— DRAFT — IASSNS-HEP-97/72

Lectures 5, 6

III. String Amplitudes and Moduli Space of Curves

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In the previous Chapter, we have quantized free bosonic strings, determined their spectrum and Hilbert space of states under certain conditions: $D = 26$ and the presence of massless gravitons and Yang-Mills particles. (The case $D \leq 25$, allowed by the no-ghost theorem, was shown to be a subcase.)

Now, we shall quantize interacting strings, and investigate quantum mechanical transition amplitudes for the scattering of string states. While operator methods were convenient for determining the spectrum of free strings, the functional integral formulation in terms of a summation over surfaces is better suited for dealing with interactions. Thus, our starting point is the transition amplitude

$$A = \sum_{\substack{\text{topologies } \Sigma \\ (h=0,\ldots,\infty)}} \int_{\mathrm{Met}(\Sigma)} Dg \frac{1}{\mathcal{N}(g)} \int_{\mathrm{Diff}(\Sigma,M)} V_1 \ldots V_N e^{-S[x,g]} \tag{3.1}$$

with Riemannian space-time M, and external string states represented by vertex operators V_i, $i = 1, \ldots, N$. This was precisely our original starting point with Euclidean signature metric on M. Minkowski amplitudes (for example, for flat $M = \mathbb{R}^D_M$) are obtained by analytic continuation in external momenta and polarization tensors. The vertex operators were determined in §II in terms of conformal primary fields of weight $(1,1)$. The summation in the transition amplitude A is over topologies of compact oriented surfaces, since we shall limit ourselves here to the study of the closed oriented string.

The strategy is as follows:

1) At fixed topology of Σ, we use the product structure of $\mathrm{Met}(\Sigma) = \mathrm{Weyl}(\Sigma) \times$ conformal classes (Σ) to decompose the measure Dg on $\mathrm{Met}(\Sigma)$ into a product of measures on the orbits of Weyl, on the orbits of $\mathrm{Diff}(\Sigma)$ and on the moduli space $\mathcal{M}_h = \mathrm{Met}(\Sigma)/\mathrm{Diff}(\Sigma) \ltimes$

Weyl(Σ). A Jacobian determinant arises, which is essentially the Faddeev-Popov determinant. This decomposition holds for any matter conformal field theory, in particular for any space-time M, defining a conformal field theory.

2) For flat Euclidean space-time M, the combined measure — including the Faddeev-Popov determinant — and the correlation functions, are found to be Weyl-invariant (i.e. Weyl(Σ) is a symmetry group) if and only if $D = 26$ and gravitons are massless on surfaces of any topology (equivalently, the $a = 1$ condition of §II holds).

3) Under the assumptions of 2) the string theory is usually called critical. In this case, the formal measure $Dg/\mathcal{N}(g)$ on Met(Σ), divided by the volume of the orbits of the symmetry group Diff(Σ) $\ltimes$ Weyl(Σ), *naturally projects down* to an *intrinsically defined measure on* $\mathcal{M}_h$. Thus, the transition amplitudes for critical strings reduce to well-defined integrals over $\mathcal{M}_h$ of the matter conformal field theory correlation functions.

4) More generally, the assumptions of 2) may be relaxed, and the matter part may be any conformal field theory of central charge $c = 26$ and with correlation functions satisfying certain Weyl invariance conditions.

5) In particular, in Lecture 9, we shall generalize the Weyl invariance conditions to the case of curved space-time manifolds M — in certain limits.

6) Under the assumptions of 2), we shall obtain in this chapter 0- and 1-loop transition amplitudes for critical bosonic string theory.

A. Finite-Dimensional Case

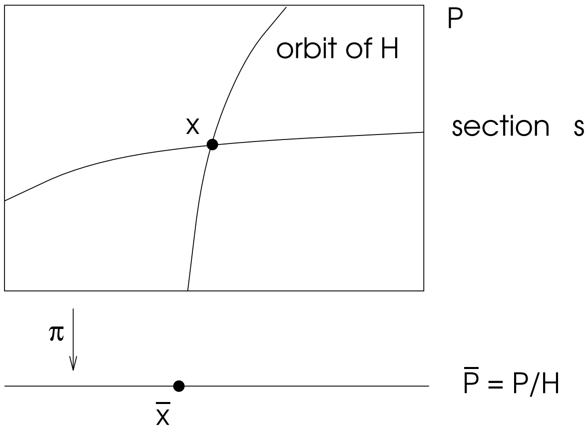
Let P be a finite-dimensional manifold, with Riemannian metric G, and associated measure $d\mu_G$. Let H be a Lie group acting on P, and assume that $d\mu_G$ is H-invariant. We wish to reduce the integrals of H-invariant functions on P to integrals on the quotient $\bar{P} = P/H$. Let $f(x)$: $P \to \mathbb{C}$ be H-invariant,

$$f(xh) = f(x) \qquad \text{for all} \qquad x \in P, \ h \in H.$$

Then for any section $s\colon \bar{P} \to P$, we have

$$\int_P d\mu_G f = \int_{\bar{P}} d\bar{\mu}\ s^*(f) J \int_H d\nu_H \ ,$$

where $d\bar{\mu}$ is the measure on $\bar{P}$ induced from $d\mu_G$, $d\nu_H$ is the left-invariant measure on H, and J is a Jacobian (or Faddeev-Popov) determinant which we define below, as follows. First, let us present a figure:



The tangent space at x to P may be decomposed into the part tangent to the orbit of H, generated by the Lie algebra $\mathcal{H}$,

$$i\colon \mathcal{H} \to T_x P$$

and a part from the push-forward of the tangent space to $T_{\bar{x}}\bar{P}$:

$$s_*\colon T_{\bar{x}}\bar{P} \to T_x(P) \ ,$$

as follows:

$$s_* \oplus i\colon T_{\bar{x}}\bar{P} \oplus \mathcal{H} \to T_x(P).$$

The Jacobian J is now obtained as

$$J = \det(T_{\bar{x}}\bar{P} \oplus \mathcal{H} \to T_x P).$$

B. Basic Notation: Tensors, Derivatives

Fix a positive definite metric g on Σ and choose local complex coordinates z, $\bar{z}$ in which

$$g = g_{z\bar{z}}(dz \otimes d\bar{z} + d\bar{z} \otimes dz) = 2g_{z\bar{z}}|dz|^2$$

Tensors decompose into one-dimensional components of weight (m, n), which are sections of a line bundle $K^{(m,n)}$:

$$\phi = \phi_{\underbrace{z\ldots z}_{m}\,\underbrace{\bar{z}\ldots\bar{z}}_{n}}(dz)^m(d\bar{z})^n \in K^{(m,n)} \ .$$

Using the metric g, we identify tensors of weight (m, n) with tensors of weight $(m-n, 0)$. The space of tensors of weight $(m, 0)$ is denoted K^m, with K^1 the space of sections of the canonical line bundle $T^*_{(1,0)}(\Sigma)$. (Generally, we do not distinguish between a bundle and the space of its sections.)

On sections of K^m we have the L^2 inner product

$$(\phi_1, \phi_2)_g = \int_\Sigma d\mu_g (g_{z\bar{z}})^{-m} \phi_1^* \phi_2 \tag{3.2}$$

as well as the covariant derivatives

$$\nabla\phi = \nabla_{\bar{z}}^{(m)}\phi + \nabla_z^{(m)}\phi$$

where $\nabla_{\bar{z}}^{(m)} = \bar{\partial}_{(m)}$ is the Cauchy Riemannn operator

$$\nabla_{\bar{z}}^{(m)}\colon K^m \to K^{(m,1)} \ \nabla_{\bar{z}}^{(m)}\phi = \frac{\partial}{\partial\bar{z}}\,\phi \otimes d\bar{z} \ . \tag{3.3}$$

Equivalently, identifying $(m, 1) \sim (m-1, 0)$, we have

$$\begin{cases} \nabla^z_{(m)}\colon K^m \to K^{m-1} & \nabla^z_{(m)}\phi = (g_{z\bar{z}})^{-1}\frac{\partial}{\partial\bar{z}}\phi \otimes (dz)^{-1} \\ \nabla_z^{(m)}\colon K^m \to K^{m+1} & \nabla_z^{(m)}\phi = (g_{z\bar{z}})^m \frac{\partial}{\partial z}(g_{z\bar{z}})^{-m}\phi \otimes dz \ . \end{cases} \tag{3.4}$$

(In general, we shall not exhibit the differentials dz and $d\bar{z}$ in ∇.) The adjoints of these operators with respect to $(\quad,\quad)_g$ are

$$\left(\nabla^z_{(m)}\right)^\dagger = -\nabla_z^{(m-1)}; \quad \left(\nabla_z^{(m)}\right)^\dagger = -\nabla^z_{(m+1)} \tag{3.5}$$

and the Laplace operators are

$$\begin{aligned}\Delta^{+}_{(m)} &= -2\nabla^{z}_{(m+1)}\nabla^{(m)}_{z} \\ \Delta^{-}_{(m)} &= -2\nabla^{(m-1)}_{z}\nabla^{z}_{(m)}\end{aligned} \tag{3.6}$$

We also make use of the Riemann-Roch-Atiyah-Singer theorem

$$\dim \mathrm{Ker}\nabla^{(m)}_{z} - \dim \mathrm{Ker}\nabla^{z}_{(m+1)} = \left(m+\frac{1}{2}\right)\chi(\Sigma) = (2m+1)(1-h) \ , \tag{3.7}$$

as well as the vanishing theorems:

$$\begin{aligned} &\bullet \quad h=0, m\geq 1, && \mathrm{Ker}\ \nabla^{(m)}_{\bar z} = 0 \\ & && \text{(no holomorphic forms on the sphere)} \\ &\bullet \quad h\geq 2, m\leq -1, && \mathrm{Ker}\ \nabla^{(m)}_{\bar z} = 0\ . \\ & && \text{(no holomorphic vector fields for } h\geq 2) \end{aligned} \tag{3.8}$$

C. Space of Metrics – Moduli Space of Riemann Surfaces

Let $\mathrm{Met}(\Sigma)$ denote the space of positive definite metrics on Σ, and $T_g\mathrm{Met}(\Sigma)$ its tangent space at g, consisting of infinitesimal deformations $\delta g \in K^{(1,1)} \oplus K^{(2,0)} \oplus K^{(0,2)}$. Following the results of §B, we have an L^2 norm $\|\delta g\|^2_g = (\delta g, \delta g)_g$ on $T_g\mathrm{Met}(\Sigma)$, which makes $\mathrm{Met}(\Sigma)$ into an infinite dimensional Riemannian manifold. The measure Dg is defined with respect to $\|\delta g\|$.

$\mathrm{Diff}(\Sigma)$ acts isometrically on $\mathrm{Met}(\Sigma)$, but $\mathrm{Weyl}(\Sigma)$ does not. The space of orbits of $\mathrm{Diff}(\Sigma) \ltimes \mathrm{Weyl}(\Sigma)$ in $\mathrm{Met}(\Sigma)$ is the *moduli space* of Riemann surfaces (or complex curves) of genus h:

$$\mathcal{M}_h \equiv \mathrm{Met}(\Sigma)/\mathrm{Diff}(\Sigma) \ltimes \mathrm{Weyl}(\Sigma)\ . \tag{3.9}$$

Actually, the group $\mathrm{Diff}(\Sigma)$ is not connected. The group of connected components is the *mapping class group* G_Σ:

$$G_\Sigma = \pi_0(\mathrm{Diff}(\Sigma))\ ,$$

which is an infinite discrete group for $h\geq 1$. The component connected to the identity in $\mathrm{Diff}(\Sigma)$ is denoted by $\mathrm{Diff}_0(\Sigma)$. Thus, there arises another natural space: J_h, which is Teichmüller space

$$\mathcal{T}_h \equiv \mathrm{Met}(\Sigma)/\mathrm{Diff}_0(\Sigma) \ltimes \mathrm{Weyl}(\Sigma)\ ,$$

which is the simply connected covering space of $\mathcal{M}_h$:

$$\mathcal{M}_h = \mathcal{T}_n / G_\Sigma \ .$$

Teichmüller space is a complex finite dimensional manifold, while moduli space is a complex finite dimensional orbifold (the action of G_Σ on $\mathcal{T}_h$ produces singularities), both of dimension

$$\dim_{\mathbb{C}} \mathcal{T}_h = \dim_{\mathbb{C}} \mathcal{M}_h = \begin{cases} 0 & h = 0 \\ 1 & h = 1 \\ 3h-3 & h \geq 2 \ . \end{cases} \tag{3.10}$$

We have explained that string transition amplitudes can be reduced to an integral over moduli space. We wish to factorize the measure Dg into its components along $\mathrm{Diff}(\Sigma)$, $\mathrm{Weyl}(\Sigma)$ and $\mathcal{M}_h$, i.e. we want to fix the gauge. To examine this factorization concretely, we decompose $T_g\mathrm{Met}(\Sigma)$ into the tangent spaces to the orbits of $\mathrm{Diff}_0(\Sigma)$ and $\mathrm{Weyl}(\Sigma)$, as explained in the finite dimensional example. We perform this factorization at an arbitrary metric $g \in \mathrm{Met}(\Sigma)$, and shall choose a section later on.

We use the fact that $\mathrm{Diff}_0(\Sigma)$ is generated by vector fields $v = v^z \oplus v^{\bar z} \in K^{(-1,0)} \oplus K^{(0,-1)}$. A Weyl transformation will in general be denoted by $e^{2\sigma}$, where σ takes real values.

Let the reference metric be $g = 2g_{z\bar z}|dz|^2$; an arbitrary metric $g + \delta g$ in the neighborhood of g may then be conveniently parametrized by a change in $g_{z\bar z}$ as well as by Beltrami differentials $\delta\eta_{\bar z}^{\,z} \in K^{(-1,1)}$ and $\delta\bar\eta_z^{\,\bar z} \in K^{(1,-1)}$.

$$g + \delta g = 2\left(g_{z\bar z} + \delta g_{z\bar z}\right)|dz + \delta\eta_{\bar z}^{\,z} d\bar z|^2 \ . \tag{3.11}$$

The space $T_g\mathrm{Met}(\Sigma)$ admits the orthogonal decomposition

$$\begin{aligned} T_g\mathrm{Met}(\Sigma) &= K^{(1,1)} \oplus K^{(2,0)} \oplus K^{(0,2)} \\ \delta g &= 2\delta g_{z\bar z}|dz|^2 + 2g_{z\bar z}\delta\eta_{\bar z}^{\,z}(d\bar z)^2 + 2g_{z\bar z}\delta\bar\eta_z^{\,\bar z}(dz)^2 \ . \end{aligned} \tag{3.12}$$

The infinitesimal action of $(\delta v^z, \delta v^{\bar z}, 2\delta\sigma)$ in the Lie aglebra of $\mathrm{Diff}_0(\Sigma) \ltimes \mathrm{Weyl}(\Sigma)$ is then simply

$$\begin{cases} \delta g_{z\bar z} = (2\delta\sigma + \nabla_z \delta v^z + \nabla_{\bar z}\delta v^{\bar z}) g_{z\bar z} \\ \delta\eta_{\bar z}^{\,z} = \nabla_{\bar z}\delta v^z \\ \delta\bar\eta_z^{\,\bar z} = \nabla_z \delta v^{\bar z} \ . \end{cases} \tag{3.13}$$

While the range of Weyl(Σ) is the full $K^{(1,1)}$, in general the range of $\mathrm{Diff}_0(\Sigma)$ need not be all of $K^{(2,0)} \oplus K^{(0,2)}$. Instead, we have the orthogonal decomposition

$$K^{(2,0)} = \mathrm{Range}\, \nabla_z^{(1)} \oplus \mathrm{Ker}(\nabla_z^{(1)})^\dagger \ . \tag{3.14}$$

The space $\mathrm{Ker}\, (\nabla_z^{(1)})^\dagger = \mathrm{Ker}\, \nabla_{\bar z}^{(2)}$ consists of holomorphic quadratic differentials on Σ, and can be identified with the holomorphic cotangent space of $\mathcal{M}_h$:

$$T^*_{(1,0)}(\mathcal{M}_h) = \mathrm{Ker}\, \nabla_{\bar z}^{(2)} \tag{3.15}$$

(Analogously, we have $T^*_{(0,1)}(\mathcal{M}_h) = \mathrm{Ker}\, \nabla_z^{(-2)}$.) To see this, we notice that holomorphic differentials $\phi_j \in \mathrm{Ker}\, \nabla_{\bar z}^{(2)}$ provide linear forms on the space of differentials $\delta\eta$:

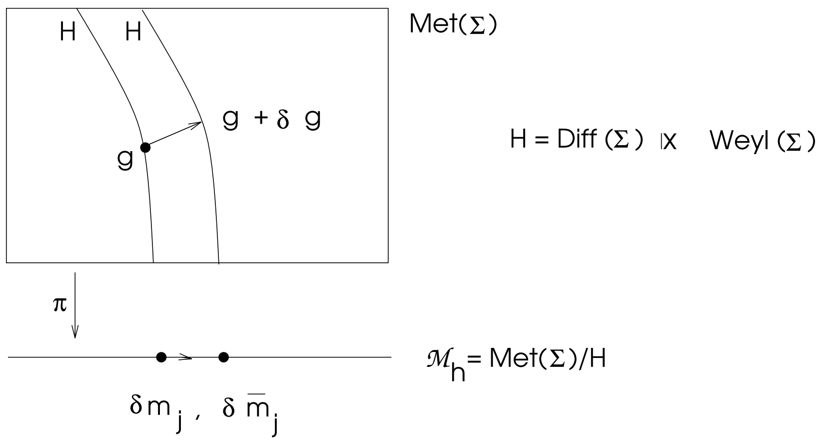
$$(\delta\eta, \phi_j) = \int_\Sigma dz\, d\bar z\ \delta\eta_{\bar z}^{z} \phi_{jzz} \tag{3.16}$$

The above pairing is Weyl(Σ)-invariant and the kernel of $(\ \cdot\ , \phi_j)$ (and $(\ \cdot\ , \phi_j^*)$) is precisely the tangent space to the orbits of $\mathrm{Diff}_0(\Sigma)$ and Weyl(Σ); hence the identification. By the Riemann-Roch theorem, and the vanishing theorems listed above, it is easy to see that the dimensions of $\mathcal{M}_h$ and of $\mathrm{Ker}\, \nabla_{\bar z}^{(2)} \oplus \mathrm{Ker}\, \nabla_z^{(-2)}$ coincide.

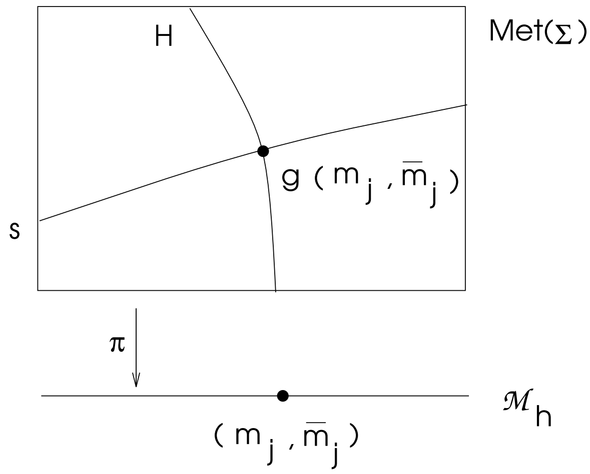
More precisely, the holomorphic quadratic differentials $\phi_j \in \mathrm{Ker}\, \nabla_{\bar z}^{(2)}$ form a complex vector space, thus providing a natural almost complex structure on the tangent space of $\mathcal{M}_h$. In fact, this almost complex structure is integrable and results in a complex structure on $\mathcal{M}_h$; for a given choice of ϕ, the above linear forms define local complex coordinates m_j, $\bar m_j$ on $\mathcal{M}_h$ given by

$$\delta m_j = (\delta\eta, \phi_j) \qquad\qquad \delta\bar m_j = (\delta\bar\eta, \bar\phi_j)\ . \tag{3.17}$$

We may represent this projection onto $\mathcal{M}_h$ by a picture analogous to the one used ot discuss the finite dimensional example at the beginning of this lecture.



Vice-versa, for a given set of complex coordinates $(m_j, \bar{m}_j)$ on $\mathcal{M}_h$, we may choose a section s of $\mathcal{M}_h$ in $\mathrm{Met}(\Sigma)$ — also called a slice — of dimension dim $\mathcal{M}_h$, and parametrized by $(m_j, \bar{m}_j)$:



The tangent space at $g(m_j, \bar{m}_j)$ then decomposes into the tangent directions to the action of $\mathrm{Weyl}(\Sigma)$, the action of $\mathrm{Diff}_0(\Sigma)$ on the differentials $\delta\eta$, $\delta\bar{\eta}$, and moduli deformation, given by a basis of *Beltrami differentials* μ_j and $\bar{\mu}_j$:

$$\left\{ \begin{aligned} \delta\eta_{\bar{z}}^{z} &= \nabla_{\bar{z}}^{(-1)}\delta v^{z} + \sum_{j=1}^{\dim \mathcal{M}_h} \delta m_j \mu_{j\bar{z}}^{z} \\ \delta\bar{\eta}_{z}^{\bar{z}} &= \nabla_{z}^{(1)}\delta v^{\bar{z}} + \sum_{j=1}^{\dim \mathcal{M}_h} \delta \bar{m}_j \bar{\mu}_{jz}^{\bar{z}} \ . \end{aligned} \right. \tag{3.18}$$

We may choose the Beltrami differentials as follows

$$\begin{cases} \mu_{j\bar z}^{z} = g^{z\bar z}\frac{\partial g_{\bar z\bar z}}{\partial m_j}(m,\bar m) \\ \bar\mu_{jz}^{\bar z} = g^{z\bar z}\frac{\partial g_{zz}}{\partial \bar m_j}(m,\bar m) \ , \end{cases} \tag{3.19}$$

so that μ_j and $\bar\mu_j$ are unchanged under Weyl transformations $g \to e^{2\sigma} g$: $\mu_j \to \mu_j$; $\bar\mu_j \to \bar\mu_j$. Thus, under a change of section s, μ_j and $\bar\mu_j$ will be shifted only by the action of a vector field in the Lie algebra of $\mathrm{Diff}_0(\Sigma)$.

Finally, for Σ with the topology of either the sphere or the torus, we also have holomorphic vector fields in Ker $\nabla_{\bar z}^{(-1)}$. These generate conformal automorphisms, and do not act on η at all. We have

$$\begin{cases} h=0 & \dim \mathrm{Ker}\nabla_{\bar z}^{(-1)} = 3 & \mathrm{SL}(2,\mathbb{C}) \\ h=1 & \dim \mathrm{Ker}\nabla_{\bar z}^{(-1)} = 1 & T^2 \\ h\geq 2 & \dim \mathrm{Ker}\nabla_{\bar z}^{(-1)} = 0 & \text{possibly finite automorphism group .} \end{cases} \tag{3.20}$$

D. Factorizing the Integration Measure

We decompose the integration measure Dg on $\mathrm{Met}(\Sigma)$ into measures on the orbits of $\mathrm{Diff}(\Sigma)$ and $\mathrm{Weyl}(\Sigma)$ and on $\mathcal{M}_h$ by using the parametrization developed above for δg in terms of Weyl changes $\delta\sigma$ and differentials $\delta\eta$, $\delta\bar\eta$:

$$\delta g = 2\delta\sigma g + 2g_{z\bar z}\delta\eta_{\bar z}^{z}(d\bar z)^2 + 2g_{z\bar z}\delta\bar\eta_{z}^{\bar z}(dz)^2 \ . \tag{3.21}$$

The relation

$$\|\delta g\|_g^2 = 4\|\delta\sigma\|_g^2 + 2\|\delta\eta\|_g^2 + 2\|\delta\bar\eta\|_g^2 \tag{3.22}$$

implies that $Dg = D\sigma\, D\eta\, D\bar\eta$, in obvious notation.

Next, we make use of the decomposition of $\delta\eta$ and $\delta\bar\eta$ into orbits of $\mathrm{Diff}_0(\Sigma)$ and moduli deformations, which we parametrize by coordinates $(\delta m_j, \delta\bar m_j)$ on $\mathcal{M}_h$ and a choice of Beltrami differentials $\mu_{j\bar z}^{z}$ and $\bar\mu_{jz}^{\bar z}$, as in (3.18). Using the orthogonal decomposition of the differentials $\delta\eta_z^{\bar z}\ g_{z\bar z}$ according to

$$K^{(2,0)} = \mathrm{Range}\nabla_z^{(1)} \oplus \mathrm{Ker}\ \nabla_{\bar z}^{(2)} \ , \tag{3.23}$$

we evaluate $\|\delta\eta\|_g^2$ and find

$$2\|\delta\eta\|_g^2 = 2\|\nabla_{\bar z}^{(-1)}\delta v^z\|_g^2 + \|\delta\eta\perp \text{ projection of } \delta\eta_z^{\bar z}\ g_{z\bar z} \text{ onto Ker } \nabla_{\bar z}^{(2)}\|_g^2 \ . \tag{3.24}$$

The second term on the right hand side is easily worked out by choosing an orthogonal basis for $\mathrm{Ker}\,\nabla_{\bar z}^{(2)}$, with basis vectors ϕ_j, $j=1,\ldots,\dim \mathcal{M}_h$:

$$\begin{aligned}
&\|\delta\eta\perp \text{ projection of } \delta\eta_z^{\bar z}\, g_{z\bar z} \text{ onto } \mathrm{Ker}\,\nabla_{\bar z}^{(2)}\|_g^2 \\
&\quad=\sum_j(\delta\eta,\phi_j)_g(\phi_j,\phi_j)_g^{-1}(\phi_j,\delta\eta)_g \\
&\quad=\sum_{j,k,\ell}\delta m_k\delta\bar m_\ell(\mu_k,\phi_j)_g(\phi_j,\phi_j)_g^{-1}(\phi_j,\mu_\ell)_g
\end{aligned} \tag{3.25}$$

The first term on the right hand side of (3.24) is conveniently rewritten in terms of the Laplace operator $\Delta^-_{(-1)}$:

$$2\|\nabla_{\bar z}^{(-1)}\delta v^z\|_g^2=(\delta v^z,\Delta^-_{(-1)}\delta v^z)_g$$

A last point must be clarified before evaluating the measure $D\eta\, D\bar\eta$. For the special cases of the sphere ($h=0$) and the torus ($h=1$), $\mathrm{Ker}\,\nabla_{\bar z}^{(-1)}\neq 0$, and there are (continuous families of) conformal automorphisms of Σ: $\mathrm{SL}(2,\mathbb{C})$ for $h=0$ and T^2 for $h=1$. The corresponding vector fields leave η and $\bar\eta$ invariant. Thus, in the change of variables, $\delta\eta\to(\delta v,\delta m_j)$, we must restrict to vector fields δv that are orthogonal to $\mathrm{Ker}\,\nabla_{\bar z}^{(-1)}$. The corresponding restricted volume element on vector fields v is denoted by $D'v$. The associated determinant of $\Delta^-_{(-1)}$ must be similarly restricted to non-zero modes.

Putting all together, we find

$$Dg=\det{}'\Delta^-_{(-1)}\frac{|\det(\mu_j,\phi_k)_g|^2}{\det(\phi_j,\phi_k)_g}\prod_j dm_j\, d\bar m_j\, D'v\cdot D\sigma\ . \tag{3.26}$$

While originally we chose $\phi_j\in\mathrm{Ker}\,\nabla_{\bar z}^{(2)}$ to form an orthogonal basis, it is now clear that the above formula holds for any basis $\mathrm{Ker}\,\nabla_{\bar z}^{(2)}$. (The finite dimensional determinants are of $\dim\mathcal{M}_h\times\dim\mathcal{M}_h$ dimensional matrices.)

It is standard practice — and extremely useful — to introduce the following combination of determinants

$$Z^\pm_{(n)}(g)\equiv\frac{\mathrm{Det}'\Delta^\pm_{(n)}}{\det(\phi_j,\phi_k)_g\det(\psi_a,\psi_b)_g} \tag{3.27}$$

where

- $\phi_j\in\mathrm{Ker}\,\nabla^z_{(n+1)}$ for $+$; $\phi_j\in\nabla_z^{(n-1)}$ for $-$
- $\psi_a\in\mathrm{Ker}\,\nabla_z^{(n)}$ for $+$; $\psi_a\in\nabla^z_{(n)}$ for $-$.

While $Z^{\pm}_{(n)}$ have non-trivial Weyl transformation properties, we shall show in the next section that this particular combination has simple transformation properties.

Putting all together, we have the following expression for the measure Dg:

Theorem:

$$Dg = D\sigma \, \widetilde{D}v \, Z^{-}_{(-1)}(g) \; |\det(\mu_j, \phi_k)|^2_g \prod_{j=1}^{\dim \mathcal{M}_h} dm_j \, d\bar{m}_j \ , \tag{3.28}$$

where we have

1) $\widetilde{D}v \equiv D'v \, \det(\psi_a, \psi_b)_g$. For $h \geq 2$, we have $\widetilde{D}v = Dv$, i.e. the measure over all vector fields of $\mathrm{Diff}_0(\Sigma)$. For $h = 0, 1$, we shall postpone the explicit construction of $\widetilde{D}v$ until §G.

2) The measure

$$|\det(\mu_j, \phi_k)_g|^2 \prod_j dm_j \, d\bar{m}_j \tag{3.29}$$

is intrinsically defined on $\mathcal{M}_h$. To see this, notice that we chose μ_j to be Weyl independent, that ϕ_j can be chosen Weyl independent, and that the pairing $(\mu_j, \phi_k)_g$ is Weyl invariant. Furthermore, we defined Beltrami differentials by

$$\mu^{z}_{j\bar{z}} = g^{z\bar{z}} \frac{\partial g_{\bar{z}\bar{z}}}{\partial m_j} \tag{3.30}$$

so that the above combination is manifestly unchanged under reparametrizations of m_j and $\bar{m}_j$.

3) $Z^{-}_{(-1)}(g)$ will be Weyl dependent, as we shall establish in the next section. This object is called the Faddeev-Popov determinant (up to the finite dimensional determinants), expressing the fact that we gauge fixed the $\mathrm{Diff}_0(\Sigma)$-invariance.

4) To complete the construction outlined in the finite-dimensional case, for the projection down onto $\mathcal{M}_h$ of the measure Dg on $\mathrm{Met}(\Sigma)$, we will have to understand the Weyl dependence of the combined $Z^{-}_{(-1)}(g)$ and matter contributions, which we do in the next section.

E. Weyl Rescalings of Functional Determinants

The determinant combinations $Z^{\pm}_{(n)}$ (including those with $n \in \frac{1}{2} + \mathbb{Z}$, which will enter when dealing with the superstring, as we shall see later on) will be considered repeatedly, and we shall need their behavior under Weyl(Σ) rescalings of the metric. Throughout, we shall adopt a regularization and renormalization scheme that preserves Diff(Σ) invariance.

Theorem.

$$Z^{\pm}_{(n)}(ge^{2\sigma}) = Z^{\pm}_{(n)}(g)e^{-2c^{\pm}_{(n)}S_L(\sigma,g)} \tag{3.31}$$

where $c^{\pm}_{(n)}$ *is given by*

$$c^{\pm}_{(n)} \equiv 6n^2 \pm 6n + 1 \tag{3.32}$$

and the Liouville action *is independent of* n *and given by*

$$S_L(\sigma,g) = \frac{1}{12\pi}\int_\Sigma d\mu_g \left[\frac{1}{2}g^{mn}\partial_m\sigma\partial_n\sigma + R_g\sigma + \beta^2(e^{2\sigma}-1)\right] . \tag{3.33}$$

The constant β *depends upon the renormalization scheme used, and will vanish when determinants are defined with* ζ*-function regularization, or with dimensional regularization.*

Since special care must be paid to the effects of the zero modes, we outline the proof. (Various intermediate steps will be proven in Problem Set IV.) First, it suffices to prove the formula for the Weyl transformation of $Z^{\pm}_{(n)}$ by infinitesimal σ; the result for general σ follows by straightforward integration (Gawędzki, prob. 8). Thus, we must prove

$$\delta_\sigma \ln Z^{\pm}_{(n)}(g) = -\frac{c^{\pm}_{(n)}}{6\pi}\int_\Sigma d\mu_g[R_g\delta\sigma + 2\beta^2\delta\sigma]$$

(Notice that from the form of this infinitesimal variation, it is clear that the determinant $Z^{\pm}_{(n)}$ defines a conformal field theory with central charge related to $\pm c^{\pm}_{(n)}$.)

We define determinants by introducing the ζ-function associated with the eigenvalues λ_α of $\Delta^{\pm}_{(n)}$:

$$\zeta(s) \equiv \sum_\alpha{}' \lambda_\alpha^{-s}$$

Here, $\sum'$ denotes the sum over all non-zero eigenvalues λ_α of $\Delta^\pm_{(n)}$. It is standard to represent this sum in terms of the heat-kernel of $\Delta^\pm_{(n)}$:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} (\mathrm{Tr}\ e^{-t\Delta^\pm_{(n)}} - N^\pm_{(n)}) \tag{3.35}$$

where $N^\pm_{(n)} = \dim \mathrm{Ker}\, \Delta^\pm_{(n)}$.

The $\zeta(s)$ function series and the heat-kernel integral representation are absolutely convergent for $\mathrm{Re}(s) > 1$. $\zeta(s)$ is holomorphic for $\mathrm{Re}(s) > 1$, but may be analytically continued throughout $\mathbb{C}$, with a simple pole at $s = 1$. The determinants are then defined by

$$\ln \mathrm{Det}' \Delta^\pm_{(n)} = -\zeta'(0) \tag{3.36}$$

The non-zero spectra of $\Delta^+_{(n)}$ and $\Delta^-_{(n+1)}$ coincide and thus their determinants are equal, so we shall henceforth focus on $\delta^-_{(n)}$.

To evaluate the Weyl dependence of determinants, we compute that of $\zeta(s)$ and analytically continue to $s = 0$. We use the following intermediate results (proven in Problem Set IV).

$$\begin{aligned}
&1)\ N^\pm_{(n)} \text{ are Weyl independent} \\
&2)\ \delta_\sigma \mathrm{Tr}\, e^{-t\Delta^-_{(n)}} = -t\tfrac{\partial}{\partial t} \mathrm{Tr}\, [2n\delta\sigma e^{-t\Delta^+_{(n-1)}} - (2n-2)\delta\sigma e^{-t\Delta^-_{(n)}}] \\
&3)\ \mathrm{Tr}\, \delta\sigma e^{-t\Delta^\pm_{(n)}} = \tfrac{1}{4\pi t} \textstyle\int_\Sigma d\mu_g \delta\sigma + \tfrac{1 \pm 3n}{12\pi} \int_\Sigma d\mu_g\, R_g\, \delta\sigma + O(t) \\
&4)\ \mathrm{Tr}\, \delta\sigma e^{-t\Delta^-_{(n)}} = -\tfrac{1}{2n-2} \delta_\sigma \ln\, \det(\phi_j, \phi_k)_g + O(e^{-\lambda_1 t}) \\
&5)\ \mathrm{Tr}\, \delta\sigma e^{-t\Delta^+_{(n-1)}} = \tfrac{1}{2n} \delta_\sigma \ln\, \det(\psi_a, \psi_b)_g + O(e^{-\lambda_1 t})
\end{aligned} \tag{3.37}$$

where λ_1 is the smallest non-zero eigenvalue of $\Delta^-_{(n)}$.

We make use of the above results and carry out the analytic continuation of $\delta_\sigma\ \zeta(s)$ to $s = 0$, and then consider $\delta\sigma\zeta'(0)$. For $\mathrm{Re}(s) > 1$, using 1) and 2), we have

$$\delta_\sigma \zeta(s) = -\frac{1}{\Gamma(s)} \int_0^\infty dt\, t^s \frac{\partial}{\partial t} \mathrm{Tr}\, [\ \cdot\ ](t) \tag{3.38}$$

where we introduced the abbreviation

$$\mathrm{Tr}\, [\ \cdot\ ](t) \equiv \mathrm{tr}[2n\delta\sigma e^{-t\Delta^+_{(n-1)}} - (2n-2)\delta\sigma e^{-t\Delta^-_{(n)}}] \tag{3.39}$$

We shall make use of the asymptotic behaviors of $\mathrm{Tr}\,[\,\cdot\,](t)$ as $t\to 0$ and $t\to\infty$, which are obtained from 3), 4), 5):

$$\begin{cases} \mathrm{Tr}\,[\,\cdot\,](t) = \frac{1}{2\pi t}\int_\Sigma d\mu_g \delta\sigma + \frac{c^-_{(n)}}{6\pi}\int_\Sigma d\mu_g \delta\sigma R_g + O(t) \\ \mathrm{Tr}\,[\,\cdot\,](t) = \delta_\sigma \ln\, \det(\phi_j,\phi_k)_g \det(\psi_a,\psi_b)_g + O(e^{-\lambda_1 t})\ . \end{cases} \tag{3.40}$$

Here, $c^-_{(n)}$ was defined in (3.32).

To separate the asymptotic behaviors as $t\to 0$, $t\to\infty$, we cut $[0,\infty]$ at $t=t_0>0$, and define $\delta_\sigma\zeta(s)=\delta_\sigma\zeta_0(s)+\delta_\sigma\zeta_\infty(s)$

$$\begin{cases} \delta_\sigma\zeta_0(s) = -\frac{1}{\Gamma(s)}\int_0^{t_0} dt\, t^s \frac{\partial}{\partial t}\mathrm{Tr}\,[\,\cdot\,](t) \\ \delta_\sigma\zeta_\infty(s) = -\frac{1}{\Gamma(s)}\int_{t_0}^{\infty} dt\, t^s \frac{\partial}{\partial t}\mathrm{Tr}\,[\,\cdot\,](t)\,. \end{cases} \tag{3.41}$$

The first term fails to converge at $s=0$, so we subtract the asymptotic behavior of $\mathrm{Tr}\,[\,\cdot\,](t)$ in $1/t$ as $t\to 0$:

$$\begin{aligned}\delta_\sigma\zeta_0(s) = &-\frac{1}{\Gamma(s)}\int_0^{t_0} dt\, t^s \frac{\partial}{\partial t}\left(\mathrm{Tr}\,[\,\cdot\,](t) - \frac{1}{2\pi t}\int_\Sigma d\mu_g\delta\sigma\right) \\ &+ \frac{1}{\Gamma(s)}\frac{1}{2\pi}\int_\Sigma d\mu_g\delta\sigma \int_0^{t_0} dt\, t^{s-2}\ .\end{aligned}$$

The first integral on the right hand side is convergent at $s=0$, while the second may be explicitly analytically continued to $s=0$. Taking the derivative at $s=0$:

$$\begin{aligned}\delta_\sigma\zeta_0'(0) &= -\int_0^{t_0} dt \frac{\partial}{\partial t}\left(\mathrm{Tr}\,[\,\cdot\,](t) - \frac{1}{2\pi t}\int_\Sigma d\mu_g\delta\sigma\right) - \frac{1}{2\pi t_0}\int_\Sigma d\mu_g\delta\sigma \\ &= -\mathrm{Tr}\,[\,\cdot\,](t_0) + \left.\left(\mathrm{Tr}\,[\,\cdot\,](t) - \frac{1}{2\pi t}\int_\Sigma d\mu_g\delta\sigma\right)\right|_{t=0} \\ &= -\mathrm{Tr}\,[\,\cdot\,](t_0) + \frac{c^-_{(n)}}{6\pi}\int_\Sigma d\mu_g R_g\delta\sigma\ .\end{aligned}$$

The term $\delta_\sigma\zeta_\infty(s)$ is given by

$$\delta_\sigma\zeta_\infty(s) = -\frac{1}{\Gamma(s)}\int_{t_0}^{\infty} dt\, t^s \frac{\partial}{\partial t}\mathrm{Tr}\,[\,\cdot\,](t)$$

As $t \to \infty$, $\mathrm{Tr}[\quad] \to$ constant $+ O(e^{-\lambda_1 t})$, $\lambda_1 > 0$, so that the integral is exponentially convergent, and analytic around $s = 0$. Hence, using the asymptotics of $\mathrm{Tr}[\,\cdot\,](t)$ as $t \to \infty$:

$$\begin{aligned}
\delta_\sigma \zeta'_\infty(0) &= -\int_{t_0}^{\infty} dt \frac{\partial}{\partial t} Tr[\quad](t) \\
&= -\delta_\sigma \ln \det(\phi_j, \phi_k)_g \det(\psi_a, \psi_b)_g + \mathrm{Tr}[\,\cdot\,](t_0) \ .
\end{aligned}$$

Putting all together, we have

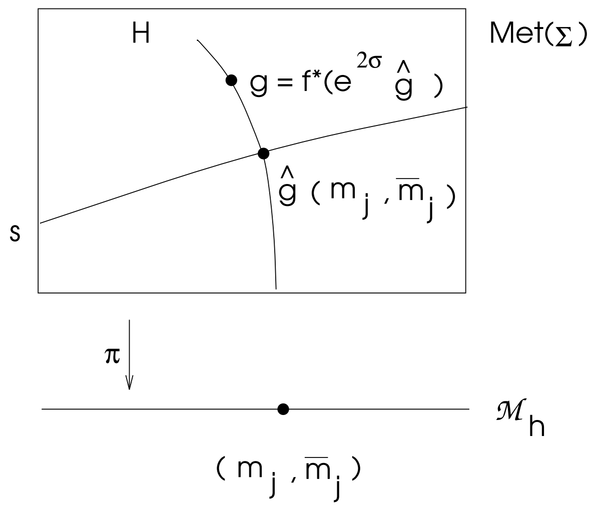
$$\begin{aligned}
\delta_\sigma \ln \mathrm{Det}' \Delta^-_{(n)} &= -\delta\sigma\zeta'(0) \\
&= \delta_\sigma \ln \det(\phi_j, \phi_k)_g \det(\psi_a, \psi_b)_g + \frac{c^-_{(n)}}{6\pi} \int_\Sigma d\mu_g R_g \delta\sigma \ ,
\end{aligned}$$

as announced previously. Notice that in ζ-function definitions of the determinants, no "area term" involving the additional constant β arises.

F. Critical Central Charge – Critical Dimension

We complete the reduction of the measure Dg to a measure on $\mathcal{M}_h$, making use of the Weyl transformation properties of functional determinants, derived in §E.

Just as we did in the case of the finite-dimensional example, we choose a section s of $\mathrm{Met}(\Sigma)$ and denote the corresponding metric $\hat{g}(m_j, \bar{m}_j)$. A general metric $g \in \mathrm{Met}(\Sigma)$ may then be parametrized in terms of the action of $\mathrm{Diff}(\Sigma)$ and $\mathrm{Weyl}(\Sigma)$ on $\hat{g}(m_j, \bar{m}_j)$: let $f \in \mathrm{Diff}(\Sigma)$ and $e^{2\sigma} \in \mathrm{Weyl}(\Sigma)$, then



The action of Diff(Σ) is by isometries, and our definition of functional determinants was invariant with respect to Diff(Σ). Thus, the effect of f^* is immaterial and may be dropped. The effect of Weyl transformations is obtained by combining the results of §D, E, and we have

$$Dg = \widetilde{D}v D\sigma Z^{-}_{(-1)}(\hat{g}) e^{-26 S_L(\hat{g},\sigma)} |\det(\hat{\mu}_j, \phi_k)_{\hat{g}}|^2 \, dm_j \, d\bar{m}_j \ . \tag{3.42}$$

We now consider a full transition amplitude A_h, evaluated on surfaces Σ of genus h. (To avoid certain complications due to conformal automorphisms, we postpone the cases $h = 0, 1$ and restrict to $h \geq 2$. The cases $h = 0, 1$ are completely analogous and will be dealt with separately in §I.) Recall that

$$A_h = \int_{\mathrm{Met}(\Sigma)} Dg \, \frac{1}{\mathcal{N}(g)} \langle V_1 \dots V_N \rangle_g \ , \tag{3.43}$$

where

$$\langle V_1 \dots V_N \rangle_g = \int_{\mathrm{Diff}(\Sigma, M)} Dx \, V_1 \dots V_N e^{-S[x,g]} \ . \tag{3.44}$$

Given the Weyl transformation properties of Dg established above, it is clear that a *matter theory* (i.e. the quantum field theory of x fields for given metric g) is singled out that makes the Weyl dependence of the amplitude trivial. We define a *critical (bosonic) string theory* as one governed by a matter conformal field theory with central charge $c = 26$, and with physical vertex operators V_i, $i = 1, \dots, N$ such that

$$\langle V_1 \dots V_N \rangle_{e^{2\sigma}\hat{g}} = e^{26 S_L(\hat{g},\sigma)} \langle V_1 \dots V_N \rangle_{\hat{g}} \ . \tag{3.45}$$

(This last condition puts restrictions on the allowed vertex operators in the theory.) For critical string theories, the amplitudes at fixed topology ($h \geq 2$) reduce to

$$\begin{aligned} A_h = \int_{\mathcal{M}_h} \prod_j dm_j \, d\bar{m}_j |\det(\hat{\mu}_j, \phi_k)_{\hat{g}}|^2 Z^{-}_{(-1)}(\hat{g}) \langle V_1 \dots V_N \rangle_{\hat{g}} \\ \cdot \int_{\mathrm{Weyl}(\Sigma)} D\sigma \int_{\mathrm{Diff}_0(\Sigma)} Dv / \mathcal{N}_0(g) \end{aligned} \tag{3.46}$$

(Recall that for $h \geq 2$, $\widetilde{D}v = Dv$.) Now, recall that the formal measure $Dg/\mathcal{N}(g)$ was the measure on Met(Σ), divided by the volume of the orbits of the group Weyl(Σ) $\times$ Diff(Σ).

Since we restricted our integration to one over $\mathcal{M}_h$ (and not Teichmüller space T_h), we effectively quotiented by the mapping class group. The remaining normalization factor is $\mathcal{N}_0(g)$, formally the volume of Weyl $\times$ $\mathrm{Diff}_0(\Sigma)$. Thus, the formal integrals over Weyl(Σ) and $\mathrm{Diff}_0(\Sigma)$ cancel $\mathcal{N}_0(g)$:

$$\int_{\mathrm{Weyl}(\Sigma)} D\sigma \int_{\mathrm{Diff}_0(\Sigma)} Dv/\mathcal{N}_0(g) = 1 \ . \tag{3.47}$$

Thus, we finally obtain an ultimate expression for amplitudes in *critical string theory* at fixed topology ($h \geq 2$); the expression reduces to integrals over moduli of Riemann surfaces:

$$A_h = \int_{\mathcal{M}_h} \prod_{\mathcal{M}_h} dm_j \, d\bar{m}_j | \det(\hat{\mu}_j, \phi_k)_{\hat{g}}|^2 Z^-_{(-1)}(\hat{g}) \langle V_1 \ldots V_N \rangle_{\hat{g}} \tag{3.48}$$

The cases for $h = 0, 1$ will be discussed in §I, J.

G. Flat Space-Time Manifold M

For flat Euclidean space-time $M = \mathbb{R}^D$, we have already established that the central charge of the conformal field theory defined by maps $x \in \mathrm{Diff}(\Sigma, M)$ is $c = D$. Thus, the flat space-time string theory is a critical string theory precisely when $D = 26$. (We now notice that this condition is the same as the one found for the free string: this is rather fortunate!) Let us work out in detail under what conditions the vertex operators also satisfy the criteria of a critical string.

To do so, it is convenient to start with a study of the Weyl transformation properties of correlation functions of unintegrated exponents:

$$\langle \prod_{i=1}^N e^{ik_i \cdot x(z_i)} \rangle_g = \int_{\mathrm{Diff}(\Sigma, M)} Dx \prod_{i=1}^N e^{ik_i \cdot x(z_i)} e^{-S_0[x,g]} \tag{3.49}$$

where S_0 is the Gaussian action, including the constant dilaton Φ_0: (We ignore the constant tachyon.)

$$S_0[x, g] = \frac{1}{8\pi} \int_\Sigma d\mu_g \, x \cdot \Delta_{(0)} x + \frac{\Phi_0}{4\pi} \int_\Sigma d\mu_g R_g \tag{3.50}$$

The Gaussian integral is carried out by separating the constant mode in x:

$$x = x_0 + x' \ , \qquad \int_\Sigma d\mu_g \, x' = 0$$

and making use of the Green functions for x':

$$\begin{cases} \Delta_{(0)} G_g(z,w) = 4\pi \delta_g(z,w) - \dfrac{4\pi}{\int_\Sigma d\mu_g} \ . \\ \displaystyle\int_\Sigma d\mu_g G_g(z,w) = 0 \end{cases} \tag{3.51}$$

The Green function at coincident points must be regularized and renormalized in a Diff(Σ)-invariant way. This may be carried out with the help of heat-kernel methods, or simply by renormalizing by the Diff(Σ)-invariant distance:

$$G_g^R(z,z) \equiv \lim_{w \to z} \left(G_g(z,w) + 2 \ln \mathrm{dist}_g(z,w) \right) \ . \tag{3.52}$$

The Weyl scaling properties of the Green function are then

$$\begin{cases} G_{e^{2\sigma}\hat g}(z,w) = G_{\hat g}(z,w) + f(z) + f(w) \\ G^R_{e^{2\sigma}\hat g}(z,w) = G^R_{\hat g}(z,z) + 2f(z) + 2\sigma(z) \ , \end{cases} \tag{3.53}$$

where $f(z)$ is a function required to maintain $\int_\Sigma d\mu_g G_g = 0$ under Weyl rescalings. We do not need its explicit form here. One finds

$$\Big\langle \prod_i e^{k_i \cdot x(z_i)} \Big\rangle_g = \delta(k) e^{-\frac{1}{2}\chi \Phi_0} \left(\frac{\mathrm{Det}'\Delta_{(0)}}{\int_\Sigma d\mu_g} \right)^{-D/2} \prod_{i,j} e^{\frac{1}{2} k_i \cdot k_j \, G_g(z_i,z_j)} \ , \tag{3.54}$$

where $k \equiv \sum_i k_i$, $\delta(k)$ is the D-dimensional δ-function, and the factor $\int_\Sigma d\mu_g$ under the determinant arises from the normalization of the zero mode x_0. As a result, under $g = e^{2\sigma}\hat g$, we have

$$\prod_{i,j} e^{-\frac{1}{2} k_i \cdot k_j \, G_g(z_i,z_j)} = \prod_{i,j} e^{-\frac{1}{2} k_i \cdot k_j \, G_{\hat g}(z_i,z_j)} e^{-k_i^2 \sigma(z_i)} \ . \tag{3.55}$$

All dependence on $f(z)$ cancels in view of momentum conservation $k = \sum_i k_i = 0$, as guaranteed by the overall $\delta(k)$ factor in (3.54). The Weyl scaling property of the determinant factor is precisely that of $Z_{(0)}(g)^{-D/2}$. ($Z_{(0)}(g)$ contains an additional factor involving holomorphic Abelian differentials, but this factor is manifestly Weyl invariant.)

Combining all of the above, we find for $D = 26$

$$\Big\langle \prod_i^N e^{ik_i \cdot x(z_i)} \Big\rangle_{\hat g e^{2\sigma}} = \prod_{i=1}^N e^{-k_i^2 \sigma(z_i)} e^{26 S_L(\hat g, \sigma)} \Big\langle \prod_{i=1}^N e^{ik_i \cdot x(z_i)} \Big\rangle_{\hat g} \ . \tag{3.56}$$

It is now straightforward to examine the Weyl covariance conditions of vertex operators for low mass cases.

1) Tachyon vertex operator

$$V_T(k) = \varepsilon(k) \int_\Sigma d\mu_g e^{ik\cdot x} \ .$$

In order to cancel the explicit Weyl dependence of the measure $d\mu_g$ on Σ:

$$d\mu_{e^{2\sigma}\hat{g}} = d\mu_{\hat{g}} e^{2\sigma}$$

against the Weyl dependence of the correlation function, we recover the familiar condition $k^2 = 2$ for the tachyon. (This condition also corresponded to $a = 1$ in §III.)

2) The massless vertex operators

$$V_\varepsilon(k) = \varepsilon_{\mu\nu}(k) \int_\Sigma d\mu_g g^{mn} \partial_m x^\mu \partial_n x^\nu e^{ik\cdot x} \ ;$$

the Weyl dependence of the part $d\mu_g g^{mn}$ is trivial, so we need the familiar condition $k^2 = 0$. In addition, Weyl dependence appears through the contraction of $\partial_m x^\mu$ with the exponential, and is cancelled by the transversality conditions

$$k^\mu \varepsilon_{\mu\nu}(k) = k^\nu \varepsilon_{\mu\nu}(k) = 0 \ .$$

Finally, from the contraction of $\partial_m x^\mu$ and $\partial_n x^\nu$, (actually only the trace-part contributes),

See Prob. Set #5.

a curvature R_g term appears, and Weyl invariance for the trace-part is achieved through the addition of a (classically non-Weyl invariant) part involving the curvature:

$$V_\varepsilon(k) = \varepsilon_{\mu\nu}(k) \int_\Sigma d\mu_g g^{mn} \partial_m x^\mu \partial_n x^\nu e^{ikx} + \alpha \varepsilon^\mu_\mu \int_\Sigma d\mu_g R_g e^{ik\cdot x} \ .$$

The physical particle corresponding to this vertex operator is the dilaton. Recall that the dilaton field Φ entered the generalized sigma model action with a Gaussian curvature term R_g as well.

3) Higher mass states: the general structure of vertex operators is given by

$$V_\varepsilon(k) = \varepsilon_{\mu_1 \ldots \mu_{2p}} \int_\Sigma d\mu_g g^{m_1 n_1} \ldots g^{m_p n_p} \partial_{m_1} x^{\mu_1} \ldots \partial_{n_p} x^{\mu_{2p}} e^{ik\cdot x}$$

plus terms involving higher derivatives on x. To cancel the overall exponential Weyl dependence, we must require $k^2 = -2p+2$, yielding the same mass spectrum as in the free string case. In complete analogy with the case 2), there will be further conditions of transversality on ε and and further terms involving R_g will have to be added to render the trace parts Weyl invariant.

H. Non-Critical Strings

Conformal matter field theories with $c \neq 26$ are called *non-critical string theories*. The amplitudes for these theories retain a non-trivial Weyl dependence through the Liouville action, and through the vertex operators. Its direct quantization requires understanding the dynamics of Liouville theory. Despite much effort, there is no complete picture to date of how this can be achieved consistently, and without the presence of a tachyon.

A tremendous number of results was obtained indirectly (by matrix model techniques) when c is less than 1 and rational.

I. Tree Level Amplitudes

Tree level amplitudes correspond to the worldsheet topology of a sphere S^2, which we stereographically project onto the complex plane $\mathbb{C}$. The moduli space $\mathcal{M}_0$ of the sphere consists of just a single point, which we may choose to represent by the *round metric*, of curvature $R_{\hat{g}} = 1$, and area 4π:

$$\hat{g} = \frac{4|dz|^2}{(1+|z|^2)^2} \qquad\qquad z \in \mathbb{C} \; . \tag{3.57}$$

The group of conformal automorphisms is $\mathrm{PSL}(2,\mathbb{C})$ and acts by Möbius transformations

$$\zeta\colon z \to \zeta(z) = \frac{az+b}{cz+d} \qquad\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C}) \; .$$

A convenient basis for the associated vector fields (called conformal Killing vectors) is

$$\psi_1^z = 1 \qquad\qquad \psi_2^z = z \qquad\qquad \psi_3^z = z^2 \; ,$$

and similarly for $\psi_a^{\bar{z}}$, $a = 1,2,3$.

To obtain the general expression for tree level transition amplitudes, we must first complete the treatments of conformal automorphisms, which we had postponed in §F, and of Weyl transformation of the measure. The starting point is our expression (3.28) for the measure Dg, considered now for $h=0$

$$Dg = D_\sigma \widetilde{D}v Z^{-}_{(-1)}(\hat{g}) e^{-26 S_L(\hat{g},\sigma)} \tag{3.58}$$

where $\widetilde{D}v = D'v \det(\psi_a, \psi_b)_g$, $\psi_a \in \mathrm{Ker}\nabla_{\bar{z}}^{(-1)}$, and the measure $D'v$ is over $\mathrm{Diff}_0(\Sigma)$ vector fields orthogonal to $\mathrm{Ker}\nabla_{\bar{z}}^{(-1)} \oplus \mathrm{Ker}\nabla_{z}^{(1)}$. Tree level amplitudes are then given by

$$A_0 = \int_{\mathrm{Weyl}(\Sigma)} D\sigma \int_{\mathrm{Diff}_0(\Sigma)/\mathrm{PSL}(2,\mathbb{C})} \widetilde{D}v/\mathcal{N}(g) Z^{-}_{(-1)}(\hat{g}) \langle V_1 \dots V_N \rangle_{\hat{g}} \tag{3.59}$$

using the same notations as in III.19.

While the group $\mathrm{PSL}(2,\mathbb{C})$ leaves the metric g invariant (up to Weyl rescaling), it does transform the positions of vertex operator insertions non-trivially. It is natural and — as we shall establish later on — required for unitarity of the transition amplitudes, to divide by the (formal) volume of the *full* $\mathrm{Diff}_0(\Sigma)$ group, and not just of $\mathrm{Diff}_0(\Sigma)/\mathrm{PSL}(2,\mathbb{C})$. To achieve this, we proceed as follows.

1) For $N \geq 3$ cases

Let the last three vertex operators be given by (of course, we may choose *any* three vertex operators)

$$V_i = \int_{\mathbb{C}} d^2 z_i \, W_i(z_i) \, , \qquad i = N-2, \, N-1, \, N \, .$$

We propose to make a change of variables from the vertex insertion points z_i, $\bar{z}_i$ $i = N-2, \, N-1, \, N$ to the group elements of $\mathrm{PSL}(2,\mathbb{C})$, thus completing the measure $D'v$ into Dv. This is possible since, given any three points $z_i \in \mathbb{C}$, there is a unique element $\zeta \in \mathrm{PSL}(2,\mathbb{C})$, such that ζ maps z_i^0 onto any triplet $z_i \in \mathbb{C}$, with

$$z_i = \zeta(z_i^0) \, .$$

to carry this out in detail, let ψ_a^z be a basis of the $\mathrm{Ker}\nabla_{\bar z}^{(-1)}$, $a=1,2,3$, chosen to be Weyl invariant. We parametrize general vector fields v in psl$(2,\mathbb{C})$ by complex parameters λ_a, $\bar\lambda_a$ with

$$v^z = \sum_{a=1}^{3} \lambda_a \psi_a^z \qquad\qquad v^{\bar z} = \sum_{a=1}^{3} \bar\lambda_a \psi_a^{\bar z} \ .$$

The measure on PSL$(2,\mathbb{C})$ vector fields is then

$$d\mu_v^{(6)} = Dv/D'v = \prod_{a=1}^{3} d\lambda_a \, d\bar\lambda_a \ \ \det(\psi_a,\psi_b)_g$$

where the determinant $\det(\psi_a,\psi_b)_g$ arises as the Jacobian of the coordinate change from v to λ. On the other hand, from the action of PSL$(2,\mathbb{C})$ on the points z_i^0: $z_i=\zeta(z_i^0)=e^v z_i^0$, we have $dz_i=\sum_a d\lambda_a\,\psi_a(z_i)$, so that the measure on d^2z_i becomes:

$$\prod_{i=N-2}^{N} d^2 z_i = \prod_{a=1}^{3} d\lambda_a \, d\bar\lambda_a \, |\det \ \psi_a(z_i)|^2 \ .$$

Clearly, all factors in this identity are Weyl-invariant. Combining all the results obtained, we have

$$\begin{aligned}\widetilde{D}v \prod_{i=N-2}^{N} d^2 z_i &= D'v \ \det(\psi_a,\psi_b)_g \prod_{a=1}^{3} d\lambda_a \, d\bar\lambda_a \, |\det \ \psi_a(z_i)|^2 \\ &= Dv \, |\det \ \psi_a(Z_i)|^2\end{aligned} \tag{3.60}$$

We may now divide by the formal volume of Weyl$(\Sigma)\ltimes$ Diff(Σ) just as in the cases $h\geq 2$, and we obtain the final expression for tree level amplitudes with $N\geq 3$:

$$A_0 = z_{(-1)}^{-}(\hat g)|\det \ \psi_a(z_i)|^2 \big\langle V_1 \dots V_{N-3} \prod_{i=N-2}^{N} W(Z_i)\big\rangle_{\hat g} \ , \tag{3.61}$$

valid for any 3-points z_i. We shall evaluate some of these amplitudes momentarily for flat space-time.

2) $N=0,\,1,\,2$ cases

Clearly, here, no three vertex operators are available to complete the integration $D'v$ into Dv. In fact, these amplitudes vanish! To justify this, notice that $D'v$ can at most be completed by the vector fields in $\mathrm{PSL}(2,\mathbb{C})/S_N$ where S_N is the stabilizer of N points on the sphere. Thus, we are left to divide out by the volume of S_N

$$\int_{\mathrm{Weyl}(\Sigma)} D\sigma \int_{\mathrm{Diff}_0/(\mathrm{PSL}(2,\mathbb{C})/S_N)} D'v/\mathcal{N}(g) \sim \frac{1}{\mathrm{Vol}(S_N)} = 0 \ . \tag{3.62}$$

For $N=2$, fix the two points at 0 and ∞, and S_2 is the group of dilations, whose volume is ∞.

An alternative argument that the amplitudes should vanish is that no conformal invariant amplitudes can be constructed, except with value 0. The physical interpretations of these vanishing amplitudes are as follows:

1)	$N=0$	tree-level vacuum energy is zero "vanishing of the cosmological constant".
2)	$N=1$	the "tadpole" amplitudes vanish, i.e. conformal field theory provides solutions (no linear terms) to string theory, at tree level.
3)	$N=2$	no mass corrections to tree level.

3) Green function and determinants

The scalar Green function on the sphere, with metric $\hat{g}$, is given by

$$G(z,z') = -\ln \frac{|z-z'|^2}{(1+|z|^2)(1+|z'|^2)} \ . \tag{3.63}$$

The determinants involving $\Delta_{(0)}$ and $\Delta^-_{(-1)}$ are just constants. They may be computed explicitly, but we shall not do so here. Instead, we abbreviate

$$\mathfrak{z} \equiv Z^-_{(-1)}(\hat{g}) \left(\frac{\mathrm{Det}'\Delta_{(0)}}{\int_\Sigma d\mu_{\hat{g}}} \right)^{-13} \ . \tag{3.64}$$

Furthermore, the only remaining finite-dimensional determinant is

$$\begin{aligned} |\det \psi_a(z_i)|^2 &= |\det \begin{pmatrix} 1 & 1 & 1 \\ z_{N-2} & z_{N-1} & z_N \\ z_{N-2}^2 & z_{N-1}^2 & z_N^2 \end{pmatrix} |^2 \\ &= |(z_{N-2}-z_{N-1})(z_{N-1}-z_N)(z_N-z_{N-2})|^2 \ . \end{aligned} \tag{3.65}$$

4) Tachyon Amplitudes

The tachyon vertex operator is particularly simple:

$$V_T(k) = \int_\Sigma d^2z\, W_T(z,k) \qquad\qquad W_T(z,k) = \varepsilon\sqrt{\hat{g}}\, e^{ik\cdot x}\ . \tag{3.66}$$

Here, ε is a constant, independent of k, and plays the same rôle as the field normalization Z-factors in the Lehmann Symanzik Zcmmerman (LSZ) formalism in quantum field theory: $\langle k|\varphi(x)|0\rangle = Z\, e^{ik\cdot x}$ for a scalar field $\varphi(x)$ and its associated 1-particle state $|k\rangle$ momentum k, and $|0\rangle$ the ground state.

We now make use of the results in p. II.23 on the correlation function of exponential operators. We introduce the renormalized ε_R by absorbing a multiplicative renormalization factor of the exponential

$$\varepsilon_R = \varepsilon\, e^{-G_R(z_i,z_i)_{\hat{g}}} \tag{3.67}$$

The amplitude now takes the form

$$A_0 = \delta(k) e^{-2\Phi_0} \varepsilon_R^N\, \mathfrak{z} \prod_{i=1}^{N-3} \int_{\mathbb{C}} d^2 z_i \prod_{i=1}^{N} \sqrt{\hat{g}(z_i)}\; e^{-\frac{1}{2}\sum\limits_{i\neq j} k_i\cdot k_j\, G(z_i,z_j)} \cdot |\det(\psi_a(z_i))|^2\ . \tag{3.68}$$

The contributions to G of the form $\ln(1+|z_i|^2)$ sum up and, using momentum conservation and k_i^2, precisely cancel the $\sqrt{\hat{g}}$ prefactors. We are left with

$$A_0(k_1\ldots k_N) = \delta(k)e^{-2\Phi_0}\varepsilon_R^N\, \mathfrak{z} |\det\ \psi_a(z_i)|^2 \prod_{i=1}^{N-3} d^2 z_i \prod_{i<j}^{N} |z_i - z_j|^{2k_i\cdot k_j} \tag{3.69}$$

Now, z_{N-2}, z_{N-1}, z_N were *any* 3 points on the space, so we may let them equal 0, 1, ∞, respectively, and we find

$$A_0(k_1,\ldots,k_N) = \delta(k)e^{-2\Phi_0}\varepsilon_R^N\, \mathfrak{z} \prod_{i=1}^{N-3}\int_{\mathbb{C}} d^2 z_i \prod_{i<j}^{N-1} |z_i - z_j|^{2k_i\cdot k_j}\ . \tag{3.70}$$

It is very instructive to examine amplitudes for small N:

(1) <u>three point function</u>

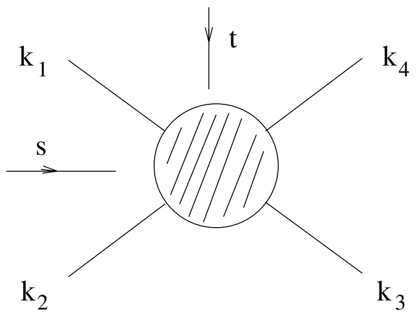
$$A_0(k_1,k_2,k_3) = \mathfrak{z}\ e^{-2\Phi_0}\varepsilon_R^3\delta(k)\ . \tag{3.71}$$

This 3-point function provides the 3-tachyon on shell coupling.

(2) <u>four point function</u>

$$A_0(k_1, \dots, k_4) = \mathfrak{z}\ e^{-2\Phi_0} \varepsilon_R^4 \delta(k) \int_{\mathbb{C}} d^2 z\ |z|^{2k_1 \cdot k_2} |1 - z|^{2k_1 \cdot k_3}\ . \tag{3.72}$$

It is standard to introduce the 3 Lorentz invariants s, t, u that characterize the kinematics of the 4-point function:



$$\begin{cases} s \equiv -(k_1 + k_2)^2 \\ t \equiv -(k_2 + k_3)^2 \\ u \equiv -(k_1 + k_3)^2\ . \end{cases} \tag{3.73}$$

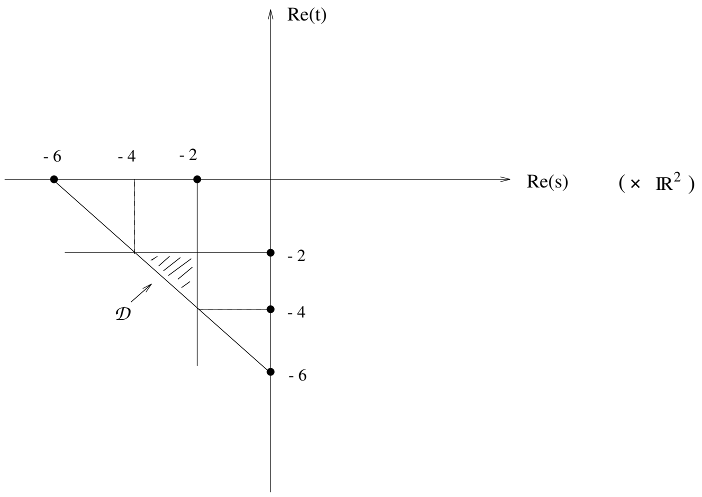
(Note that $s + t + u = -8$.) In terms of s, t, u the four point function is

$$A_0(k_1, \dots, k_u) = \mathfrak{z}\ e^{-2\Phi_0} \varepsilon_R^4 \delta(k) \int_{\mathbb{C}} d^2 z |z|^{-s-4} |z - 1|^{-u-4}\ . \tag{3.74}$$

This integral expression is absolutely convergent in the following region $\mathcal{D}$, defined by

$$\begin{cases} \mathrm{Re}(s) < -2 \\ \mathrm{Re}(u) < -2 \\ \mathrm{Re}(t) < -2 \end{cases} \qquad \text{or} \qquad \begin{cases} \mathrm{Re}(s) < -2 \\ \mathrm{Re}(t) < -2 \\ \mathrm{Re}(s+t) > -6\ . \end{cases} \tag{3.75}$$

Taking just the real parts:

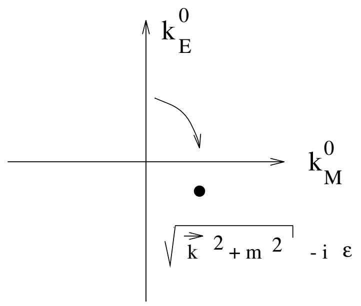


Let $s, t \in \mathfrak{D}$, then we may evaluate the amplitude by elementary methods, and we find the *Virasoro-Shapiro* amplitude:

$$A_0(s,t) = \mathfrak{z} e^{-2\Phi_0} \varepsilon_R^4 \delta(k) \pi \, \frac{\Gamma(-1-s/2)\Gamma(-1-t/2)\Gamma(-1-u/2)}{\Gamma(2+s/2)(\Gamma(2+t/2)\Gamma(2+u/2)} \ . \tag{3.76}$$

Since we know the analytic continuation of $\Gamma(z)$ throughout $\mathbb{C}$, with poles only at $z \in -\mathbb{N}$, we automatically have an analytic continuation of the full amplitude $A_0(s,t)$, throughout $(s,t) \in \mathbb{C} \times \mathbb{C}$. As a byproduct, we also immediately have the amplitude in Minkowski space-time momenta, by letting

$$k_i^0 \to -ik_i^0$$



$$\begin{cases} s \to s + i\epsilon \\ t \to t + i\epsilon \\ u \to u + i\epsilon \end{cases} \qquad \delta(k) \to i\delta(k)$$

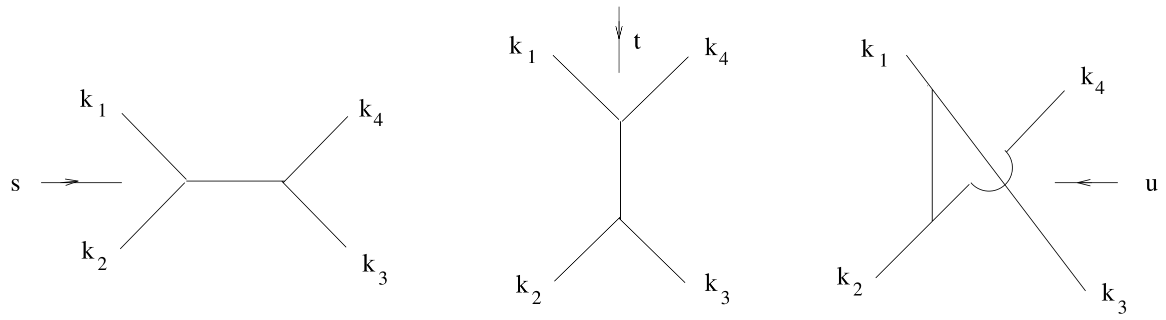
where it is understood that $\epsilon > 0$ and that we only consider the limit in which $\epsilon \to 0$. Thus, the 4-point amplitude in Minkowski space-time is given by

$$A_0(s,t) = \mathfrak{z} e^{-2\Phi_0} \epsilon_R^4 \delta(k) \pi i \frac{\Gamma(-1-2/2-i\epsilon)\Gamma(-1-t/2-i\epsilon)\Gamma(-1-u/2-i\epsilon)}{\Gamma(2+s/2+i\epsilon)\Gamma(2+t/2+i\epsilon)\Gamma(2+u/2+i\epsilon)} \ . \tag{3.77}$$

The amplitude has only pole singularities — as expected at tree level — and their location is easily identified. The Γ-function contributions from the denominator are entire. From the numerator, we have poles at

$$\begin{cases} s = -2 + 2n_s & n_s \in \mathbb{N} \\ \\ t = -2_2 n_t & n_t \in \mathbb{N} \\ \\ u = -2 + 2n_u & n_u \in \mathbb{N} \ . \end{cases} \tag{3.78}$$

These poles precisely correspond to the creation of intermediate string states whose mass is the above value for s, t or u. The states may occur either in the *s, t or u channel*:

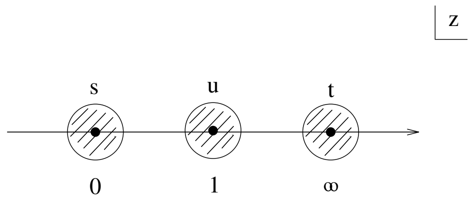


It is instructive to identify where these singularities arise from in the integral representation for the amplitude:

s poles: $z \to 0$

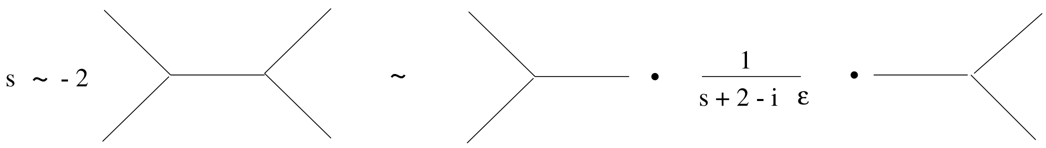
t poles: $z \to \infty$

u poles: $z \to 1\,.$



The fact that three QFT diagrams arise from a single string diagram is the simplest example of a general phenomenon mentioned in the introduction.

Let us concentrate on the $s = -2$ pole, which corresponds to a tachyon intermediate state. The diagram there *factorizes* into a propagator, and 2 three point functions for external tachyon:



Insisting upon this factorization, we obtain a non-linear equation between the 3-point function and 4-point function, expressing factorization — or unitarity — of the transition amplitudes (or so-called S-matrix elements). This relation fixes ε_R:

$$\varepsilon_R^2 = \frac{2\pi}{\mathfrak{z}}\, e^{2\Phi_0}\ . \tag{3.79}$$

J. One loop amplitudes

At one loop, the worksheet has the topology of a torus, and there is a single complex modulus $\tau \in \mathbb{C}$. We represent Σ by

$$\Sigma = \mathbb{C}/(\mathbb{Z} + \tau Z) \qquad\qquad \mathrm{Im}\,\tau > 0$$

with the flat metric $\hat{g} = 2|dz|^2$. Teichmüller space is $\mathcal{T}_1 = \{\tau \in \mathbb{C},\ \mathrm{Im}\,\tau > 0\}$, the mapping class group is $G_\Sigma = \mathrm{SL}(2,\mathbb{Z})$, acting on τ as $\mathrm{PSL}(2,\mathbb{Z})$ by

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \qquad\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{3.80}$$

Moduli space is conveniently taken as $\mathcal{T}_1/\mathrm{PSL}(2,\mathbb{Z}) = \mathcal{M}_1$, but we shall have to recall that the full mapping class group also divides out by $\{\pm I\}$ in $\mathrm{SL}(2,\mathbb{Z})$. It is standard to take the fundamental domain

$$\mathcal{M}_1 = \left\{ \tau = \tau_1 + i\tau_2, \tau_{1,2} \in \mathbb{R},\ \tau_2 > 0,\ |\tau_1| \le \frac{1}{2},\ |\tau| \ge 1 \right\} . \tag{3.81}$$

Determinants of Laplace operators on tensors of any integer weight are given by the case of Laplacians on functions, which was computed by Gawędzki:

$$\begin{aligned} \mathrm{Det}'\Delta_{(0)} &= \tau_2^2\, |\eta(\tau)|^4 \\ \eta(\tau) &\equiv e^{i\pi\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) . \end{aligned} \tag{3.82}$$

The Green function on the scalar field x is given by

$$G(z,w|\tau) = -\ln \left| \frac{\vartheta_1(z-w|\tau)}{\vartheta_1'(0|\tau)} \right|^2 - \frac{\pi}{2\tau_2}(z - w - \bar{z} + \bar{w})^2 , \tag{3.83}$$

where $\vartheta_1(z|\tau)$ is the Jacobi theta function that has a zero at $z = 0$.

The N-string transition amplitude is given by the following expression

$$A_1(k_1,\ldots,k_N) = \int_{\mathcal{M}_1} \frac{d^2\tau}{2\tau_2^2} \frac{1}{(4\pi^2\tau_2)^{12}} \frac{1}{|\eta(\tau)|^{48}} \ll V_1,\ldots,V_N \gg \tag{3.84}$$

where $\ll V_1 \ldots V_N \gg$ stands for the *normalized* correlation function of the vertex operators $V_1 \ldots V_N$: $\ll V_1 \ldots V_N \gg = \langle V_1 \ldots V_N \rangle \,/\, \langle 1 \rangle$.

For example, considering only tachyon vertex operators, we have

$$\ll V_1 \ldots V_N \gg = \delta(k) \varepsilon_R^N \prod_{i=1}^{N} \int d^2 z_i \exp \left(-\frac{1}{2} \sum_{i \neq j} k_i \cdot k_j G(z_i, z_j | \tau) \right) \tag{3.85}$$

There is no dependence on the dilaton constant Φ_0 here, because the Euler number on the torus vanishes. Other correlation functions may be similarly derived.

To one loop order, we may now also have non-zero $N = 0, 1, 2$ point functions. Let us consider $N = 0$ first: $\ll 1 \gg = 1$.

$$A_1(\ \cdot\ ) = \mathrm{Vol}(M) \int_{\mathcal{M}_1} \frac{d^2\tau}{2\tau_2^2} \frac{1}{(4\pi^2\tau_2)^{12}} \frac{1}{|\eta(\tau)|^{48}} \ . \tag{3.86}$$

Using the asymptotics for $\mathrm{Im}\, \tau \to \infty$,

$$\eta(\tau) \sim e^{i\pi\tau/12} = e^{i\pi\tau_1/12} e^{-\pi\tau_2/12} \tag{3.87}$$

we see immediately that this amplitude diverges for large τ_2. In fact, this $\eta(\tau)$ factor is common to all amplitudes, for any type of vertex operators, and produces a divergence in any amplitude, for any arrangement of external momenta. We shall not show this here in general, but leave this point to be checked by the reader. Thus, bosonic closed oriented string theory is divergent in flat space-time.

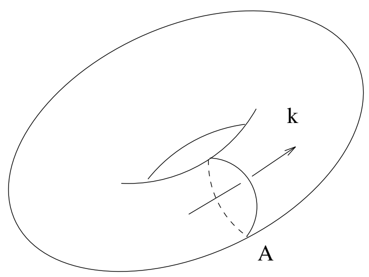
Nature of Divergence

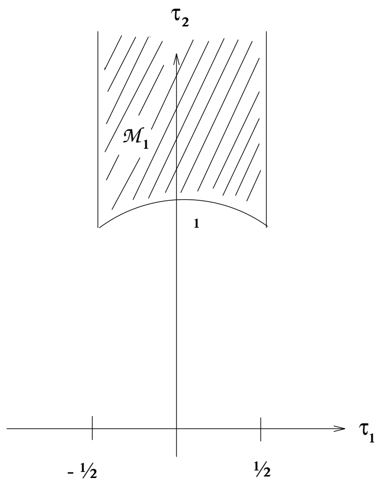
Let us try to gain additional insight into the physical origin of this divergence. To do so, it is helpful to exhibit the dependence on internal momenta by inserting a factor:

$$1 = \int d^{26}k \ \delta \left(k - \frac{1}{2\pi i} \oint_A dz \, \partial_z x \right) \ , \tag{3.88}$$

where A is the cycle in Σ whose pre-image in $\mathbb{C}$ runs from $z = 0$ to $z = 1$. We find

$$A_1(\ \cdot\ ) = \mathrm{Vol}(M) \int_{\mathbb{R}^{26}} d^{26}k \int_{\mathcal{M}_1} \frac{d^2\tau}{2\tau_2} \frac{1}{(4\pi^2)^{12}} \frac{1}{|\eta(\tau)|^{48}} e^{-2\pi\tau_2 k^2} \ . \tag{3.89}$$





Substituting the asymptotics of the $\eta(\tau)$-function

$$A_1(\,\cdot\,) = \mathrm{Vol}(M) \int\limits_{\mathbb{R}^{26}} d^{26}k \int_{\mathcal{M}_1} \frac{d^2\tau}{2\tau_2}\, \frac{1}{(4\pi^2)^{12}} e^{-2\pi\tau_2(k^2-2)}(1 + \# e^{-2\pi\tau_2} + \cdots\ )\ , \qquad (3.90)$$

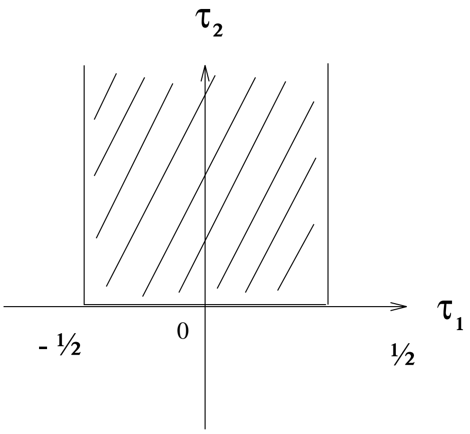
it is now immediately apparent why a divergence occurs, and where it occurs. For $\tau_2 \to \infty$, the momentum range $0 \le k^2 < 2$ contributes an exponentially growing factor. In fact, integrating out τ_2 for large τ_2:

$$\int \frac{d\tau_2}{\tau_2} e^{-2\pi\tau_2(k^2-2)} \sim \frac{1}{k^2-2}$$

This contribution arises from the tachyon! It is an I.R. problem related to the stability of flat space-time.

It is equally instructive to see where the integral is convergent: at large k^2, since the range of τ_2 is cut off from below by the choice of $\mathcal{M}_2$, the integral is exponentially suppressed for large momenta k^2. Thus, this loop amplitude is superconvergent at high energy, in contrast with field theory.

In fact, we can use this simple example to exhibit the distinction between what this partition function would look like in quantum field theory and what it looks like in string theory. In field theory, we may imagine the partition function for the same states, given by the same generating function. The only difference is the region $\mathcal{M}_1$:



Integration region for QFT
with the same states as string theory.

Now, since τ_2 reaches down to $\tau_2 = 0$, the superconvergence disappears and the integrand is power behaved in momenta. Thus, we see here that a crucial difference between string theory and quantum field theory arises from modular invariance, producing ultraviolet finite answers in the case of strings