the exponential may also be combined with derivatives for $k^2 = 0$

$$V_{G,B,\Phi}(k) = \int_\Sigma d^2 z\, \varepsilon_{\mu\bar\mu}(k)\, \partial_z x^\mu \partial_{\bar z} x^{\bar\mu} \ :e^{ik\cdot x}: \tag{2.54}$$

which reproduces the states for the graviton, the antisymmetric tensor, and the dilaton as symmetric traceless, anti-symmetric, and trace part respectively, with $k_\mu \varepsilon^{\mu\bar\mu}(k) = k_{\bar\mu}\varepsilon^{\mu\bar\mu}(k) = 0$. For higher powers of $\partial_z x^\mu$ and $\partial_{\bar z} x^{\bar\mu}$, we will have masses of the form $M^2 = 2n - 2$ in general. However, higher derivatives $\partial_z^2 x^\mu$ may also appear. While these are not primary fields (they mix with $\partial_z x$ under conformal reparametrizations), they may combine with other operators and still yield primary fields.

When we obtained the Virasoro conditions on the polarization tensors $\varepsilon_{\mu\bar\mu}$, we established an invariance of the states in $\mathcal{F}^{\text{phys}}$ under the transformations

$$\varepsilon_{\mu\bar\mu} \to \varepsilon_{\mu\bar\mu} + k_\mu \bar\gamma_{\bar\mu} + k_{\bar\mu}\gamma_\mu \tag{2.55}$$

(We shall shortly interpret this invariance as a manifestation of Diff(M)-invariance for the closed string.) This invariance is also easily established for the vertex operators: for example a shift by $\bar\gamma$ yields

$$\begin{aligned}\int_\Sigma d^2z\, k_\mu \bar\gamma_{\bar\mu} \partial_z x^\mu \partial_{\bar z} x^{\bar\mu} \ : e^{ikx} : &= -i \int_\Sigma d^2z \bar\gamma_{\bar\mu} \partial_{\bar z} x^{\bar\mu} (\partial_z e^{ikx}) \\ &= i \int_\Sigma d^2z \bar\gamma_{\bar\mu} (\partial_z \partial_{\bar z} x^{\bar\mu}) e^{ik\cdot x}\end{aligned} \tag{3.56}$$

But we have $\partial_z \partial_{\bar z} x^\mu = 0$ by the field equations — away from any other singularities — and thus the integral vanishes. (The effects of the singularities may also be taken into account, as we shall show later on and do not modify the conclusion above.)

J) Identifying the graviton: vertex operators from background fields

One of the most important discoveries in string theory is that the $k^2 = 0$, $\varepsilon_{\mu\bar\mu}(k)$-symmetric traceless physical state, found above, should be identified with the graviton. This is the particle in quantum field theory that represents excitations of the space-time metric and that mediates the force of gravity. Here, we provide further evidence for this identification. Along the same lines, we provide evidence for the identification of the open string $k^2 = 0$, $\varepsilon_\mu(k)$-state with the Yang-Mills particle in this section and in §K.

Consider a space-time M which is almost flat with the following metric

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + \varepsilon_{\mu\nu}(x) \tag{2.57}$$

where $\varepsilon_{\mu\nu}(x)$ is everywhere small compared to $\eta_{\mu\nu}$. The tensor $\varepsilon_{\mu\nu}(x)$ may be viewed as a superposition of plane waves by Fourier transform:

$$\varepsilon_{\mu\nu}(x) = \int d^{26}k\ \varepsilon_{\mu\nu}(k) e^{ik\cdot x} \tag{2.58}$$

Now, consider string theory in this background: the action on the worldsheet is given by

$$S_G = \frac{1}{8\pi} \int_\Sigma d\mu_g\, g^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x) \tag{2.59}$$

and the transition amplitude involves the exponential of S_G. Since $\varepsilon_{\mu\nu}$ is small, we may expand in powers of $\varepsilon_{\mu\nu}$:

$$e^{-S_G} = e^{-S_\eta} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} S_n \frac{1}{(8\pi)^n} \tag{2.60}$$

with S_η the action corresponding to the flat metric and S_n defined by

$$S_n = \int d^{26}k_1 \ldots d^{26}k_n V(k_1) \ldots V(k_n) \tag{2.61}$$

where

$$V(k) = \varepsilon_{\mu\nu}(k) \int_\Sigma d\mu_g g^{mn} \partial_m x^\mu \partial_n x^\nu e^{ikx}$$

The operator $V(k)$ is precisely the vertex operator we constructed for the graviton — written now in general metric g — for a definite momentum k.

Thus, we see that strings couple to the fluctuations in the background metric precisely by the graviton vertex operator. The plane waves may be regarded as creating excitations in the background space-time metric, which make it deviate from the flat metric. (A completely analogous correspondence holds for the other massless states, the dilaton and the anti-symmetric tensor.)

We have however established more precise information: in order to maintain conformal invariance and eliminate negative norm states, the momentum k and the polarization tensor $\varepsilon_{\mu\nu}$ had to satisfy *on-shell conditions*

$$\begin{aligned} k^2\varepsilon_{\mu\nu}(k) = 0 &\Longrightarrow \Box\varepsilon_{\mu\nu}(x) = \Box G_{\mu\nu}(x) = 0 \\ k^\mu\varepsilon_{\mu\nu}(k) = 0 &\Longrightarrow \partial^\mu\varepsilon_{\mu\nu}(x) = \partial^\mu G_{\mu\nu}(x) = 0 \\ \varepsilon^\mu_\mu(k) = 0 &\Longrightarrow \varepsilon^\mu_\mu(x) = 0 \ , \end{aligned} \tag{2.62}$$

where we have listed to the right of the double arrows the restatement of the conditions on the metric.

Now, since $G_{\mu\nu}$ is almost flat (ε being small), corrections quadratic and higher in ε may be neglected. Considering the Ricci tensor in this approximation, we have

$$\begin{aligned} R_{\mu\nu} &= R^\lambda_{\mu\lambda\nu} = \partial_\nu\Gamma^\lambda_{\mu\lambda} - \partial_\lambda\Gamma^\lambda_{\mu\nu} + O(\varepsilon^2) \\ 2R_{\mu\nu} &= \partial_\mu\partial_\nu\varepsilon^\lambda_\lambda - \partial_\lambda\partial_\mu\varepsilon^\lambda_\nu - \partial_\lambda\partial_\nu\varepsilon^\lambda_\mu + \Box\varepsilon_{\mu\nu} \end{aligned} \tag{2.63}$$

The conditions on $G_{\mu\nu}$ obtained from the on-shell conditions of $\varepsilon_{\mu\nu}(k)$ now imply that

$$R_{\mu\nu} = 0 \ , \tag{2.64}$$

within the almost flat (or linearized) approximation.

In addition, the fact that shifts by null states are physically unobservable implies the gauge invariance of the physical states in $\mathcal{F}^+$ under the shifts

$$\varepsilon_{\mu\nu}(k) \longrightarrow \varepsilon_{\mu\nu}(k) + k_\mu \gamma_\nu + k_\nu \gamma_\mu \tag{2.65}$$

(with $k^\mu \gamma_\mu = 0$ for on-shell states). In terms of the metric, this gauge invariance translates into

$$\varepsilon_{\mu\nu}(x) \to \varepsilon_{\mu\nu}(x) + \partial_\mu \gamma_\nu(x) + \partial_\nu \gamma_\mu(x). \tag{2.66}$$

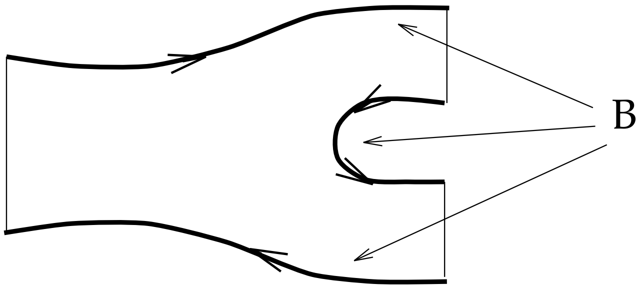
We recognize this transformation as an infinitesimal diffeomorphism generated by the vector field $\gamma^\mu(x)$, in the approximation where $O(\varepsilon^2)$ terms are neglected. $R_{\mu\nu} = 0$ is invariant under these transformations, as can be checked explicitly.

In fact $R_{\mu\nu} = 0$ is the only Diff(M)-invariant equation that reduces to $\Box G_{\mu\nu}(x) = 0$ in the linearized limit. Thus, we find that — to leading order in metric fluctuations – the conformal invariance constraints translate into Ricci flatness, i.e. Einstein's equations in the absence of matter. This discovery supports our proposed identification of $\varepsilon_{\mu\nu}(k)$, $k^2 = 0$ string states with gravitons.

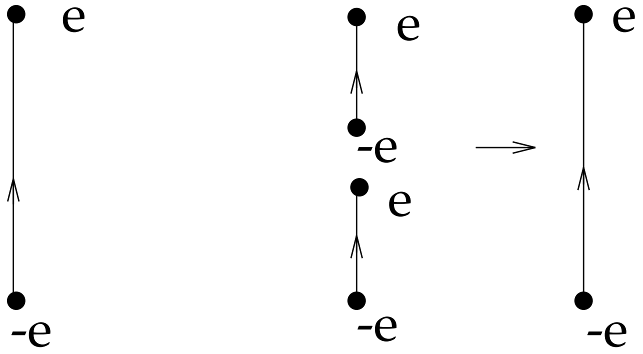
Later on (Lecture VIII), we shall further confirm this identification in another important limit, that of small variations in the derivatives of the metric, the so-called low energy limit.

K) **Internal degrees of freedom of open strings: Chan-Paton rules**

The end points of the open string are distinguished points. It is consistent with Poincaré and 2-d conformal invariance to put degrees of freedom or *charges* at the end points. During interactions of several strings, the charges simply flow along the boundary, and should thus be *conserved.*



The simplest case corresponds to adding *electric charge* to the end points of oriented strings:



Consistency with the interactions requires that the charges be opposite (i.e. corresonding to conjugate representatives) at the two ends. An external $U(1)$ gauge field A_μ couples to these charges just in the same way as it couples to point particles with electric charge $\pm e$, moving on worldlines:

$$S_A = ie \int_B d\tau\, \dot{x}^\mu(\tau) A_\mu(x(\tau)). \tag{2.67}$$

Following the derivation of vertex operators from the coupling to background fields that we used for the closed string, we easily obtain the vertex operators for the open string gauge states, by decomposing A_μ into Fourier modes

$$A_\mu(x) = \int d^D k\, \varepsilon_\mu(k) e^{ik\cdot x} \ , \tag{2.68}$$

where $\varepsilon_\mu(k)$ is the polarization tensor. Thus, the vertex operators are

$$V(\varepsilon, k) = \varepsilon_\mu(k) \int_B d\tau\, \dot{x}^\mu(\tau) e^{ik\cdot x(\tau)} \tag{2.69}$$

with conformal invariance for $k^2 = 0$.

We may add further degrees of freedom to the end points of the string in the following way. Let λ^i, $i = 1, \ldots, N$ be Grassmann valued fields that live only on the boundary B of the string worldsheet (recall that Neumann b.c. were imposed on B for the map x: $\Sigma \to M$). A natural candidate for the action for such degrees of freedom is

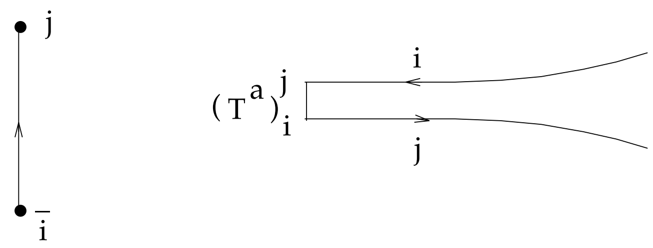
$$S_0(\lambda) = \int_B d\tau\, \bar\lambda^i \frac{d}{d\tau}\lambda_i \ . \tag{2.70}$$

This action is Diff(B) invariant, and does not depend upon the Σ metric. The equation of motion for λ_i simply requires that λ_i be τ-indepenent, so that λ_i may indeed be interpreted as conserved charges. The action S_0 automatically has SO(N) symmetry when λ_i are real (and $\bar\lambda^i = \delta^{ij}\lambda_j$) and SO($2N$) symmetry when λ_i is complex. As a result, quantization automatically produces a Hilbert space of states on which a spinor representation of SO(N) (for λ_i real) or SO($2N$) (for λ_i complex) acts.

To restrict to a subgroup G of SO(N) or SO($2N$), and/or a subrepresentation Λ of the corresponding spinor representations, it is necessary to restrict the Hilbert space of states. This is conveniently achieved by introducing a Lagrange multiplier, σ: $B \to \mathbb{R}$. For $G = U(N)$, for example, the restriction to the rank p $(0 \le p \le N)$ totally anti-symmetric tensor representation of $U(N)$, we use the following modification of S_0:

$$S(\lambda) = \int_B d\tau \left(\bar\lambda^i \frac{d}{d\tau}\lambda_i + \sigma(\tau)(\bar\lambda^i\lambda_i - p) \right) \tag{2.71}$$

Analogous modifications may be used for $G =$ SO(N) and Sp(N). Let us now restrict to $G = U(N)$, and $p = 1$, so that Λ equals the defining representation. Quantization of λ_i generates N conserved charges attached to the boundaries of the string



The string for $U(N)$ gauge group should be *oriented*, since the charges at both ends of the string transform under inequivalent representations of $U(N)$. The string state now transforms under the representation $\bar{\Lambda} \otimes \Lambda$ of $U(N)$, which is the adjoint of $U(N)$. The representation matrices T^a, with $a = 1, \ldots, \dim U(N) = N^2$ describe the string state belonging to the representation $\bar{\Lambda} \otimes \Lambda$, by the index a. The matrix elements $(T^a)_i^j$ of T^a specify which λ_i and $\bar{\lambda}^j$ are created at the boundary B of the open string. The full Fock space of the open string is given by $\mathcal{F}_{\text{open}} \otimes \bar{\Lambda} \otimes \Lambda$, and the physical state space is $\mathcal{F}_{\text{open}}^{\text{phys}} \otimes \bar{\Lambda} \otimes \Lambda$.

We observe that for $G = U(N)$ and Λ the defining representation, $\bar{\Lambda} \otimes \Lambda$ transforms in the adjoint representation and the open string states are thus natural candidates for Yang-Mills particles with gauge group $G = U(N)$. Had we considered any other representation for Λ (other than the defining or its complex conjugate), the string states would not have tranformed precisely under the adjoint representation of $U(N)$, and could not have been interpreted as Yang-Mills particles. Thus the defining representation is singled out by its relevance to describing Yang-Mills particles.

The strings for $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ should be unoriented, since the charges at both ends transform under equivalent representations of $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$, respectively. Considering Λ to be the defining representation of $\mathrm{SO}(N)$, we obtain string states in the adjoint of $\mathrm{SO}(N)$ by the anti-symmetrized tensor product of Λ by itself $(\Lambda \otimes \Lambda)_A$. On the other hand, for Λ the defining representation of $\mathrm{Sp}(N)$, we obtain the string states in the adjoint of $\mathrm{Sp}(N)$ by the symmetrized tensor product of Λ by itself $(\Lambda \otimes \Lambda)_S$. In each case we recover precisely the string states suitable for Yang-Mills particles.

For exceptional groups, no construction is known that produces string states in the adjoint representation only, and it is widely believed that it is impossible to obtain Yang-Mills particles from open strings with these groups.

A non-Abelian gauge field can couple to the boundaries of the open string, much in the same way as this happens for Abelian gauge fields. Here, however, we must include the coupling of the charges of the gauge field A_μ^a to those of the string. Let us restrict again to the defining representation of $U(N)$. The coupling of A^a_μ is given as follows:

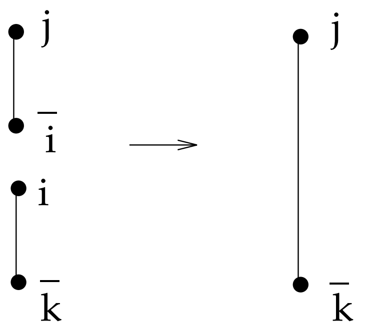
$$S_A(\lambda) = \int_B d\tau \bar{\lambda}^i \left(\delta^j_i \frac{d}{d\tau} + i A^a_\mu \dot{x}^\mu (T^a)^j_i \right) \lambda_j + \sigma(\tau)(\bar{\lambda}^i \lambda_i - 1) \ . \tag{2.72}$$

The coupling to the gauge field A^a_μ may again be used to construct the vertex operators for the creation and annihilation of gauge states of the string. One finds upon quantization that the vertex opertor for the $U(1)$ case is simply to be multiplied by the matrices $(T^a)^j_i$:

$$V^a(\varepsilon, k) = T^a V(\varepsilon, k) \ . \tag{2.73}$$

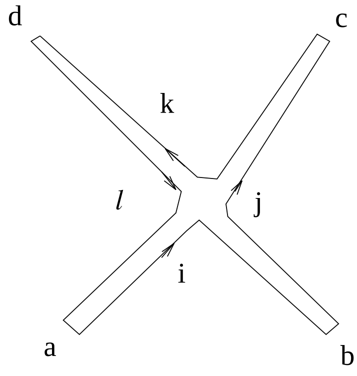
As the internal degree of freedom and the matrices T^a commute with Virasoro, the physical state conditions on V^a are satisfied whenever they are on $V(\varepsilon, k)$.

It is instructive to examine the effects of these degrees of freedom on interactions as well. For simplicity, we restrict to $A^a_\mu = 0$. Charges then simply continue to be translated along the boundary B in a conserved way, as shown schematically in the figure below.



Interactions are only allowed if charges are conserved.

In a full string diagram, the effect will be to include a trace over the matrices T, just as in Yang-Mills Feynman rules, as shown schematically below.



$\sim \operatorname{tr} T^a \, T^b \, T^c \, T^d$

With the help of the Yang-Mills vertex operators V^a of (2.73) and (2.69), one recovers Yang-Mills scattering amplitudes in the limit where momenta are small compared to the Planch length.