Lecture 7

IV. Faddeev–Popov Ghosts — BRST Quantization

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It is standard practice to recast the Faddeev-Popov determinant, which arose from fixing Diff(Σ) gauge invariance, in terms of ghost fields. (See also Faddeev's lectures.) One introduces Grassmann-valued *ghost fields* $(c, \bar{c}) = (c^z (dz)^{-1}, c^{\bar{z}}(d\bar{z})^{-1})$ and *anti-ghost* fields $(b, \bar{b}) = (b_{zz}(dz)^2, b_{\bar{z}\bar{z}}(d\bar{z})^2)$ transforming as the gauge transformation vector fields $(v, \bar{v})$ and as the metric deformations $(\delta g_{zz}, \delta g_{\bar{z}\bar{z}})$, respectively. It is convenient to study such b-c systems for general weight first, since we shall make use of such systems as well: $c \in K^{1-n}; b \in K^n$. For now n is an integer, but later n will also be allowed to be a half-integer.

A. Determinants and b-c systems

We therefore generalize by considering a system of a pair of Grassmann-valued fields $(c, \bar{c}) = (c(dz)^{1-n}, \bar{c}(d\bar{z})^{1-n})$ and $(b, \bar{b}) = (b(dz)^n, \bar{b}(d\bar{z})^n)$, so that the above case is $n = 2$. In this notation, b and c are independent whereas $\bar{b}$, $\bar{c}$ are their complex conjugates. The action is first order in derivatives:

$$S_n(b, c) = \frac{1}{2\pi} \int_\Sigma d\mu_g \left(b \nabla^z_{(1-n)} c + \bar{b} \nabla^{\bar{z}}_{(n-1)} \bar{c} \right) \ . \tag{4.1}$$

We have the following *Theorem*:

$$\begin{aligned} &Z^-_{(n)}(g) |\det \phi_j(z_k)|^2 \ |\det \psi_a(w_b)|^2 \\ &\qquad = \int D(b\bar{b}) D(c\bar{c}) e^{-S_n(b,c)} \prod_{j=1}^{N^-_{(n)}} b(z_j)\bar{b}(z_j) \quad \prod_{a=1}^{N^-_{(n-1)}} c(w_a)\bar{c}(w_a) \ , \end{aligned} \tag{4.2}$$

where the measures $D(b\bar{b})$ and $D(c\bar{c})$ are obtained from the L^2 norms on K^n, $\bar{K}^n$, $K^{(1-n)}$ and $\bar{K}^{(1-n)}$; $\phi_j \in \text{Ker}\, \nabla^z_{(n)}$, $\psi_a \in \text{Ker}\, \nabla^z_{(1-n)}$ and $N^\pm_{(n)} = \dim \text{Ker}\, \Delta^\pm_{(n)}$. Note that for $n \geq 2$ there are no ψ_α's whereas for $h = 0$ there are no ϕ_j's.

We begin the proof by recalling a finite-dimensional Grassmann integral, over variables $b_i,\ c_i,\ i=1,\ldots,N$, such that $\{b_i,b_j\}=\{c_i,c_j\}=\{b_i,c_j\}=0$. We use the standard Grassmann integration rules:

$$\int dc_i\, c_j=\delta_{ij} \qquad\qquad \int db_i\, b_j=\delta_{ij} \tag{4.3}$$

and $\int dc_i\,1=\int db_i\,1=0$. Then, for any matrix M_{ij}, we have

$$\prod_{i=1}^{N}\int db_i dc_i\, e^{b_i M_{ij} c_j}=\det\ M\ . \tag{4.4}$$

To prove the Theorem, we begin by expanding b and c in an orthonormal basis of eigenfunctions of $\Delta^-_{(n)}$ and $\Delta^-_{(1-n)}$ as follows. Let Ψ_α be an orthonormal basis of eigenfunctions of $\Delta^-_{(1-n)}$ corresponding to the non-zero eigenvalues λ^2_α, and let ψ_a be an orthonormal basis of zero modes of $\Delta^-_{(1-n)}$. We have

$$\begin{cases}\Delta^-_{(1-n)}\Psi_\alpha=\lambda^2_\alpha\Psi_\alpha\\ \Delta^-_{(1-n)}\psi_a=0\ ,\end{cases} \tag{4.5}$$

and we define Φ_α by

$$\begin{cases}\sqrt{2}\,\nabla^z_{(1-n)}\Psi_\alpha=\lambda_\alpha\Phi_\alpha\\ -\sqrt{2}\nabla^{(1-n)}_z\Phi_\alpha=\lambda_\alpha\Psi_\alpha\ ,\end{cases} \tag{4.6}$$

so that Φ_α provides an orthonormal basis of eigenfunctions of $\Delta^+_{(-n)}$ for non-zero eigenvalues. Let ϕ_j be an orthonormal basis for Ker $\nabla^{(-n)}_z$. As a result, $(g_{z\bar z})^n\Phi^*_\alpha$ together with the zero modes $(g_{z\bar z})^n\phi^*_j$ provides an orthonormal basis of eigenfunctions for $\Delta^-_{(n)}$. Thus, we have the decompositions

$$\begin{aligned}c&=\sum_\alpha c_\alpha\Psi_\alpha+\sum_a c_a\psi_a\\ b&=\sum_\alpha b_\alpha(g_{z\bar z})^n\Phi^*_\alpha+\sum_j b_j(g_{z\bar z})^n\phi^*_j\ .\end{aligned} \tag{4.7}$$

The L^2 norm then induces the following measures:

$$\begin{cases}D(b\bar b)=\prod_\alpha db_\alpha\, d\bar b_\alpha\prod_j d\,b_j\ \ d\,\bar b_j\\ D(c\bar c)=\prod_\alpha dc_\alpha\, d\bar c_\alpha\prod_a dc_a\, d\bar c_a\end{cases} \tag{4.8}$$

and it is straightforward to evaluate the action:

$$
\begin{aligned}
S_n(b,c) &= \frac{1}{2\pi}\sum_{\alpha,\beta} b_\alpha c_\beta \int_\Sigma d\mu_g (g_{z\bar{z}})^n \Phi^*_\alpha \nabla^z_{(1-n)} \Psi_\beta + c.c. \\
&= \frac{1}{2\pi\sqrt{2}}\sum_{\alpha,\beta} b_\alpha c_\beta \lambda_\beta \int_\Sigma d\mu_g (g_{z\bar{z}})^n \Phi^*_\alpha \Phi_\beta + c.c \\
&= \frac{1}{2\pi\sqrt{2}}\sum_{\alpha} (b_\alpha c_\alpha \lambda_\alpha + \bar{b}_\alpha \bar{c}_\alpha \lambda_\alpha)\ .
\end{aligned} \tag{4.9}
$$

Notice that the zero mode coefficients b_j, c_a do not enter the action. Modulo a constant $\left(\frac{1}{2\pi\sqrt{2}}\right)^{\zeta(0)}$ coming from the ζ-function regularization,

$$
\int \prod_\alpha db_\alpha d\bar{b}_\alpha dc_\alpha d\bar{c}_\alpha \, e^{-S_n(b,c)} = \prod_\alpha \lambda^2_\alpha = \mathrm{Det}'\Delta^-_{(1-n)}\ . \tag{4.10}
$$

The remaining finite product of factors can only involve the zero modes, since the number of factors of b and c precisely equals the number of zero-mode integrations.

$$
\begin{aligned}
\int \prod_a dc_a d\bar{c}_a \prod_j db_j d\bar{b}_j \prod_{j=1}^{N^-_{(n)}} b(z_j)\bar{b}(z_j) \prod_{a=1}^{N^-_{(1-n)}} c(w_a)\bar{c}(w_a) \\
= |\det\ \phi_j(z_k)|^2\ |\det\ \psi_a(w_b)|^2\ .
\end{aligned} \tag{4.11}
$$

Recall that our bases ϕ_j, ψ_a had been chosen orthonormal. In an arbitrary basis, the result would be

$$
\frac{|\det\ \phi_j(z_k)|^2}{\det(\phi_j,\phi_k)}\ \frac{|\det(\psi_a(w_b))|^2}{\det(\psi_a,\psi_b)}\ . \tag{4.12}
$$

Putting all together, we obtain the result in the Theorem by using the definition of the determinant combination $Z^-_{(n)}(g)$.

Notice that the functional integral without b or c insertions,

$$
\int D(b\bar{b})D(c\bar{c})e^{-S_n(b,c)}\ ,
$$

vanishes because of the existence of zero modes which do not enter the action at all.

B. Ghost representation of the Faddeev-Popov determinant

For bosonic string theory, $n = 2$ is the suitable weight, and we have the following general form for the amplitudes

$$A = \sum_{h=0}^{\infty} e^{-\chi\Phi_0} \int_{\mathcal{M}_h} dm_j d\bar{m}_j \int D(b\bar{b})D(c\bar{c})e^{-S_2(b,c)} \prod_j |(\mu_j, b)|^2 \int_{\mathrm{Map}(\Sigma;M)} Dx\, \mathcal{O}\, e^{-S_m(x)} \tag{4.13}$$

where

$$\mathcal{O} = \begin{cases} \bullet & \prod_{a=N-2}^{N} c(z_a)\bar{c}(z_a)V_1 \ldots V_{N-3}W(z_{N-2})W(z_{N-1})W(z_N) & (N \geq 3;\ h = 0) \\ \bullet & 0 & (N = 0,1,2;\ h = 0) \\ \bullet & c(z_N)\bar{c}(z_N)V_1 \ldots V_{N-1}W(z_N) & (N \geq 1;\ h = 1) \\ \bullet & V_1 \ldots V_N\ , & (h \geq 2) \end{cases} \tag{4.14}$$

and $S_2(b,c)$ is the action for the ghost fields in the case $n = 2$. This representation is valid for any matter action S_m, and any set of vertex operators V_i. The fields c and b are called the Faddeev-Popov ghosts and anti-ghosts.

C. Conformal Field Theory of the b-c system

From the Weyl transformation law of the determinant combinations $Z^{-}_{(n)}(g)$, and the Weyl invariance of det $\phi_j(z_k)$ and det $\psi_a(w_b)$, we know that the b-c system is a conformal field theory with central charge

$$-2c^{-}_{(n)} = -2(6n^2 - 6n + 1)\ .$$

In particular, we confirm that for the ghost system of the bosonic string (i.e. $n = 2$) the central charge is $c_{\mathrm{ghosts}} = -26$, cancelling the central charge of the matter part. We now study this conformal field theory for any $n \in \frac{1}{2}\mathbb{Z}$, assuming that a spin structure has been chosen on Σ. In particular, for $n = \frac{1}{2}$, we have a Dirac fermion $\tilde{\psi}_+ = b$, $\psi_+ = c$; $\tilde{\psi}_- = \bar{b}$, $\psi_- = \bar{c}$ (in which case b, c are in the same space of Dirac spinors), or two Majorana spinors: $(\psi_+, \tilde{\psi}_+)$ and $(\psi_-, \tilde{\psi}_-) = (\psi_+, \tilde{\psi}_+)^*$. (See Witten's lectures.)

From the classical action, we obtain the stress tensor by varying with respect to δg^{zz} and using

$$\delta\nabla^z_{(1-n)} = \frac{1}{2}\delta g^{zz}\nabla_z^{(1-n)} + \frac{1-n}{2}\nabla_z(\delta g^{zz}) \ . \tag{4.15}$$

We find for the classical stress tensor:

$$\begin{aligned} T^{(bc)}(z) = T^{(bc)}_{zz} &= (1-n)\nabla_z bc - nb\nabla_z c \\ &= (1-n)(\partial_z b - n\Gamma b)c - nb(\partial_z c - (1-n)\Gamma c) \\ &= (1-n)(\partial_z b)c - nb\partial_z c \ . \end{aligned} \tag{4.16}$$

Using Weyl rescaling, we may study the system on a surface with locally flat metric $\hat{g} = 2|dz|^2$. At the same time, covariant derivatives become just Cauchy-Riemann operators: $\nabla^z_{(n)}$ becomes $\bar{\partial}_{(n)}$ and $\nabla_z^{(n)}$ becomes $\partial_{(n)}$ ($\bar{\partial}$ and ∂ for short). The action is then

$$S_n(b,c) = \frac{1}{2\pi}\int_\Sigma d^2\, z(b\bar{\partial}c + \bar{b}\partial\bar{c}) \ . \tag{4.17}$$

The field equations are $\bar{\partial}c = \bar{\partial}b = 0$, $\partial\bar{c} = \partial\bar{b} = 0$. The OPE of b and c is given by the Green function of $\bar{\partial}$:

$$b(z)c(w) \sim \frac{1}{z-w} \sim c(z)b(w) \ . \tag{4.18}$$

The quantum stress tensor may now be defined by subtracting the OPE singularities from the classical stress tensor. The holomorphic stress tensor is defined by

$$T^{(bc)}(z) = \lim_{w\to z}\left\{(1-n)\partial b(z)c(w) - nb(z)\partial c(w) + \frac{1}{(z-w)^2}\right\} \ . \tag{4.19}$$

One immediately verifies

$$\begin{aligned} T^{(bc)}(z)T^{(bc)}(w) &\sim -\frac{(6n^2-6n+1)}{(z-w)^4} + \frac{2}{(z-w)^2}T^{(bc)}(w) + \frac{1}{z-w}\partial_w T^{(bc)}_{(w)} \\ T^{(bc)}(z)c(w) &\sim \frac{1-n}{(z-w)^2}c(w) + \frac{1}{z-w}\partial_w c(w) \\ T^{(bc)}(z)b(w) &\sim \frac{n}{(z-w)^2}b(w) + \frac{1}{(z-w)}\partial_w b(w) \ . \end{aligned} \tag{4.20}$$

(There is no singularity in the expansion of $b(z)b(w)$ or $c(z)c(w)$.) The classical action S_n is invariant under $U(1)_{bc}\times U(1)_{\bar{b}\bar{c}}$ phase rotations:

$$\begin{cases} c \to e^{i\theta}c \\ b \to e^{-i\theta}b \end{cases} \qquad \begin{cases} \bar{c} \to e^{-i\theta'}\bar{c} \\ \bar{b} \to e^{i\theta'}\bar{b} \end{cases} \tag{4.21}$$

Only the diagonal subgroup with $\theta = \theta'$ leaves the measure $D(b\bar{b})D(c\bar{c})$ invariant, and results in a quantum symmetry.

The current associated with $U(1)_{bc}$ is given by

$$j_z^{(bc)} = :cb: = \lim_{w\to z}\left\{c(w)b(z) - \frac{1}{w-z}\right\} , \tag{4.22}$$

and we find the OPE's:

$$j_z^{(bc)}c(w) \sim -\frac{1}{z-w}\,c(w) \; ; \qquad\qquad j_z^{(bc)}b(w) \sim \frac{1}{z-w}\,b(w) \; . \tag{4.23}$$

The current $j_z^{(bc)}$ has an anomaly, as can be established from the fact that $j_z^{(bc)}$ does not transform as a tensor (for $n \neq \frac{1}{2}$):

$$T^{(c)}(z)j_w^{(bc)} \sim \frac{1-2n}{(z-w)^3} + \frac{1}{(z-w)^2}j_w^{(bc)} + \frac{1}{z-w}\partial_w j_w^{(bc)} \; . \tag{4.24}$$

Thus, under analytic diffeomorphisms $z \to z' = f(z)$, we have

$$j'^{(bc)}(z')dz' = j^{(bc)}(z)dz + \left(n - \frac{1}{2}\right) d(\ln\, f'(z)) \tag{4.25}$$

Note that there is no anomaly for the case of spinors when $n = 1/2$. Also, on a worldsheet with general metric g, $j^{(bc)}$ is not conserved; instead $\nabla^z j_z^{(bc)} = \left(\frac{1}{2} - n\right) R_g$.

D. Bosonization of the b-c system

In Witten's lectures, a Bose-Fermi correspondence was established between a 2-dimensional Dirac fermion and a real scalar boson field. The Dirac fermion corresponds here to the case $n = \frac{1}{2}$. We will generalize the Bose-Fermi correspondence to the case of any $n \in \frac{1}{2}\mathbb{Z}$. We begin by noticing that the b-c stress tensor can be expressed in terms of the $U(1)_{bc}$ current only:

$$T^{(bc)}(z) = \frac{1}{2}j^{(bc)}(z)^2 + \left(n - \frac{1}{2}\right)\partial_z j^{(bc)}(z) \; , \tag{4.26}$$

where the product of operators is always renormalized à la OPE. Using

$$j^{(bc)}(z)j^{(bc)}(w) \sim \frac{1}{(z-w)^2} \; , \tag{4.27}$$

it is straightforward to verify that $T^{(bc)}$ satisfies all identities listed above.

We now introduce a scalar field ϕ, with action (for general metric on Σ)

$$S_Q(x) = \frac{1}{4\pi} \int_\Sigma d\mu_g \left\{ \frac{1}{2} g^{mn} \partial_m \phi \partial_n \phi + Q R_g \phi \right\} . \tag{4.28}$$

This system was studied extensively in Problem Set #1. Let us consider it only on locally flat metrics. Its central charge is $c = 1 + 3Q^2$, and there is a current $\partial_z \phi$. Identifying the two systems will require that they have equal central charge:

$$1 + 3Q^2 = -12n^2 + 12n - 2 \qquad \text{or} \qquad Q = i(2n-1) . \tag{4.29}$$

(The second solution $Q = -i(2n-1)$ is equivalent upon changing the sign of ϕ.) The identification of the currents gives

$$\begin{aligned} j^{(bc)}(z) &= i\partial_z \phi \\ T^{(bc)}(z) &= -\frac{1}{2}(\partial_z \phi)^2 + \frac{1}{2} Q \partial_z^2 \phi . \end{aligned} \tag{4.30}$$

The fields b, c and $\bar b$, $\bar c$ may be identified in terms of the *chiral boson fields* $\phi_+(z)$,$\phi_-(\bar z)$, with $\phi(z, \bar z) = \phi_+(z) + \phi_-(\bar z)$ by

$$\begin{cases} c(z) = e^{i\phi_+(z)} \\ b(z) = e^{-i\phi_+(z)} \end{cases} \qquad \begin{cases} \bar c(\bar z) = e^{+i\phi_-(\bar z)} \\ \bar b(\bar z) = e^{-i\phi_-(\bar z)} . \end{cases} \tag{4.31}$$

The $U(1)_{bc} \times U(1)_{\bar b \bar c}$ transformations now simply shift ϕ_+ and ϕ_- by θ and $-\theta'$, respectively.

It is now easy to see that the number (N_b, N_c) of b and c operator insertions yielding non-zero correlation functions is constrained by the Riemann-Roch-Atiyah-Singer theorem:

$$N_b - N_c - \frac{i}{2} Q \chi = 0 \qquad \text{or} \qquad N_b - N_c = \left(\frac{1}{2} - n \right) \chi ,$$

where χ is the Euler number of Σ.

E. The b-c Fock Space

The full Fock space is obtained as a tensor product of the bc Fock space $\mathcal{F}^{(bc)}$ and the $\bar b \bar c$ Fock space $\mathcal{F}^{(\bar b \bar c)}$:

$$\mathrm{Fock}(b, c, \bar b, \bar c) = \mathcal{F}^{(bc)} \otimes \mathcal{F}^{(\bar b \bar c)} . \tag{4.32}$$

We proceed to construct $\mathcal{F}^{(bc)}$, and $\mathcal{F}^{(\bar{b}\bar{c})}$ is constructed analogously. To do so, we consider the system on an annulus Σ, centered at $z=0$, with flat metric $g=2|dz|^2$, and we expand the holomorphic fields $b(z)$ and $c(z)$ in a Laurent series:

$$
\begin{aligned}
c(z) &= \sum_r c_r\, z^{-r-(1-n)} \\
b(z) &= \sum_r b_r\, z^{-r-n}
\end{aligned}
\qquad\qquad r\in\mathbb{Z}\ . \tag{4.33}
$$

The shifts in the powers of z reflect the conformal weights of the fields. The OPE of b and c yields

$$
\{b_r, b_s\} = \{c_r, c_s\} = 0 \qquad \{b_r, c_s\} = \delta_{r+s,0}\ . \tag{4.34}
$$

The stress tensor (renormalized as in IV.7) yields the Virasoro generators

$$
\begin{aligned}
L_m^{(bc)} &= \sum_{r\in\mathbb{Z}} (mn-r) b_r c_{m-r} \qquad m\neq 0 \\
L_0^{(bc)} &= \sum_{r=1}^{\infty} r(b_{-r}c_r + c_{-r}b_r) - \frac{1}{2}\,n(n-1)\ .
\end{aligned} \tag{4.35}
$$

The $U(1)_{bc}$ current gives the bc number operator:

$$
U = \oint \frac{dz}{2\pi i} j_z^{(bc)} = \sum_{r=1}^{\infty} (c_{-r}b_r - b_{-r}c_r) + \frac{1}{2}(c_0 b_0 - b_0 c_0) + n - \frac{1}{2}\ . \tag{4.36}
$$

(The constant shifts in $L_0^{(bc)}$ and U_0 require great care, and are fundamental in the sequel.)

The polarization here is again inherited from Minkowski worldsheet signature:

$$
b_r^\dagger = b_{-r} \qquad c_r^\dagger = c_{-r}\,; \qquad\qquad r\in\mathbb{Z}\ . \tag{4.37}
$$

The ground state of the b-c system is doubly degenerate, since b_0 and c_0 commute with L_0 (in fact they do not enter into the expession for L_0), and since b_0, c_0 generate an algebra $\{b_0, c_0\} = 1$, $b_0^2 = c_0^2 = 0$ of which the ground state is a two-dimensional representation. We define the basis vectors by $|\uparrow\rangle$ and $|\downarrow\rangle$, with

$$
b_r, c_r\ |\uparrow\rangle \quad |\downarrow\rangle = 0 \qquad\qquad r>0\ . \tag{4.38}
$$

We choose the states so that

$$\begin{aligned} c_0\,|\uparrow\rangle &= 0 \\ b_0\,|\downarrow\rangle &= 0\ . \end{aligned} \tag{4.39}$$

It follows that

$$\begin{aligned} b_0\,|\uparrow\rangle &= |\downarrow\rangle & \qquad U\,|\uparrow\rangle &= n\,|\uparrow\rangle \\ c_0\,|\downarrow\rangle &= |\uparrow\rangle & \qquad U\,|\downarrow\rangle &= (n-1)\,|\downarrow\rangle\ . \end{aligned} \tag{4.40}$$

It is natural to choose the following inner products

$$\langle\downarrow|\uparrow\rangle = \langle\uparrow|\downarrow\rangle = 1 \tag{4.41}$$

(Notice that $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 0$ by their properties under b_0 and c_0.)

The b-c Fock space is naturally obtained by applying the creation operators $\{b_{-r}, c_{-r}\}_{r\in\mathbb{N}}$ to either one of the ground states $|\uparrow\rangle$ or $|\downarrow\rangle$. The inner product thus inherited on $\mathcal{F}^{(bc)}$ is not positive definite.

F. BRST Quantization

We now return to string theory, where the relevant b-c system is the one with $n = 2$, in the notation used above. The combined matter-ghost action is (in a locally flat g)

$$S = \frac{1}{2\pi}\int d^2z \left(\frac{1}{2}\partial x\cdot\bar\partial x + b\bar\partial c + \bar b\partial\bar c\right)\ . \tag{4.42}$$

The combined stress tensor (we denote the x stress tensor by $T^{(x)}$)

$$T(z) = T^{(x)}(z) + T^{(bc)}(z) \tag{4.43}$$

has zero central charge in $D = 26$ and is thus a primary field of weight $(2,0)$. The associated Virasoro generators are

$$\begin{cases} L_m = \sum\limits_{r\in\mathbb{Z}}(2m-r)b_r c_{m-r} + \dfrac{1}{2}\sum\limits_{r\in\mathbb{Z}} x_r\cdot x_{m-r} \quad (m\neq 0) \\ L_0 = \dfrac{1}{2}p^2 - 1 + \sum\limits_{r=1}^{\infty} r(x_{-r}x_r + b_{-r}c_r + c_{-r}b_r) \end{cases} \tag{4.44}$$

The second term (-1) in the second equation arises from the ghost stress tensor on p. IV. 8 for $n = 2$. The Virasoro algebra of L_m reduces to the Witt algebra (i.e. with zero central charge)

$$[L_\ell, L_m] = (\ell - m)L_{\ell+m} \tag{4.45}$$

Recall the ghost number

$$U = 1 + c_0 b_0 + \sum_{r=1}^{\infty}(c_{-r}b_r - b_{-r}c_r) \tag{4.46}$$

and recall that the round states $|\uparrow\rangle$ and $|\downarrow\rangle$ have ghost number 2 and 1, respectively.

The full Fock space of states of the closed bosonic string, including the ghost states, is

$$\mathcal{F}_{\text{closed}} = \left(\bigoplus_{k\in\mathbb{R}^{26}} \mathcal{F}_k \otimes \widetilde{\mathcal{F}}_k\right) \otimes \mathcal{F}^{(bc)} \otimes \mathcal{F}^{(\bar{b}\bar{c})} \tag{4.47}$$

We now wish to identify, inside $\mathcal{F}$, the subspace of physical states. We shall concentrate on $\mathcal{F}_k \otimes \mathcal{F}^{(bc)}$, the treatment of $\widetilde{\mathcal{F}}_k \otimes \mathcal{F}^{(\bar{b}\bar{c})}$ is analogous.

Since the Virasoro algebra of L_ℓ generators holds with vanishing central charge, it is natural to impose

$$L_\ell \mid \phi\rangle = 0 \qquad\qquad \forall\ \ell \in \mathbb{Z}\ .$$

However, since the Fock space now also contains ghost states, this condition is insufficient. One would like to supplement it with a condition of "no ghosts". But from the treatment above, we see that ghost number changes under conformal mappings, so the no ghost condition is not invariant. The proper framework for imposing physical state conditions, when ghosts are present, is BRST quantization. It permits to recast the physical state conditions in terms of cohomology.

We begin by discussing a finite-dimensional case, with finite-dimensional Lie group G and associated Lie algebra $\mathcal{G}$. Let Λ be a representation of $\mathcal{G}$, with representation matrices Λ_i, $i = 1, \ldots, \dim G$, and let f_{ij}^k be the structure constants of $\mathcal{G}$:

$$[\Lambda_i, \Lambda_j] = f_{ij}^k \Lambda_k \tag{4.48}$$

One introduces anti-ghosts b_i in the adjoint representation of $\mathcal{G}$ and ghosts c^i in the dual to the adjoint representation of $\mathcal{G}$. (For compact G, these are identified, but in general, it may not be possible to do so.) These satisfy

$$\{b_i, c^j\} = \delta_j^j \qquad\qquad \{c^i, c^j\} = \{b_i, b_j\} = 0 \ . \tag{4.49}$$

The number operator is the analogue of the ghost number

$$U = c^i b_i \qquad\qquad [U, c^i] = c^i \ ; \ \ [U, b_i] = -b_i \tag{4.50}$$

and is invariant under $\mathcal{G}$. The Fock space $\mathcal{F}$ is determined by choosing a polarization. The natural polarization here is as follows

$$b_i \mid 0\rangle = 0 \qquad\qquad \mathcal{F} = \{c^i\} \mid 0\rangle \ , \tag{4.51}$$

and $\mathcal{F}$ contains $2^{\dim\ G}$ states.

This model may be realized concretely by taking c^i, $i = 1, \ldots, \dim\ G$ to be a basis of left-invariant 1-forms on G, and b_i to be the *contraction* with the form c^i: $b_i = \partial/\partial c^i$. The operator U gives the degree of the form and the ground state $\mid 0\rangle$ corresponds to forms of degree 0, i.e. constants. The Fock space $\mathcal{F}$ is the vector space of all left-invariant forms on G: $\Omega^{\mathrm{inv}}(G)$.

In BRST theory, one introduces the *BRST operator* (still for our finite-dimensional group G):

$$Q \equiv c^i \Lambda_i - \frac{1}{2} f_{ij}^k c^i c^j b_k \ . \tag{4.52}$$

This operator acts on $\mathcal{F} \otimes \Lambda$, where Λ is the representation vector space of Λ. This operator is *nilpotent.* To see this, first show that $Q^2 = 0$ when $\Lambda_i = 0$; the remaining terms with Λ present are

$$Q^2 = \frac{1}{2} c^i c^j \left([\Lambda_i, \Lambda_j] - f_{ij}^k \Lambda_k\right) \ , \tag{4.53}$$

which vanishes in view of the structure relations on Λ.

In terms of left-invariant forms on the group G, the operator Q has a natural interpretation as well. This is understood by examining how a differential acts on a left-invariant differential form in $\Omega^{(p)}(G)$: if $\omega \in \Omega^{(p)}(G)$ then $\omega = \omega_{i_1} \ldots {}_{i_p}\, c^{i_1} \wedge \ldots \wedge c^{i_p}$, with constant $\omega_{i_1 \ldots i_p}$. The differential on c^i is given by the Cartan structure equations

$$dc^i + \frac{1}{2}\, f^i_{jk} c^j \wedge c^k = 0 \tag{4.54}$$

and thus its action on ω may be recast in a purely algebraic form

$$d\omega = -\frac{1}{2}\, f^i_{jk} c^j c^k b_i \ . \tag{4.55}$$

Furthermore, when ω transforms under the representation Λ of $\mathcal{G}$, then Q is nothing but the covariant derivative acting on $\mathcal{F} \otimes \Lambda$, and its square vanishes in view of the flatness of the connection c^i.

We now consider states in the Fock space $\mathcal{F}_{(m)}$ of degree $m = 0, 1, \ldots,$ dim G. The BRST operator Q has degree 1, and maps Q: $\mathcal{F}_{(m)} \to \mathcal{F}_{(m+1)}$ for $m \leq$ dim $G - 1$, and Q: $\mathcal{F}_{(\mathrm{dim}\ G)} \to 0$. A state $|\psi\rangle$ is *BRST invariant* if

$$Q\ |\psi\rangle = 0 \qquad\qquad \text{(i.e. } Q\text{-closed)}$$

and *BRST trivial* if for some $|x\rangle \in \mathcal{F}_{(m-1)}$, we have

$$|\psi\rangle = Q\ |\chi\rangle \qquad\qquad \text{(i.e. } Q\text{-exact)}$$

BRST cohomology is defined by the following equivalence relation: $|\psi\rangle \sim |\psi'\rangle$ provided there is some $|\chi\rangle$ such that

$$|\psi'\rangle = |\psi\rangle + Q\ |\chi\rangle \ .$$

BRST cohomology classes of degree m are denoted by $H^{(m)}(Q)$. The BRST cohomology of degree 0 is easily computed. First, $|\psi\rangle \in \mathcal{F}_{(0)}$ means that $b_j\ |\psi\rangle = 0$ for all j, so that the condition of BRST invariance becomes

$$Q\ |\psi\rangle = c^i \Lambda_i\ |\psi\rangle = 0 \qquad \Rightarrow \qquad \Lambda_i\ |\psi\rangle = 0$$

Here, we have used the fact that, since $\{b_i, c^j\} = \delta^j_i$, we have $c^i\ |\psi\rangle \neq 0$ for all i, and these vectors are linearly independent. Thus, all BRST invariant states of ghost number 0 are

$\mathcal{G}$-invariant. Clearly no states in $\mathcal{F}_{(0)}$ are BRST trivial. Vice versa, all $\mathcal{G}$-invariant states of ghost number 0 are BRST invariant.

In string theory, the BRST formalism is implemented as follows. The Lie aglebra $\mathcal{G}$ is the infinite-dimensional Virasoro algebra, with structure constants

$$[L_i, L_j] = (i-j)L_{i+j} + \text{ central}; \quad f_{ij}^k = (i-j)\delta_{i+j}^k \ . \tag{4.56}$$

We do not specify the central term, since we shall not infer its implications from the finite-dimensional model. The representation Λ of $\mathcal{G}$ is given by the matter fields x, with generators $L_i^{(x)}$, and central charge 26. The ghost and anti-ghost modes are

$$c^\ell = c_{-\ell} \qquad \text{and} \qquad b_\ell \qquad\qquad \ell \in \mathbb{Z} \ . \tag{4.57}$$

The polarization for the string theory BRST is different from that for the finite-dimensional model

$$b_\ell, c_\ell \mid\downarrow\rangle \ , \quad \mid\uparrow\rangle = 0 \qquad \ell > 0 \ . \tag{4.58}$$

The BRST operator is defined by the structure constants f_{ij}^k, but the precise ordering prescription, and the effects of the central terms cannot be obtained from the finite-dimensional model. It will turn out (see next paragraph) that the correct prescription is given by

$$Q \equiv \sum_{r\in\mathbb{Z}} (L_r^{(x)} - \delta_{r,0})c_{-r} - \frac{1}{2}\sum_{r,s\in\mathbb{Z}} (r-s) \ : c_{-r}c_{-s}b_{r+s} : \tag{4.59}$$

The correct prescription may be derived most efficiently from the fact that the BRST charge derives from a conserved (in fact analytic) BRST current, which arises as the Noether current of BRST symmetry of the action S. The transformation rules are (they may be derived from the commutation of the operator λQ, for some constant Grassmann parameter λ, with the fields x^μ, b, c):

$$\begin{cases} \delta x^\mu = \lambda c \partial x^\mu \\ \delta c = \lambda c \partial c \\ \delta b = 2\lambda (T^{(x)} + T^{(bc)}) \ . \end{cases} \tag{4.60}$$

The resulting BRST current is

$$
\begin{aligned}
j^{\mathrm{BRST}}(z) &= -\frac{1}{2}\, c\partial x \cdot \partial x + bc\partial c + \frac{3}{2}\, \partial^2 c \\
&= cT^{(x)} + \frac{1}{2}\, cT^{(bc)} + \frac{3}{2}\, \partial^2 c \ .
\end{aligned} \tag{4.61}
$$

The last term on the right hand side ensures that j^{BRST} is a $(1,0)$ form as a quantum operator. The quantum (renormalized) current is defined as usual by OPE singularity subtractions. The BRST charge is then

$$
Q = \oint \frac{dz}{2\pi i}\, j^{\mathrm{BRST}}(z) \ , \tag{4.62}
$$

and agrees with the expression of (4.59).

Using the OPE algebra for the b-c system, derived in §C, we find that for $D = 26$ we have $Q^2 = 0$. Conversely, if $Q^2 = 0$, then the Virasoro algebra of $L_m = L_m^{(x)} + L_m^{gh)}$ must be satisfied with zero central charge. This is easily established from the fact that

$$
L_m = \{Q, b_m\} \qquad \Longrightarrow \qquad [L_m, Q] = 0 \tag{4.63}
$$

and hence

$$
[L_m, L_n] = \{Q, [L_m, b_n]\} = (m-n)\{Q, b_{m+n}\} = (m-n)L_{m+n} \ .
$$

The BRST invariance condition $Q\,|\psi\rangle = 0$ now reproduces the physical state conditions $L_m\,|\psi\rangle = 0$, $m \in \mathbb{Z}$, in the following way. First, we define a map i from $\mathcal{F}_k^{\mathrm{phys}}$ into $\mathcal{F}_k^{\mathrm{phys}} \otimes \mathcal{F}^{(bc)}$. which maps physical states in $\mathcal{F}_k^{\mathrm{phys}}$ into states of ghost number 1 as follows:

$$
i\colon \mathcal{F}_k^{\mathrm{phys}} \to \mathcal{F}_k \otimes \mathcal{F}^{(bc)}, \ \ |\varepsilon, k\rangle \to |\varepsilon, k, \downarrow\rangle = |\varepsilon, k\rangle \otimes |\downarrow\rangle \tag{4.64}
$$

Since $b_m\,|\downarrow\rangle = 0$ for all $m \in \mathbb{N}$ and $c_n\,|\downarrow\rangle = 0$ for all $n \in \mathbb{N}^+$, it follows that if $|\epsilon, k\rangle$ is physical, then $Q\,|\varepsilon, k, \downarrow\rangle = 0$. Thus, i is really a map into $\mathrm{Ker}\, Q$

$$
i\colon \mathcal{F}_k^{\mathrm{phys}} \to \mathrm{Ker}\, Q \ . \tag{4.65}
$$

Next, let us consider a state $|\psi\rangle \in \mathcal{F}_k \otimes \mathcal{F}^{(bc)}$, of the form

$$
|\psi\rangle = |\epsilon, k\rangle \otimes |\downarrow\rangle \tag{4.66}
$$

with ghost number 1, and belonging to Ker Q: $Q\,|\psi\rangle = 0$. Then, in view of $b_m\,|\downarrow\rangle = 0$ for $m \in \mathbb{N}$ and $c_n\,|\downarrow\rangle = 0$ for $n \in \mathbb{N}^+$, we have

$$\begin{aligned} Q\,|\psi\rangle &= \sum_{r\in\mathbb{Z}} (L_r^{(x)} - \delta_{r,0}) c_{-r}\,|\varepsilon, k\rangle \otimes |\downarrow\rangle \\ &= \left\{ (L_0^{(x)} - 1) c_0 + \sum_{r=1}^{\infty} L_r^{(x)} c_{-r} \right\} |\varepsilon, k\rangle \otimes |\downarrow\rangle \ , \end{aligned} \tag{4.67}$$

The condition $Q\,|\psi\rangle = 0$ implies $(L_0^{(x)} - 1)\,|\varepsilon, k\rangle = 0$ and $L_r^{(x)}\,|\varepsilon, k\rangle = 0$ for $r \in \mathbb{N}^+$, which are the correct physical state conditions on $\mathcal{F}_k$. (The ground state $|\uparrow\rangle$ would miss the conditions $(L_0^{(x)} - 1)\,|\varepsilon, k\rangle = 0$.)

Which of these states are BRST trivial? Again, from the definition of Q and the fact that $b_m\,|\psi\rangle = c_m\,|\psi\rangle = 0$ for $m \in \mathbb{N}^+$, it follows that if $|\psi\rangle = Q\,|\lambda\rangle$, with $|\lambda\rangle$ of ghost number 0 then

$$|\psi\rangle = \sum_{r=1}^{\infty} L_{-r}^{(x)}\,|\lambda_r\rangle \tag{4.68}$$

for some states $|\lambda_r\rangle$. But then $|\lambda\rangle$ is spurious.

In fact, one can show that any state $|\psi\rangle$ in Ker Q, of ghost number 1, is of the form

$$|\psi\rangle = |\varepsilon, k\rangle \otimes |\downarrow\rangle + Q\,|x\rangle$$

where $|\varepsilon, k\rangle \in \mathcal{F}_k^{\text{phys}}$ and $|x\rangle$ is some state of ghost number 0. Thus, the cohomology of Q of ghost number 1, i.e. space of states of ghost number 1 which are BRST invariant, modulo BRST trivial states, is isomorphic to the physical state space $\mathcal{F}^+$ defined previously:

$$\mathcal{F}^+ = \mathcal{F}^{\text{phys}} / \mathcal{F}^{\text{spurious}} \sim H^{(1)}(Q) \tag{4.69}$$