— DRAFT — IASSNS-HEP-97/72

Lecture 8

V. Moduli Dependence of Determinants and Green Functions

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In §III and §IV, we have reduced the calculation of string transition amplitudes to integrals of combinations involving certain determinants and Green functions over the finite-dimensional moduli spaces $\mathcal{M}_h$. In this section, we complete the analysis by producing explicit formulas for these determinants and Green functions. We shall begin by collecting results on surfaces with constant curvature metrics. Then, we proceed and exploit the complex structure of $\mathcal{M}_h$ to obtain explicit expressions in terms of Jacobi theta functions. (Derivations of the results stated in this section will not be given here; they may be found in E. D'Hoker, D. H. Phong, Rev. Mod. Phys. Vol. 60 (1988), p. 917–1065, and references therein.)

A) Worldsheets with constant curvature metric ($h \geq 2$)

Let Σ be a compact surface of genus $h \geq 2$. By the uniformization theorem, the simply connected cover of Σ is the upper half plane H, and $\Sigma = H/\Gamma$, where $\Gamma = \pi_1(\Sigma)$ is the fundamental group of Σ. For any metric g on Σ, there is a unique Weyl transformation σ such that $g = e^{2\sigma}\hat{g}$ with $\hat{g}$ of constant curvature $R_{\hat{g}} = -1$. The metric $\hat{g}$ lifts to the Poincaré metric on H and Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ acting on H, with compact quotient $\Sigma = H/\Gamma$. Moduli space may now be identified as follows:

$$\mathcal{M}_h = \{\hat{g}, R_{\hat{g}} = -1\} / \mathrm{Diff}(\Sigma) \ . \qquad (5.1)$$

We now have the following results for determinants and for the measure on $\mathcal{M}_h$ arising in string amplitudes:

- $$\mathrm{Det}\,\Delta^+_{(n)} = S(n+1)\,; \quad \mathrm{Det}'\Delta_{(0)} = S'(1) \qquad (5.2)$$
- $$\prod_j dm_j d\bar{m}_j |\det(\mu_j,\phi_k)|^2 \det(\phi_j,\phi_k)^{-1} = d(\mathrm{Weil} - \mathrm{Peterson}) \ . \qquad (5.3)$$

Here $S(s)$ is the *Selberg zeta function*, defined by

$$S(s) \equiv \prod_{\gamma \text{ primitive}} \prod_{k=1}^{\infty} (1 - e^{-(s+k)\ell_\gamma}) \tag{5.4}$$

for ℓ_γ the lengths of the *simple closed* (*i.e. primitive*) *geodesics* γ on $\Sigma = H/\Gamma$.

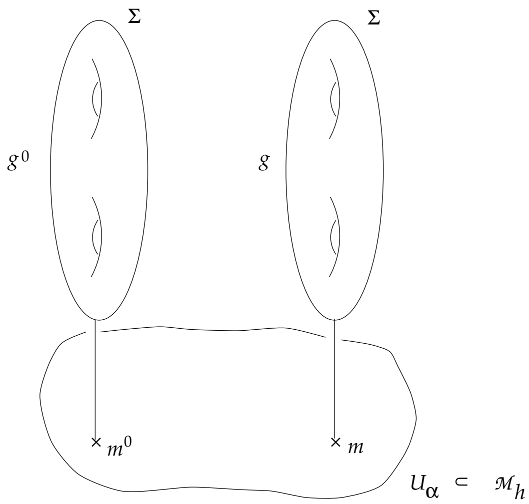
B) **Holomorphicity in Moduli**

The most powerful results on Green functions and determinants are obtained by exploiting the fact that $\mathcal{M}_h$ is a complex orbifold, and that Green functions and determinants primarily involve complex analytic dependence on $\mathcal{M}_h$.

To examine this complex dependence, we first work locally on $\mathcal{M}_h$, in a neighborhood of a point $m^0 \in \mathcal{M}_h$, represented by a metric g^0. We now parametrize the neighborhood U_α of m^0 and thus of g^0 by Beltrami differentials $\mu = \mu_{\bar{z}}{}^{z}$, which provide complex coordinates on $\mathcal{M}_h$, as established in §III. We have a general metric g in the neighborhood of g^0:

$$g = 2g_{z\bar{z}}|dz + \mu d\bar{z}|^2 \qquad \text{where} \qquad g^0 = 2g_{z\bar{z}}|dz|^2 \ . \tag{5.6}$$

We retain the Weyl factor $g_{z\bar{z}}$ for now, since we shall also study the interplay of Weyl dependence and moduli dependence.



The dependence of the functional determinant combinations $Z^{\pm}_{(n)}(g)$ on the complex coordinates of $\mathcal{M}_h$ is simple, and may be evaluated explicitly. This result is the content of the so-called *Belavin-Knizhnik theorem*; or holomorphic anomaly theorem:

$$\delta_\mu \delta_{\bar\mu} \ln Z^\pm_{(n)}(g) = -\frac{c^\pm_{(n)}}{12\pi} \int_\Sigma d\mu_g \, (\nabla_z \delta\mu \nabla_{\bar z} \delta\bar\mu - R_g \delta\mu\delta\bar\mu) \tag{5.7}$$

where the coefficients $c^\pm_{(n)} = 6n^2 \pm 6n + 1$ are the same as those entering the Weyl anomaly.

It follows immediately from the Belavin-Knizhnik theorem that Weyl invariant combinations of $Z^\pm_{(n)}(g)$'s are also especially simple in their dependence on moduli. Indeed, the $\delta_\mu\delta_{\bar\mu}$ variation of any Weyl invariant combination vanishes, so that such a combination is a sum of a holomorphic and an anti-holomorphic part. We have, e.g.

$$Z^\pm_{(n)}(g)\, Z_{(0)}(g)^{-c^\pm_{(n)}} = |\mathcal{Z}^\pm_{(n)}(m_j)|^2 \; . \tag{5.8}$$

where $\mathcal{Z}^\pm_{(n)}(m_j)$ is holomorphic inside $\mathcal{M}_h$. In particular, the combination entering the integrand of the bosonic string amplitudes is precisely of this form, for superscript $-$ and $n = -1$:

$$Z^-_{(-1)}(g)\, Z_{(0)}(g)^{-13} = |\mathcal{Z}^-_{(-1)}(m_j)|^2 \; . \tag{5.9}$$

We shall make use of this powerful result in string theory in §C.

The BK Theorem may be understood geometrically in the language of determinant line bundles over $\mathcal{M}_h$, where the right hand side of BK — the holomorphic anomaly — arises as a non-zero curvature of the determinant line bundles $\mathrm{Det}(\nabla^z_{(n)})$ and $\mathrm{Det}(\nabla_z^{(n)})$ with certain hermitian metrics, the so-called Quillen metrics. The vanishing of the BK anomaly $\mathcal{Z}^\pm_{(n)}$ then indicates that the corresponding line bundles

$$\mathrm{Det}\nabla^z_{(n)} \otimes (\mathrm{Det}\nabla^z_{(1)})^{-\otimes c^-_{(n)}} \; , \tag{5.10}$$

are flat. In string theory, the case is $n = 2$, and this case is related to the flatness of $\mathcal{K} \otimes \lambda^{-13}$, where $\mathcal{K}$ is the canonical bundle of $\mathcal{M}_h$ and λ is the Hodge bundle.

We shall not prove the BK Theorem here (it can be proven using heat-kernel methods, just as in the derivation of the Weyl anomaly), but only indicate the origin of the holomorphic anomaly. The key observation is that the covariant derivatives $\nabla^z_{(n)}$ depend only on $\delta\mu$, but not on $\delta\bar\mu$ to first order in $\delta\mu$ and $\delta\bar\mu$; thus $\nabla^z_{(n)}$ is holomorphic in $\delta\mu$ to first order.

$$\left\{\begin{array}{l}\delta_\mu \nabla^z_{(n)} = \delta\mu \nabla^{\bar z}_{(n)} + n(\nabla^{\bar z}\delta\mu) \\ \delta_{\bar\mu}\nabla^z_{(n)} = 0\ .\end{array}\right. \tag{5.11}$$

Functional determinants of Laplace operators would then naively be expected to be products of a holomorphic times anti-holomorphic factor:

$$\mathrm{Det}\Delta^-_{(n)} \overset{?}{\sim} \mathrm{Det}\nabla_z^{(n-1)}\mathrm{Det}\nabla^z_{(n)}\ , \tag{5.12}$$

with $\mathrm{Det}\nabla^z_{(n)}$ holomorphic on $\mathcal{M}_h$ and $\mathrm{Det}\nabla_z^{(n-1)}$ anti-holomorphic. Actually, determinants must be properly defined (e.g. using ζ-function techniques, as in §III.), in a fashion that preserves their Diff(Σ) invariance. When this analysis is carried out, anomalies, such as the Weyl anomaly, may appear. Here, the above holomorphic factorization of determinants is also beset by an anomaly. In fact, these anomalies are intimately related, as can be seen from the argument below. As was explained in §III, the variations of μ, $\bar\mu$ and Weyl rescalings are independent. Hence, using the Weyl anomaly of $Z^\pm_{(n)}(g)$:

$$\begin{aligned}\delta_\sigma\delta_\mu\delta_{\bar\mu}\ln\, Z^\pm_{(n)}(g) &= \delta_\mu\delta_{\bar\mu}\delta_\sigma \ln\, Z^\pm_{(n)}(g) \\ &= -\frac{c^\pm_{(n)}}{6\pi}\,\delta_\mu\delta_{\bar\mu}\int_\Sigma d\mu_g\, R_g\ .\end{aligned} \tag{5.13}$$

But, the variation of the Gaussian curvature is easily computed

$$\delta_\mu\delta_{\bar\mu}(d\mu_g\, R_g) = (\nabla_z\delta\mu\nabla_{\bar z}\delta\bar\mu)d\mu_g \neq 0\ . \tag{5.14}$$

Thus, clearly, the holomorphic anomaly $\delta_\mu\delta_{\bar\mu}\ln\, Z^\pm_{(n)}(g)$ cannot cancel unless also the Weyl anomaly cancels.

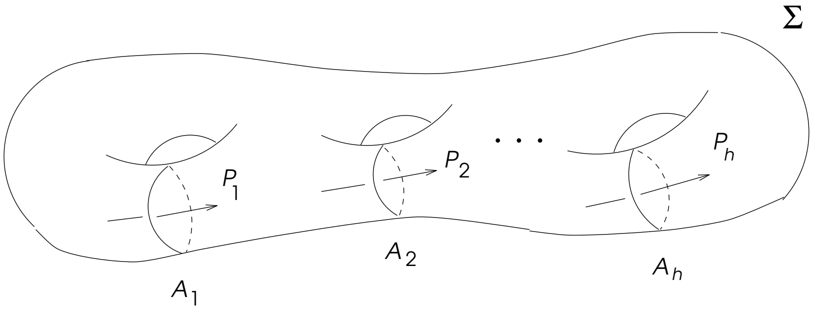
C) The Chiral Splitting Theorem

In critical string theory, we also have strong results on the holomorphicity properties of the correlation functions for vertex operators of physical on-shell states, given by the Chiral Splitting Theorem. We begin by assembling the data we need to formulate this theorem.

Let $V_i = \int_\Sigma d^2 z_i \, W_i(z_i, \bar z_i)$ be vertex operators of physical on-shell states whose polarization tensors are factorized in left and right movers. Let A_I, B_I, $I = 1, \dots, h$ be a canonical basis of homology 1-cycles on Σ, with

$$\#(A_I, A_J) = \#(B_I, B_J) = 0 \qquad\qquad \#(A_I, B_J) = \delta_{IJ} \; . \tag{5.15}$$

We define internal loop momenta $p_I \in \mathbb{R}^{26}$ associated with each A_I cycle. Because the A_I cycles have trivial interaction numbers, these internal loop momenta are independent.



The correlation function of unintegrated vertex operators $W_i(z_i, \bar z_i)$ at fixed internal loop momenta p_I are defined by

$$\begin{aligned} \langle W_1 \dots W_N \rangle_g (p_I) \equiv \int_{\mathrm{Map}(\Sigma, M)} & Dx \, W_1(z_1, \bar z_1) \dots W_N(z_N, \bar z_N) \\ & \prod_{I=1}^h \delta \left(p_I^\mu - \int_{A_I} \frac{dz}{2\pi i} \, \partial_z x^\mu \right) e^{-S[x,g]} \end{aligned} \tag{5.16}$$

The *Chiral Splitting Theorem* states that

$$\langle W_1 \dots W_N \rangle_g (p_I) = \delta(k) Z_0(g)^{-13} \mathcal{F} \widetilde{\mathcal{F}} \tag{5.17}$$

where $\mathcal{F}(z_i, m_j;\, p_I, k_i)$ is a complex analytic function of z_i and m_j, and $\widetilde{\mathcal{F}}$ is its complex conjugate

$$\widetilde{\mathcal{F}}(\bar z_i, \bar m_j;\, p_I, k_i) = \mathcal{F}(z_i, m_j;\, p_I, k_i)^* \; . \tag{5.18}$$

Here, we have suppressed the dependence of W and $\mathcal{F}$ on polarization tensors occurring in the construction of W. Also, the function $\mathcal{F}$ depends upon the choice of curves representing the A_I boundary cycles, and this dependence has been subsumed in the definition of the internal momenta p_I.

An outline of the proof of this Theorem will be given in §E. Let us simply remark here that the proper conformal field theory interpretation of the integration over loop momenta is the following. At fixed values of the internal loop momenta, p_I through all A_I cycles, only a single conformal family traverses each A_I-cycle. The contributions from left and right movers are factorized, so that the entire amplitude — at fixed momenta p_I — is factorized.

Mathematically, we have gained considerably through the Belavin-Knizhnik and Chiral Splitting Theorems, since we may now apply techniques of complex function theory, theta-functions and algebraic geometry to study amplitudes.

Physically, we have uncovered a method for separating the contributions of left and right movers in the integrand of transition amplitudes, via the holomorphic versus anti-holomorphic dependence on $\mathcal{M}_h$ and Σ. (This will be crucial in the study of superstrings.)

An immediate result from the combination of the Belavin-Knizhnik and Chiral Splitting Theorems is that the full transition amplitude at genus h is an integral over a product of a holomorphic times anti-holomorphic function.

$$A_h = \delta(k) \int_{\mathbb{R}^{26h}} \prod_I d\, p_I^{26} \int_{\mathcal{M}_h} \prod_j d\, m_j \bar{m}_j \int_\Sigma d^2 z_1 \dots \int_\Sigma d^2 z_N \, C \, \widetilde{C} \,, \tag{5.19}$$

where

$$C(z_i, m_j;\, p_I, k_i) = \det(\mu_j, \phi_k) \mathcal{Z}^-_{(-1)}(m_j) \mathcal{F}(z_i, m_j;\, p, k) \tag{5.20}$$

and

$$\widetilde{C}(\bar{z}_i, \bar{m}_j;\, p_I, k_i) = C(z_i, m_j;\, p_I, k_i)^* \ . \tag{5.21}$$

(Of course, for $h \geq 1$, these expressions are only formal, since the integration ultimately diverges. The generalization to the case of the superstring on the other hand will be well-defined.)

D) **Holomorphic and Meromorphic Differentials** (a brief review of basics)

We begin by recalling some basic tools of complex fucntion theory on Riemann surfaces. The space of holomorphic line bundles of degree d is the *Picard variety*

$$\mathrm{Pic}_d(\Sigma) = \{\text{holomorphic line bundles } L \text{ on } \Sigma\,,\ c_1(L) = d\}\ .$$

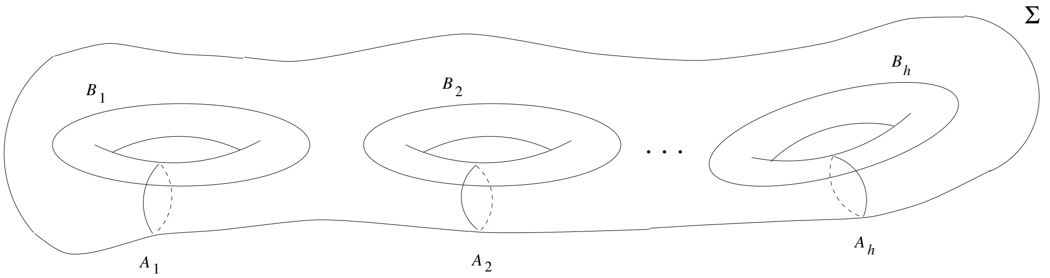
The subspace of line bundles L of degree 0 is the *Jacobian variety*: $J(\Sigma) = \mathrm{Pic}_0(\Sigma)$, which may also be viewed as

$$J(\Sigma) = H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})\ . \tag{5.22}$$

Choosing a reference $L_0 \in \mathrm{Pic}_d(\Sigma)$, any other $L \in \mathrm{Pic}_d(\Sigma)$ may be obtained as a tensor product $L = L_0 \otimes \ell$, $\ell \in J(\Sigma)$.

A completely explicit realization is obtained by making a choice of homology basis (A_I, B_I), $I = 1, \ldots, h$, with canonial intersection matrix:

$$\#(A_I, A_J) = \#(B_I, B_J) = 0 \qquad\qquad \#(A_I, B_J) = -\#(B_J, A_I) = \delta_{IJ}\ . \tag{5.23}$$



To this choice corresponds a unique basis of holomorphic Abelian differentials (holomorphic sections of K), $(\omega_1, \ldots, \omega_h) = \vec{\omega}$ or denoted simply ω by

$$\oint_{A_I} \omega_J = \delta_{IJ}\,, \qquad\qquad I, J = 1, \ldots, h\ . \tag{5.24}$$

The *period matrix* is then defined by

$$\oint_{B_I} \omega_J = \Omega_{IJ} \tag{5.25}$$

and the Jacobian is a complex torus of dimension h:

$$J(\Sigma) = \mathbb{C}^h/(\mathbb{Z}^h + \Omega\,\mathbb{Z}^h)\ . \tag{5.26}$$

The intersection from $\#(\ ,\ )$ is left invariant under the action of the *modular group* on the homology cycles

$$\begin{pmatrix} B \\ A \end{pmatrix}_I \longrightarrow \begin{pmatrix} B' \\ A' \end{pmatrix}_I = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{IJ}}_{\in \mathrm{Sp}(2h;\mathbb{Z})} \begin{pmatrix} B \\ A \end{pmatrix}_J \tag{5.27}$$

under which

$$\Omega \to \Omega' = (a\Omega + b)(c\Omega + d)^{-1} \ . \tag{5.28}$$

The Riemann bilinear relations guarantee that Ω lies in the *Siegel upper half space*: $\Omega \in \mathcal{H}_h$, where

$$\mathcal{H}_h = \{h \times h \text{ complex } \Omega;\ \Omega_{IJ} = \Omega_{JI};\ \mathrm{Im}\,\Omega > 0\} \ . \tag{5.29}$$

In fact, Torelli's theorem guarantees that to each Riemann surface in $\mathcal{M}_h$ there corresponds a unique point in the fundamental domain of the modular group

$$\mathcal{A}_h \equiv \mathcal{H}_h / \mathrm{Sp}(2h;\ \mathbb{Z}) \ , \tag{5.30}$$

(the converse holds for the torus, but not for $h \geq 2$.)

$\mathcal{H}_h$ carries a $\mathrm{Sp}(2h;\ \mathbb{Z})$-invariant metric

$$ds^2 = \mathrm{Tr}\,((\mathrm{Im}\,\Omega)^{-1} d\Omega (\mathrm{Im}\,\Omega)^{-1} d\Omega^*) \ . \tag{5.31}$$

Since ω_I depends holomorphically on $\mathcal{M}_h$, so does Ω_{IJ}; in fact, we have an explicit formula for the variation of Ω_{IJ} with holomorphic Beltrami differentials $\mu_{\bar z}^{z}$ associated with a Riemann surface deformation:

$$\delta\Omega_{IJ} = -i \int_\Sigma d^2z\, \mu_{\bar z}^{z}\, \omega_{Iz}\, \omega_{Jz} \tag{5.32}$$

The induced complex structure on $\mathcal{M}_h$ is the one associated with the metric ds^2 of (5.31).

The *Jacobi ϑ-function*, with characteristics $\delta = (\delta', \delta'') \in [0,1]^{2h}$, is defined as a function on $\mathbb{C}^h \times \mathcal{H}_h$:

$$\vartheta[\delta](\zeta, \Omega) \equiv \sum_{n \in \mathbb{Z}^h + \delta'} e^{\pi i n^T \Omega n + 2\pi i n^T (\zeta + \delta'')} \ . \tag{5.33}$$

It obeys, for any $M, N \in \mathbb{Z}^h$:

$$\vartheta[\delta](\zeta + M + \Omega N, \Omega) = \vartheta[\delta](\zeta, \Omega) e^{-i\pi\, N^T \Omega\, N - 2\pi i\, N^T(\zeta+\delta'') + 2\pi i \delta' M} \ , \tag{5.34}$$

and this property may be used to define ϑ as the unique (up to constant factor) holomorphic section of a holomorphic line bundle on $J(\Sigma)$, with the above transition functions. One also often uses $\vartheta(\zeta, \Omega) \equiv \vartheta[0](\zeta, \Omega)$.

Using the Jacobi ϑ-functions, and the Riemann vanishing theorem (which yields the zeros of ϑ as a function of ζ, but which we shall not discuss here), one may uniquely (up to constant factor) reconstruct a meromorphic function on Σ from its divisor $D = z_1 + \cdots + z_d - w_1 - \cdots - w_d$:

$$f_D(z) = \prod_{i=1}^d \vartheta \left(\zeta + \int_{z_i}^z \omega, \Omega \right) \vartheta \left(\zeta + \int_{w_i}^z \omega, \Omega \right)^{-1} \ . \tag{5.35}$$

Here ζ is such that $\vartheta(\zeta, \Omega) = 0$, but so that the above ϑ functions do not vanish identically; $f_D(z)$ is then independent of ζ.

Spin bundles on Σ belong to $\mathrm{Pic}_{h-1}(\Sigma)$, and may be described in terms of their spin structure, of which there are 2^{2h}. A choice of homology basis singles out a particular spin bundle S_0, and all other spin structures may be labeled by points in the Jacobian, as indicated earlier:

$$S = S_0 \otimes \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \qquad\qquad \delta'' + \Omega \delta' \in J(\Sigma) \ , \tag{5.36}$$

where δ' and δ'' are half-points, i.e. $\delta'_I,\ \delta''_I \in \{0,\ \frac{1}{2}\}$. Spin structures are partitioned into *even* or *odd spin structures*, according to the parity of the ϑ-function

$$\vartheta[\delta](-\zeta, \Omega) = (-)^{4\delta' \cdot \delta''} \vartheta[\delta](\zeta, \Omega) \ . \tag{5.37}$$

Thus, there are $2^{h-1}(2^h \pm 1)$ even/odd spin structure with $4\delta' \cdot \delta''$ even/odd, and the number of holomorphic sections is then even/odd as well, in fact generally 0/1. The generic holomorphic section $h_\delta(z)$ for an odd spin structure δ is obtained from the square root of an Abelian differential with exactly $h-1$ double zeros, (this $h_\delta(z)$ is a Dirac zero mode)

$$h_\delta(z) \equiv \left(\sum_I \partial_I \vartheta[\delta](0, \Omega) \quad \omega_I(z) \right)^{1/2} \ . \tag{5.38}$$

Finally, we define one of the fundamental objects that enter Green functions and determinants in string theory. The *prime form* is a function on the universal cover of $\Sigma \times \Sigma$ and $\mathcal{H}_h$, given by

$$E(z,w) \equiv \frac{\vartheta[\delta]\left(\int_w^z \omega, \Omega\right)}{h_\delta(z)\ h_\delta(w)} \tag{5.39}$$

where δ is any *odd* half characteristic. $E(z,w)$ is a holomorphic $\left(-\frac{1}{2},0\right)$ differential form in both z and w, with a single zero at $z=w$. As such, $E(z,w)$ generalizes the function $z-w$ on $\mathbb{C}$ to higher genus surfaces. On the torus, we have the well-known expression

$$E(z,w) = \frac{\vartheta_1(z-w,\Omega)}{\vartheta_1'(0,\Omega)} \ . \tag{5.40}$$

The monodromies of $E(z,w)$ around A_I homology cycles are simple: $E(z,w)$ picks up a $-$ sign; the monodromy as z moves around B_I once is:

$$E(z,w) \to -\exp\left[-i\pi\Omega_{II} - 2\pi i \int_z^w \omega_I\right] E(z,w) \ . \tag{5.41}$$

Under modular transformations (represented by their action on the homology cycles as in (5.27)), we have

$$E(z,w) \to \exp\left[\pi i \int_z^w \omega^T (c\Omega+d)^{-1} c \int_z^w \omega\right] E(z,w) \ . \tag{5.42}$$

Meromorphic Abelian differentials (meromorphic sections of K) are readily obtained

$$\begin{aligned} \omega_{xy}(z) &= \partial_z \ln \frac{E(z,x)}{E(z,y)}\, dz \\ \omega_w(z) &= \partial_z \partial_w \ln E(z,w) dz \ . \end{aligned} \tag{5.43}$$

Meromorphic $\frac{1}{2}$ differentials (meromorphic sections of S_δ) are

$$\begin{aligned} &\delta \text{ even} \qquad S_\delta(z,w) = \frac{1}{E(z,w)} \quad \frac{\vartheta[\delta]\left(\int_z^w \omega, \Omega\right)}{\vartheta[\delta](0,\Omega)} \\ &\delta \text{ odd} \qquad S_\delta(z,w) = \frac{1}{E(z,w)} \quad \frac{\partial_I \vartheta[\delta]\left(\int_z^w \omega, \Omega\right)\omega_I(y)}{\partial_I \vartheta[\delta](0,\Omega)\omega_I(y)} \end{aligned} \tag{5.44}$$

For δ even, S_δ is the Szegö kernel, while for δ odd, $\S_\delta$ is not single-valued on Σ as defined.

E) **Green Functions, Determinants and Chiral Splitting**

The Green function of scalar fields on Σ is simply expressed in terms of the prime form and Abelian differentials

$$G(z,w) = -\ln |E(z,w)|^2 + 2\pi \,\mathrm{Im} \int_w^z \omega_I (\mathrm{Im}\,\Omega)^{-1}_{IJ}\, \mathrm{Im} \int_w^z \omega_J \tag{5.45}$$

The second term on the right hand side is required to render $G(z,w)$ single-valued on $\Sigma\times\Sigma$, as well as to render it modular invariant. This expression is readily seen to generalize the 1-loop result (and of course the 0-loop result as well.)

We use this result to prove the Chiral Splitting theorem for the case of vertex operators for tachyon states. (Higher mass states and their vertex operators may be obtained by factorization or equivalently by using the OPE for exponential operators.) To restrict the loop momenta and bring out an integration over them, we can insert the identity (as we did in §III for one loop amplitudes)

$$1 = \prod_I \int_{\mathbb{R}^{26}} dp_I \, \delta\left(p_I^\mu - \frac{1}{2\pi i} \int_{A_I} dz\, \partial_z x^\mu \right) \tag{5.46}$$

into the correlation function $\langle W_1(z_1,\bar z_1)\ldots W_N(z_N,\bar z_N)\rangle_g$. In fact, here, we may use a short-cut, yielding the same result, carried out as follows. If $W_i(z_i,\bar z_i) = \exp ik_i\cdot x(z_i,\bar z_i)$ then we have

$$\begin{aligned} \left\langle \prod_{i=1}^N e^{ik_i\cdot x(z_i,\bar z_i)} \right\rangle_g &= \delta(k) \left(\frac{\mathrm{Det}'\Delta_{(0)}}{\int_\Sigma d\mu_g} \right)^{-13} \prod_{i<j} e^{-k_i\cdot k_j\, G(z_i,z_j)} \\ &= \delta(k) Z_{(0)}(g)^{-13} \prod_{i<j} |E(z_i,z_j)|^{2k_i\cdot k_j} X \end{aligned} \tag{5.47}$$

where

$$X = \det(\omega_I,\omega_J)^{-13} \prod_{i<j} e^{-2\pi k_i\cdot k_j \,\mathrm{Im} \int_w^z \omega\, (\mathrm{Im}\,\Omega)^{-1}\, \mathrm{Im}\int_w^z \omega} \,. \tag{5.48}$$

Using the Riemann bilinear relations, we have

$$\det(\omega_I,\omega_J) = \det(2\mathrm{Im}\,\Omega_{IJ}) \tag{5.49}$$

and we may now represent X by a single Gaussian integral over real parameters p_I^μ, which are just the internal loop momenta:

$$X = \prod_I \int_{\mathbb{R}^{26}} d^{26} p_I \big| e^{i\pi p_I^\mu \Omega_{IJ} p_J^\mu + 2\pi i p_I^\mu k_i^\mu \int_P^{z_i} \omega_I} \big|^2 \ . \tag{5.50}$$

It is now straightforward to read off the function $\mathcal{F}$ that enters the Chiral Splitting Theorem:

$$\mathcal{F}(z_i, m_j; p_I, k_i) = \prod_{i<j} E(z_i, z_j)^{k_i \cdot k_j} \cdot \exp\left[i\pi p_I^\mu \Omega_{IJ} p_j^\mu + 2\pi i p_I^\mu k_i^\mu \int_P^{z_i} \omega_I \right] \tag{5.51}$$

The point P is arbitrary in view of the momentum conservation $\delta(k)$-function factor. $\mathcal{F}$ above is determined only up to phases. In particular, $\mathcal{F}$ is multiple-valued, with monodromies that may be easily computed from the monodromies of $E(z,w)$, as given in (5.41). $\mathcal{F}$ also has branch cuts along the homology cycles: the effect of letting vertex insertion point z_i cross a cycle A_I is to add the external momentum k_i to the loop momentum p_I. Modular transformation properties of $\mathcal{F}$ are also non-trivial, and may be read off from (5.42) and (5.28).

It remains to express the functional determinant combinations $\mathcal{Z}^-_{(n)}(m_j)$ in terms of ϑ-functions and differentials. This may be achieved by a variety of techniques, inducing bosonization and integration over $\mathcal{M}_h$ of the stress tensor. We shall not develop these techniques here but limit ourselves to stating the result for $\mathcal{Z}^-_{(-1)}$:

$$\mathcal{Z}^-_{(-1)}(m_j) = \frac{\vartheta(3\Delta_w|\Omega)}{\det \partial_w^i \phi_j(w)} \left(\frac{\partial_w^h \vartheta(\Delta_w|\Omega)}{\det \partial_w^I \omega_J(w)} \right)^{-9} \ .$$

Here, w is an arbitrary point on Σ, ∂_w^α stands for the derivative of order α in w and Δ_w is the Riemann vector

$$\Delta_w^I = \frac{1}{2} - \frac{1}{2}\Omega_{II} + \sum_{J \neq I} \oint_{A_J} \omega_J(z) \int_w^z \omega_I \ .$$