Lectures 9 and 10

VI. Strings on General Manifolds

Eric D'Hoker

In the preceding lectures, we have quantized strings in flat Minkowski space-time. The existence of a Hilbert space of physical string states (i.e. the consistent elimination of negative norm states) requires Weyl(Σ) and Diff(Σ) invariance of the full quantum theory. This condition fixes the dimension of space-time to be $D = 26$ and results in the appearance of massless spin 1 and spin 2 string states which behave like Yang-Mills particles and gravitons, respectively. The spectrum also contains massless dilatons and anti-symmetric tensor states.

In the present lecture, we wish to extend the analysis of the consistent elimination of negative norm states to the case of arbitrary space-time manifolds, with general metric $G_{\mu\nu}$, anti-symmetric tensor $B_{\mu\nu}$ and dilaton Φ, and possible other background fields. On physical grounds, we shall assume that the space-times of interest have Minkowski signature $(-,+,\cdots,+)$. For simplicity, we assume that the dimension of space-time D is such that $D \geq 4$ in order to avoid dealing with certain peculiarities that arise in $D = 2, 3$. We expect again that the space-time dimension D will be fixed by the condition that all negative norm states be consistently eliminated. The precise value of D where this happens may, however, depend on the background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, so we shall leave D to be determined later on. We shall also assume that space-time M has at least 2 non-compact dimensions, including the time direction, so that scattering can be defined in M.

Given the signature of the space-time manifold M, the Fock space of states produced by the quantization of the map x will contain again negative norm states. These states are to be eliminated before a Hilbert space of string states can be constructed. To achieve elimination of negative norm states, it is clearly necessary that the worldsheet quantum field theory of the map x, defined in terms of the background $G_{\mu\nu}$, $B_{\mu\nu}$, Φ (and perhaps other fields), together with the Faddeev-Popov ghost fields b, c, $\bar{b}$, $\bar{c}$, be invariant under

Diff(Σ) and Weyl(Σ) transformations. In this case, the BRST quantization scheme of §IV may be set up, just as in the case of flat Minkowski M.

For certain classes of space-time manifolds M, this condition can be shown to be also sufficient. When $M = \mathbb{R}^2 \times M_{D-2}$, where $\mathbb{R}^2$ includes the time direction, so that the metric G on M_{D-2} is positive, we have the usual Hilbert space construction for the states arising from the M_{D-2} part of space-time, since these states have positive norm. One can show that the BRST quantization scheme for all of M leads to a Hilbert space. The proof proceeds by passing to the lightcone gauge variables, which may be defined for the $\mathbb{R}^2$ part of M just as in the case of flat M. In lightcone gauge, only positive norm, physical states survive, and may be mapped onto BRST cohomology classes in a one-to-one fashion, just as for flat M. For more general M, I am not aware of a proof that a Hilbert space may be constructed, but it is generally believed to be the case, at least "under certain conditions". For example, a space-time M with a black hole is expected to cause problems.

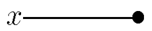
Henceforth, we shall restrict to space-time manifolds M and background fields, such that Weyl(Σ) $\ltimes$ Diff(Σ) invariance of the x, b, c, $\bar{b}$, $\bar{c}$ system leads to a Hilbert space. Equivalently, the quantum field theory of the x field is a conformal field theory of central charge $c = 26$. Thus, we have reduced the conditions of consistent string propagation in space-time manifolds M background fields to a problem in quantum field theory: that of identifying and parametrizing conformal field theories of the map x with central charge $c = 26$.

A) Perturbation Theory Around General Field Configurations

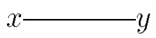
Before launching into a study of the conditions for conformal invariance of the field theory of the map x with general space-time M, general $G_{\mu\nu}$, $B_{\mu\nu}$, Φ, (and possibly other background fields), we wish to clarify the physical significance of string quantization in non-flat background field configurations. We proceed by comparison with perturbation theory in quantum field theory (QFT).

In QFT, we start with canonical fields ϕ and a classical action $S[\phi]$ which we assume to be independent of $\hbar$. Perturbation theory (in the loop expansion) is carried out by choosing a field configuration ϕ_0, and expanding the field ϕ in powers of $\sqrt{\hbar}$: $\phi = \phi_0 + \sqrt{\hbar}\,\varphi$. The Feynman rules are then obtained by an expansion of the action $S[\phi]$ in powers of $\sqrt{\hbar}$:

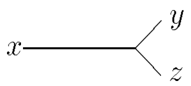
$$\frac{1}{\hbar}\,S[\phi] = \frac{1}{\hbar}\,S[\phi_0] \; + \frac{1}{\sqrt{\hbar}} \int \varphi \frac{\delta S}{\delta \phi}\,[\phi_0] + \tfrac{1}{2} \iint \varphi\varphi \frac{\delta^2 S}{\delta \phi^2}\,[\phi_0] + \frac{1}{6}\sqrt{\hbar} \iiint \varphi\varphi\varphi\,\frac{\delta^3 S}{\delta\phi^3}\,[\phi_0] + \cdots$$



$$\frac{1}{\sqrt{\hbar}}\frac{\delta S}{\delta\phi(x)}\,[\delta_0] \tag{6.A.1}$$



$$\left(\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\,[\phi_0]\right)^{-1}$$



$$\sqrt{\hbar}\,\frac{\delta^3 S}{\delta\phi(x)\delta\phi(y)\delta\phi(z)}\,[\phi_0]$$

$$\vdots$$

Both the Feynman rules for the φ propagator and for the φ interaction vertices depend upon the background field ϕ_0 in a non-trivial way, as is evident from the Feynman rule expansion in (6.A.1).

In string theory, we have no equivalent of the canonical field ϕ and of the classical action $S[\phi]$ that is as well understood as in QFT. (Many attempts at constructing a string field theory formulation have been made over the years, but have met with only limited success.) In string theory, our starting point consists only of the Feynman rules, which are given in terms of a summation over surfaces Σ, and maps $\Sigma \to M$, and which yield on-shell transition amplitudes only. The rôle of ϕ_0 in QFT is played in string theory by the background fields $G_{\mu\nu}$, $B_{\mu\nu}$ Φ, and possibly other fields. And indeed, the Feynman rules of string theory depend upon these background fields, just as the Feynman rules of QFT depended upon ϕ_0.

Fortunately, much crucial physical information may be reconstructed from the Feynman rules: we already know that they yield the transition amplitudes for physical states, i.e. the S-matrix (at least in perturbation theory). In particular, it is clear from QFT that certain transition amplitudes yield direct information on the background ϕ_0. The most important one for our purposes is that *ϕ_0 is a classical solution*, i.e. $\delta S/\delta\phi[\phi_0] = 0$ if and only if all *tadpole graphs are zero to tree level.* The string theory equivalent of this criterion is that a given background field configuration $G_{\mu\nu}$, $B_{\mu\nu}$, Φ (and possibly other fields) are a *solution to the string equations of motion to tree level* (i.e. classically) *if and only if all tadpole graphs vanish.* This criterion gives us a powerful tool to investigate whether a certain configuration is a classical solution or not. In QFT, we do not usually investigate the Feynman rules to decide whether a configuration ϕ_0 is a solution, because we can directly check whether $\delta S/\delta\phi[\phi_0]$ vanishes or not. In string theory, this alternative is not available, at present.

We have established previously that the existence of a Hilbert space of string states requires the map x to define a conformal field theory of central charge $c = 26$. It is a simple consequence of conformal invariance on the sphere (established for x in flat space-time M in §III) that the 1-point function of any primary conformal field of weight $(1, 1)$, describing an on-shell physical state, must vanish to tree level. Combining this result with that of the preceeding paragraph, we discover that if x defines a conformal field theory of central charge 26, for a certain background configuration $G_{\mu\nu}$, $B_{\mu\nu}$, Φ, (and possibly other fields), then this configuration is a classical (tree level) solution to the string equation! In other words, demanding that the backgrounds $G_{\mu\nu}$, $B_{\mu\nu}$, Φ , ... define a $c = 26$ CFT will enforce on $G_{\mu\nu}$, $B_{\mu\nu}$, Φ (and possibly other fields), the string equations of motion and guarantee the existence of a Hilbert space of physical states. While in QFT, we knew the equations of motion for ϕ_0 directly from the action, in string theory the equations on the background fields will be novel.

We finish this subsection by noticing that other useful information may be obtained from the Feynman rules. First, a configuration $G_{\mu\nu}$, $B_{\mu\nu}$, Φ , ... is a solution to the full (perturbative) quantum string equations if the sum of all tadpoles cancels to all orders. This is in analogy with QFT, where a solution to the full (perturbative) quantum fields equations is an extremum of the full effective action, so that the sum of all tadpole graphs cancels. Secondly, the presence of tachyons in the spectrum signals that the background around which the perturbative expansion was formulated is not a local minimum.

B) Renormalization of Generalized Non-linear sigma Models

Let M be a Riemannian manifold of dimension D, with metric $G_{\mu\nu}$, and fields $B_{\mu\nu}$, Φ defined in §I. The starting point for string dynamics in the background of these fields is the generalized non-linear sigma model action for the maps x: $\Sigma \to M$, given by $S = S_{G,B} + S_\Phi$ and

$$\begin{aligned} S_{G,B}[x,g] &= \frac{1}{8\pi\ell^2} \int_\Sigma d\mu_g \partial_m x^\mu \partial_n x^\nu \{ g^{mn} G_{\mu\nu}(x) + \varepsilon^{mn} B_{\mu\nu}(x) \} \\ S_\Phi[x,g] &= \frac{1}{2\pi} \int_\Sigma d\mu_g R_g \Phi(x) \end{aligned} \tag{6.B.1}$$

Recall that the term involving $B_{\mu\nu}$ is the pullback of a two form $B = B_{\mu\nu} dx^\mu \wedge dx^\nu$ under the map x, written out in coordinates. Here, ε^{mn} is the tensor defined by

$$\varepsilon^{mn} \equiv \bar\varepsilon^{mn} / \sqrt{\det\, g_{pq}} \qquad\qquad \bar\varepsilon^{01} = -\bar\varepsilon^{10} = 1 \ . \tag{6.B.2}$$

We have re-exhibited the dependence on the string tension κ, introduced in §I via the parameter ℓ. It is standard notation to express κ in terms of the *Regge slope parameter* α'; we find it convenient to use the Planck length scale ℓ, instead, with the following relations

$$\kappa = \frac{1}{4\pi\alpha'} = \frac{1}{8\pi\ell^2} \ , \qquad\qquad \alpha' = 2\ell^2 \ , \qquad \ell > 0 \ . \tag{6.B.3}$$

It is possible to include a further term in S corresponding to the dynamics of the tachyon state, to which we shall associated the field U:

$$S_U[x,g] = \frac{1}{4\pi} \int_\Sigma d\mu_g U(x) \tag{6.B.4}$$

Recall from Lecture 1 that the classical action S is invariant under

(a) Diff(Σ)

(b) Diff(M)

(c) $U(1)_B$ gauge invariance, acting by $B \to B + d\gamma$, for some $\gamma \in \Omega^{(1)}(M)$.

Weyl(Σ) transformations on g leave $S_{G,B}$ invariant, but will develop anomalies upon quantization. S_Φ and S_U fail to be Weyl invariant, even classically.

We naturally have two notions of dimension in the problem: the worldsheet or Σ-dimension, and the space-time or M-dimension. The Σ-dimension of x, g, G, B, Φ and ℓ vanishes, while that of ∂ and R_g is 1 and 2, respectively. The M-dimension of x and ℓ is -1, and that of g, G, B, Φ vanishes.

We begin by discussing the renormalization of the quantum field theory defined by S, for general fields $G_{\mu\nu}$, $B_{\mu\nu}$, Φ. We shall not discuss here the effects of including the tachyon field in S. We shall see that in certain regularization schemes, such as dimensional regularization, its contribution can be ignored. Also, in the superstring, the tachyon will be absent anyway.

1) *Low Energy or α'-expansion.*

The parameter ℓ is a very small length-scale (see Prob. 5), compared to scales accessible at present (whose order of magnitude we denote by $L_{\rm expt}$)

$$\ell \sim 10^{-33}\text{cm} \qquad\qquad L_{\rm expt} \sim 10^{-17}\text{cm} \ . \tag{6.B.5}$$

Thus, we consider the quantum theory defined by S in an expansion in powers of ℓ (or equivalently in powers of α'), and we are guaranteed that this approximation will be reliable if we just retain the leading contributions. The $\alpha' = 2\ell^2$ expansion is an expansion in increasing powers of M-derivatives on the fields, hence the name *low-energy expansion.*

If it were only for $S_{G,B}$, the parameter ℓ^2 would play precisely the same rôle as $\hbar$ plays in quantum field theory, and the expansion is powers of ℓ would yield a loop expansion. S_Φ has a different dependence in ℓ though, as each factor of Φ effectively contributes an extra factor of ℓ^2, and this will rearrange the orders of the loop expansion. Perturbation theory may be carried out as usual by letting $x = x_0 + \ell y$ for some fixed reference configuration x_0. Expansion of the fields G, B, Φ, for example of

$$G_{\mu\nu}(x) = G_{\mu\nu}(x_0) + \ell \partial_\kappa G_{\mu\nu}(x_0) y^\kappa + \frac{1}{2}\ell^2 \partial_\kappa \partial_\lambda G_{\mu\nu}(x_0) y^\kappa y^\lambda + \cdots \tag{6.B.6}$$

reveals that S is a quantum field theory with an infinite number of couplings. It is seen from (6.B.6) that the independent couplings of this quantum field theory correspond to the successive derivatives of the fields G, B and Φ at the expansion point x_0.

2) *Regularization schemes.*

It is possible to choose regularization schemes that preserve the classical symmetries ($\mathrm{Diff}(\Sigma), \mathrm{Diff}(M), U(1)_B$) of S. We shall always assume that such a regulator is used, and give here two examples. It is of course not possible to choose a regularization scheme that also preserves Weyl(Σ) invariance. This failure precisely results in the appearance of the Weyl anomaly as was already discovered for flat M and vanishing B and Φ in §IV.

a) Dimensional regularization consists in replacing the dimensionlity of Σ by an arbitrary complex number $2-\varepsilon$, and multiplying the action S by a factor of $\Lambda^{-\varepsilon}$. Here Λ is the dimensional regularization renormalization scale with Σ-dimension 1. Dimensional regularization has the advantage that it never generates quadratic divergences, so that if we start with $S_U = 0$ (no contribution of the tachyon to the action), then no tachyon dependence will ever be generated upon renormalization. Dimensional regularization has the disadvantage that the ε^{mn} tensor of (6.B.2) is specifically 2-dimensional, so that an additional ad hoc prescription for its regularization must be provided, and the results should be verified using the Ward identities to make sure that the ad hoc prescription is consistent with all symmetries.

b) Heat-kernel methods have already been used in the definition of determinants resulting from Gaussian quantum theories. They may also be used for QFT with non-trivial interactions by replacing its propagator by the heat-kernel regularization propagator. Let Δ_0 be the second order operator on the field y at x_0, and Δ_0^{-1} the associated propagator for y. We define the heat-kernel propagator by using a small real constant $\eta > 0$:

$$\Delta_\eta^{-1} = \Delta_0^{-1} \cdot e^{-\Delta_0 \eta^2} \ . \tag{6.B.7}$$

3) $\mathrm{Diff}(M)$ *and* $U(1)_B$ *invariant renormalization.*

Since x and y have zero Σ-dimension, all terms listed in (6.B.1) for S have Σ-dimension 2 and thus S is power counting renormalizable. Under the renormalization group flow, the change in each one of the infinite number of coupling constants (i.e. the Taylor coefficients of the expansion of G, B, Φ in powers of y as shown for G in (6.B.6)) will be given by a beta function associated with that coupling constant. An infinite number of renormalization conditions is required to specify the initial condition of the theory completely. These conditions amount to providing the functions $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$ and $\Phi(x)$ at some reference renormalization scale which we denote by Λ_0. For the metric, for example, the renormalization group equations will read:

$$\begin{cases} \Lambda \frac{d}{d\Lambda} G_{\mu\nu}(x) = \beta_{\mu\nu}(x) \\ \\ G_{\mu\nu}(x)\big|_{\Lambda_0} = {G_{\mu\nu}}^0(x) \end{cases} \tag{6.B.8}$$

Here $\beta_{\mu\nu}(x)$ depends on x only through the dependence of $\beta_{\mu\nu}$ on the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, as well as the derivatives of these fields.

Now, we impose upon the renormalization procedure that it preserve the Diff(M) and $U(1)_B$ invariance of S. The existence of a regulator that preserves these symmetries guarantees that there will be no anomalies in these symmetries. This implies that $\beta_{\mu\nu}$ must be a $U(1)_B$-invariant, symmetric rank 2 Diff(M) tensor. Similarly for the β-functions of the fields B and Φ.

We conclude with a remark on the nature of the renormalizability of the action S. Customarily, renormalizable QFT's are power counting renormalizable, but depend only upon a finite number of input parameters, which may be viewed as initial conditions on the renormalization group equations. In four dimensional QFTs of particle dynamics, the fact that only a finite number of input parameters is needed renders the theory physically predictive. Indeed, once those parameters have been determined (even approxiamtely), the theory is specified and all physical quantities are in principle calculable. Here, however, the number of input parameters in the theory of action S is infinite. In string theory, this is precisely what is needed in order to be able to formulate string theory in infinite dimensional families of backgrounds, which will correspond to solutions to string equations of motion.

4) *Irrelevance of Higher Mass Couplings.*

The fields $G_{\mu\nu}$, $B_{\mu\nu}$, Φ are associated with the graviton, anti-symmetric tensor and dilaton states in string theory, respectively. Those are precisely all the *massless states* of the bosonic string. A natural question arises as to whether fields associated with *massive string states* should also be included in S. In other words, we wish to find out whether S is the most general action possible for the dynamics of bosonic string theory.

Using the correspondence (established in §II) between variations in the background fields (such as $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ) and vertex operators for the associated physical states, we may infer the structure of possible contributions to S from massive states, directly from the corresponding vertex operators. A vertex operator of a string state of square mass $M^2 = -2 + 2p$ is of the form

$$V(\varepsilon, k) = \int_\Sigma d\mu_g \varepsilon_{\mu_1 \dots \mu_{2p}}(k) P^{\mu_1 \dots \mu_{2p}}(\partial x, \partial^2 x, \dots; k) e^{ik\cdot x} \tag{6.B.9}$$

where ε is a polarization tensor characterizing the string state and P is a polynomial of degree $2p$ in derivatives on x, so that P has Σ-dimension $2p$. Denoting the field associated with this state by $E_{\mu_1 \dots \mu_{2p}}(x)$, the contribution to the action S will take the form

$$S_E[x, g] = \int_\Sigma d\mu_g \, E_{\mu_1 \dots \mu_{2p}}(x) \widetilde{P}^{\mu_1 \dots \mu_{2p}}(\partial x, \partial^2 x, \dots) \,. \tag{6.B.10}$$

Here $\widetilde{P}$ is again a polynomial of worldsheet dimension $2p$. For $p = 0$, we recover the tachyon action, which we omit. For $p = 1$, we recover the action S, associated with massless string states.

For $p \geq 2$, S_E produces a non-renormalizable, or *irrelevant*, contribution to the world-sheet action. Addition of S_E to S yields a quantum field theory that can also be defined solely from the action S, but with modified background field values G, B, Φ. In other words, the family of renormalized QFT's defined by $S + S_E$ coincides with the family of renormalized QFT's defined solely by S.

This means that bosonic string theory is completely specified by the background fields associated with massless states only (we continue to ignore the tachyon). Once those have been specified, the dynamics of all massive states is a consequence of string dynamics and is completely determined. Of course, as argued in the beginning of this lecture, the condition of consistent string propagation will further limit the possible backgrounds to those yielding a conformal field theory of central charge $c = 26$.

C) General Structure of Weyl Dependence

We now determine the general structure of the Weyl dependence of Green functions in the quantum field theory specified by the action S in terms of fixed background fields $G_{\mu\nu}$, $B_{\mu\nu}$, Φ. We consider unnormalized correlation functions of operators $\phi_1 \ldots \phi_n$, inserted at points $\xi_1, \ldots, \xi_n \in \Sigma$.

In §E, we shall provide a concrete and useful reformulation of correlation functions in terms of the effective action, via the covariant background field method. From this reformulation, it will be immediate to establish in perturbation theory the properties that we shall use formally in this section and in the next one (§D).

Under a Weyl(Σ) transformation $g \to e^{2\delta\sigma} g$, with support away from $\xi_1 \ldots \xi_n$, we have by definition of the stress tensor T_{mn}:

$$\delta_\sigma \langle \phi_1 \ldots \phi_n \rangle_g = \frac{1}{2\pi} \int_\Sigma d\mu_g \delta\sigma \langle T_m{}^m \phi_1 \ldots \phi_n \rangle_g \ . \qquad (6.C.1)$$

We assume throughout that correlation functions are covariant under Diff(Σ), so that

$$\nabla^m T_{mn} = 0 \ . \qquad (6.C.2)$$

$T_m{}^m$ receives contributions from the explicit Weyl-non-invariance of S_Φ, from Weyl anomalies, and from the interplay between non-invariance of S_Φ and Weyl anomalies. All such contributions to $T_m{}^m$ are *local functions of* x, (i.e. dependent on x and on finite order derivatives on x) and *polynomial in the derivatives of* x (see Witten's lectures).

Collecting the various criteria, we find that $T_m{}^m$ has to be

1) local in x and polynomial in derivatives of x;
2) Diff(Σ) scalar;
3) of Σ-dimension 2;
4) of M-dimension 0;
5) Diff(M)-invariant;
6) $U(1)_B$ invariant;
7) invariant under shifts in Φ by a constant.

Only the last point may need justification: a shift in Φ by a constant yields a shift in the action S by a term proportional to the Euler number, which is Weyl independent.

Making use of a combination of these criteria, we find that $T_m{}^m$ must involve precisely 2 derivatives on Σ, and must be determined in terms of 4 unknown local functions β^G, β^B, β^Φ and β^V of x:

$$\begin{aligned} T_m{}^m = \partial_m x^\mu \partial_n x^\nu \Big(\beta^G_{\mu\nu}(x) g^{mn} + \beta^B_{\mu\nu}(x)\varepsilon^{mn}\Big) \\ + \beta^\Phi(x) R_g + \beta^V_\mu(x) g^{mn} D^*_m \partial_n x^n \ . \end{aligned} \tag{6.C.3}$$

Here, D^*_m is the covariant derivative on $T^*\Sigma \otimes TM$ pulled back to Σ by the map x, and is defined by*

$$D^*_m \partial_n x^\mu = \nabla_m \partial_n x^\mu + \Gamma^\mu_{\nu\rho} \partial_m x^\nu \partial_n x^\rho \ , \tag{6.C.4}$$

where ∇_m is the standard covariant derivative on $T^*\Sigma$, as defined in §II, and Γ is the affine connection of the metric $G_{\mu\nu}$. The functions β^G, β^B, β^Φ, β^V are called *beta functions*, even though in this context they govern a change of Weyl scale rather than a change of renormalization scale.

The structure of $T_m{}^m$ reveals a puzzle: the action S involves only the three fields G, B, Φ, but the Weyl variation of the correlation functions produces four beta functions. Indeed, there is an additional beta function β^V. If only *constant* Weyl transformations δ_σ are performed in (6.C.1), the term involving β^V may be integrated by parts and effectively absorbed into β^G. For general Weyl transformations, however, the term involving β^V is truly independent.

The resolution of this puzzle lies in the fact that upon performing a Weyl transformation on g, we are free to simultaneously transform x. On general grounds, the form of this transformation is given by $\delta_\sigma x^\mu = \ell^2 V^\mu(x)\delta\sigma$, where $V^\mu(x)$ is a vector field on M. We shall make it clear later on how to implement this additional effect of a Weyl transformation in actual calculations. Using the freedom to make this additional transformation, we see that $V^\mu(x)$ may be chosen so as to compensate for the β^V-function. The remaining β-functions

* We shall denote the pullback under the map x of covariant derivatives D on TM by D^*.

$\beta^G_{\mu,\nu}$, $\beta^B_{\mu\nu}$, β^Φ then have precisely as many components as the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ whose Weyl variation they describe.

Assuming now that the β^V_μ term has been cancelled by a suitable transformation $\delta_\sigma x^\mu$, we have the most general expression for $T_m{}^m$ as follows (we continue to use the same notation for β's):

$$T_m{}^m = \partial_m x^\mu \partial_n x^\nu \left(\beta^G_{\mu\nu}(x) g^{mn} + \beta^B_{\mu\nu}(x)\varepsilon^{mn}\right) + \beta^\Phi(x) R_g \ . \tag{6.C.5}$$

(This phenomenon of augmenting Weyl transformations on g by further transformations on the fields was already encountered in Problem Set #1. There, we studied a special case of action S where $D=1$, $G=1$, $B=0$, $\Phi = Qx$ and

$$S[x,g] = \frac{1}{4\pi}\int_\Sigma d\mu_g \left(\frac{1}{2} x\Delta_g x + QR_g x\right) \tag{6.C.6}$$

and discovered that upon a Weyl transformation, $\delta_\sigma x = Q\delta\sigma$.)

Classical contribution to $T_m{}^m$.

The contribution to $T_m{}^m$ arising from the variation of the classical action with respect to $\delta_\sigma x^\mu$ is given by

$$T_m{}^m(\delta_\sigma x^\mu) = g^{mn}\mathcal{D}^*_m \partial_n x^\mu V^\nu(x) G_{\mu\nu}(x) \ . \tag{6.C.7}$$

To define the covariant derivative $\mathcal{D}^*_m$, we need the *torsion tensor*, $H_{\mu\nu\rho}$, which is the field strength of the field $B_{\mu\nu}$, given by $H = dB$, or in component notation

$$H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \ . \tag{6.C.8}$$

Then, we have

$$\mathcal{D}^*_m \partial_n x^\mu \equiv D^*_m \partial_n x^\mu - \frac{1}{2} H^\mu_{\nu\rho} g_{mp}\varepsilon^{qp}\partial_q x^\nu \partial_n x^\rho \tag{6.C.9}$$

so that $\mathcal{D}^*_m$ is a covariant derivative with respect to a connection with torsion $\Gamma + \frac{1}{2}H$ on TM.

It is now clear that a vector field $V^\mu(x)$ may be chosen so as to cancel β^V_μ in the expression for $T_m{}^m$ in (6.C.3). To lowest order in ℓ^2, we have

$$V^\mu(x) G_{\mu\nu}(x) = -\beta^V_\nu(x) + O(\ell^2) \ . \tag{6.C.10}$$

Notice that this cancellation of the β^V term produces modifications in the functions β^G and β^B and possibly also β^Φ.

D) General Structure of Weyl Anomaly in Low Energy Expansion

An exact computation of β^G, β^B and β^Φ is possible only if G, B and Φ exhibit a high degree of symmetry. (e.g. flat $G_{\mu\nu}$, $B_{\mu\nu} = \Phi = 0$.) For general G, B and Φ, only approximate expressions can be obtained. Fortunately, the constant ℓ is exceedingly small compared to L_{expt}, so an expansion in power of ℓ should be very reliable. On general grounds, the expansion of the functions β^G, β^B and β^Φ has the following structure

$$\beta(x) = \sum_{p=0}^{\infty} \ell^{2p} \beta^{(p)}(x) \tag{6.D.1}$$

where the coefficients $\beta^{(p)}(x)$ are

1) local in x and function of x, but not of derivatives of x;
2) independent of g;
3) of Σ dimension 0;
4) of M dimension $2+2p$ for β^G, β^B; $2p$ for β^Φ;
5) Diff(M) tensors;
6) $U(1)_B$ invariant, dependent upon B only through $H = dB$;
7) invariant under shifts in Φ by a constant.

Expanding up to and including terms with two derivatives (total) on the fields G, B, Φ requires expansion up to order $p = 0$ for β^G, β^B and orders $p = 0, 2$ for β^Φ. Up to this order, we find the most general possible expression satisfying the above criteria:

$$\begin{aligned}
\beta^G_{\mu\nu} &= a_1 R^G_{\mu\nu} + a_2 G_{\mu\nu} + a_3 G_{\mu\nu} R^G + a_4 H_{\mu\rho\sigma} H_\nu^{\rho\sigma} + a_5 G_{\mu\nu} H_{\rho\sigma\tau} H^{\rho\sigma\tau} \\
&\qquad + a_6 D_\mu D_\nu \Phi + a_7 G_{\mu\nu} D^2 \Phi + a_8 G_{\mu\nu} D^\rho \Phi\, D_\rho \Phi \\
\beta^B_{\mu\nu} &= b_1 D^\kappa H_{\kappa\mu\nu} + b_2 D^\kappa \Phi H_{\kappa\mu\nu} \\
\beta^\Phi &= c_0 + \ell^2 \left\{ c_1 R^G + c_2 D^2 \Phi + c_3 D^\kappa \Phi D_\kappa \Phi + c_4 H_{\rho\sigma\tau} H^{\rho\sigma\tau} \right\}
\end{aligned} \tag{6.D.2}$$

Here, R^G and $R^G_{\mu\nu}$ are the Ricci scalar and tensor of the metric G, indices are raised and lowered with the help of the metric G, and D_μ is the covariant derivative with respect to the affine connection Γ. On a vector field V^μ, D_μ acts by

$$D_\mu V^\kappa \equiv \partial_\mu V^\kappa + \Gamma^\kappa_{\mu\nu} V^\nu \ . \tag{6.D.3}$$

The coefficients $a_1, \ldots, a_8$, b_1, b_2, $c_0, c_1, \ldots, c_4$ are now the only remaining unknowns, and we shall determine them shortly.

It is very useful to investigate the ℓ-dependence of the graphs contributing to each coefficient. This counting is given by the usual loop expansion, except that each factor of Φ contributes a factor of ℓ^2 additionally. Within this counting, a_8 would arise from a graph with -1 loops, and thus must be absent: $a_8 = 0$. We have furthermore

$$\begin{aligned} &0-\text{loops:} && a_6, a_7, b_2, c_3 \\ &1-\text{loop:} && a_1, \ldots, a_5, b_1, c_0, c_2 \\ &2-\text{loops:} && c_1, c_4 \ . \end{aligned}$$

Two of these coefficients may be obtained from the Weyl anomaly calculation in flat $G_{\mu\nu}$, with $B_{\mu\nu} = \Phi = 0$. We have

$$\begin{aligned} &\bullet \quad a_2 = 0 : && \text{no field renormalization in free } x\text{-field theory;} \\ &\bullet \quad c_0 = +D/6 : && \text{the central charge of } D \text{ scalars } x^\mu. \end{aligned} \tag{6.D.4}$$

The tree-level coefficients may be obtained from classical calculations, and are thus also easily gotten. One finds (see Problem Set #7)

$$\begin{aligned} &\bullet \quad a_6 = 1 \,, \qquad a_7 = 0 \\ &\bullet \quad b_2 = 1/2 \ . \\ &\bullet \quad c_3 = 2 \ . \end{aligned} \tag{6.D.5}$$

It remains to obtain the 1-loop coefficients a_1, a_3, a_4, a_5, b_1, c_2 and the 2-loop coefficients c_1 and c_4. To do so, we use the background field method.

E) Background Field Quantization Method

The background field quantization method is a powerful tool for obtaining the effective action, which is the generating functional for 1-particle irreducible (1PI) Feynman diagrams, in terms of a functional integral. When local symmetries are present, the method may be organized in a manifestly covariant way, thereby greatly enhancing the effectiveness of perturbative expansion methods. We start by explaining the background field method for a general scalar field ϕ, and then adapt the method to the case of sigma models.

Let S be the classical action for the classical field ϕ, given in terms of classical coupling constants. We shall designate by coupling constants all the parameters entering the definition of S. Their dependence will not be exhibited. By replacing classical field and coupling constants by *bare* quantities, we obtain the *bare action* $S_0[\phi_0]$ in terms of the *bare field* ϕ_0 and *bare coupling constants*. We shall assume that the field ϕ_0 is multiplicatively renormalized by a field renormalization factor Z_ϕ, which depends only upon the coupling constants and the renormalization scale Λ, but not on the field ϕ. Let J be a source function. Then the generating functional $W[J]$, which is the generating functional for *renormalized connected Feynman diagrams*, is defined by

$$e^{W[J]} \equiv \int D\phi_0 \, e^{-S_0[\phi_0]+Z_\phi \int J\phi_0} \ . \tag{6.E.1}$$

Alternatively, the above functional integral may be expressed directly in terms of the *renormalized field* $\phi \equiv Z_\phi \phi_0$. It is standard to introduce the *renormalized action* $S_R[\phi]$ by

$$S_0[\phi \, Z_\phi^{-1}] \equiv S_R[\phi] \tag{6.E.2}$$

(S_R consists of the classical action, plus renormalization counterterms.) Then, we have

$$e^{W[J]} = \int D\phi \, e^{-S_R[\phi]+\int J\phi} \ . \tag{6.E.3}$$

The effective action for connected, $1PI$ renormalized Feynman diagrams is denoted by $\Gamma[\varphi]$ and is defined as the Legendre transform of $W[J]$:

$$\Gamma[\varphi] \equiv \int J\varphi - W[J] \qquad\qquad J(x) = \frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} \ . \tag{6.E.4}$$

Expressing $W[J]$ in terms of $\Gamma[\varphi]$, eliminating J, and shifting ϕ by φ in (6.E.3), we obtain the final formula for $\Gamma[\varphi]$:

$$e^{-\Gamma[\varphi]} = \int D\phi \, \exp\left\{ -S_R[\varphi+\phi] + \int \phi \frac{\delta\Gamma[\varphi]}{\delta\varphi} \right\} \tag{6.E.5}$$

At first sight, it would appear that we have gained little: the right hand side depends upon $\Gamma[\varphi]$ itself, and we have only obtained an implicit equation for $\Gamma[\varphi]$. However, a loop expansion (in powers of $\hbar$ in QFT, which we shall perform in terms of ℓ^2 here, for later use) reveals that the above expression yields a powerful recursive equation for $\Gamma[\varphi]$. To see this, redefine $\phi \to \ell\phi$, and expand S_R and $\Gamma[\varphi]$ around the classical action. We make use of the fact that $\Gamma[\varphi]$ admits a Laurent expansion in powers of ℓ^2 (not just ℓ). We have

$$S_R[\varphi + \ell\phi] = \frac{1}{\ell^2} S[\varphi] + \frac{1}{\ell} \int \phi \, \frac{\delta S}{\delta\varphi} [\varphi] + \bar{S}[\varphi;\phi](\ell) \tag{6.E.6}$$

and

$$\Gamma[\varphi] = \frac{1}{\ell^2} S[\varphi] + \bar{\Gamma}[\varphi](\ell) \tag{6.E.7}$$

Here, $\bar{S}$ and $\bar{\Gamma}$ admit a Taylor series expansion in ℓ and ℓ^2, respectively. Substituting these general forms into (6.E.5), we find:

$$e^{-\bar{\Gamma}[\varphi](\ell)} = \int D\phi \, \exp\left\{ -\bar{S}[\varphi;\phi](\ell) + \ell \int \phi \, \frac{\delta\bar{\Gamma}[\varphi](\ell)}{\delta\varphi} \right\} \tag{6.E.8}$$

from which it is transparent that $\bar{\Gamma}$ is determined by a nice recursive equation. For example, to order ℓ^0, we have a standard formula for the 1-loop effective action

$$e^{-\bar{\Gamma}[\varphi](0)} = \int D\phi \, e^{-\bar{S}[\varphi;\phi](0)} \; . \tag{6.E.9}$$

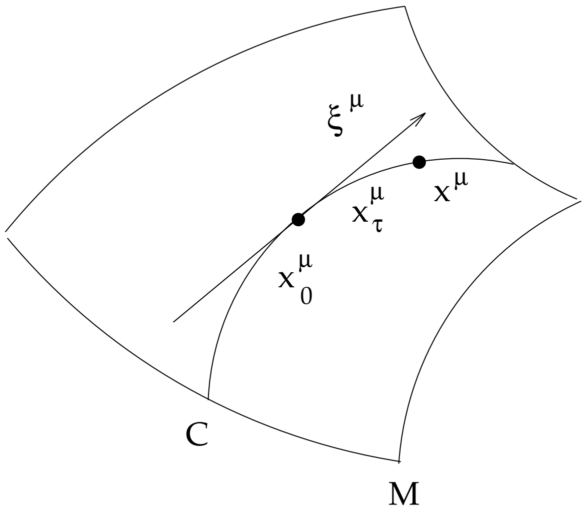
F) Covariant Expansion Methods

Applying the background field method to the non-linear sigma model with action S, given in (6.B.1), the quantum field ϕ becomes the coordinate x^μ of a point on the Riemannian manifold M, while the classical field φ becomes a reference point ${x_0}^\mu$ on M. Addition of coordinates — as would apparently be needed in the background field method developed above — is not covariant under Diff(M). It is beneficial to make use of the Riemann normal coordinate expansion instead, since this method yields Diff(M) covariant results.

For x_0^μ and x^μ sufficiently close to one another, there is a unique shortest geodesic curve C, parametrized by x_τ^μ, that interpolates between x_0^μ for $\tau = 0$ and x^μ for $\tau = \ell$. The function x_τ^μ obeys the geodesic equation

$$\frac{D}{D\tau}\,\dot{x}_\tau^\mu = \ddot{x}_\tau^\mu + \Gamma^\mu_{\nu\rho}(x_\tau)\dot{x}_\tau^\nu\,\dot{x}_\tau^\rho = 0 \tag{6.F.1}$$

The tangent vector to C at $\tau = 0$ is a vector field $\xi^\mu \equiv \dot{x}_0^\mu$ and x^μ may be parametrized (in a neighborhood of x_0^μ) by x_0^μ and ξ^μ: $x^\mu = e^{\ell\xi}x_0^\mu$.



Diff(M)-tensors admit covariant expansions in powers of ℓ. For example, the expansion of rank 0 tensors (scalars, such as Φ) is given by

$$\Phi(x) = \Phi(x_0) + D_\kappa\Phi(x_0)\ell\xi^\kappa + \frac{1}{2}D_\kappa\, D_\lambda\Phi(x_0)\ell^2\xi^\kappa\xi^\lambda + O(\ell^3). \tag{6.F.2}$$

while the expansion of rank 2 tensors (such as G, B) is given by

$$\begin{aligned} T_{\mu\nu}(x) = T_{\mu\nu}(x_0) + D_\kappa T_{\mu\nu}(x_0)\ell\xi^\kappa + \frac{1}{2}\{&D_\kappa D_\lambda T_{\mu\nu}(x_0) \\ &-\frac{1}{3}R^\rho_{\kappa\mu\lambda}T_{\rho\nu}(x_0) - \frac{1}{3}R^\rho_{\kappa\nu\lambda}T_{\mu\rho}(x_0)\}\ell^2\xi^\kappa\xi^\lambda + O(\ell^3)\ . \end{aligned} \tag{6.F.3}$$

Here, D_κ is the covariant derivative with respect to the affine connection $\Gamma^\mu_{\nu\rho}$, and $R^\rho_{\kappa\mu\lambda}$ is the associated Riemann curvature.

As a result, the generalized non-linear sigma model action $S[x,g]$ of §B may be similarly expanded around x_0, in a power series in ℓ, by setting $x = e^{\ell\xi}x_0$.

$$\begin{aligned} S[x,g] =& S[x_0,g] + \ell \int_\Sigma d\mu_g \, \xi^\mu \, S^{(1)}_\mu[x_0,g] + \bar S[x_0,\xi;g] \\ S^{(1)}_\mu[x_0,g] =& \frac{1}{\sqrt{\det\, g}} \frac{\delta S[e^{\xi'}x_0,g]}{\delta \xi'^\mu}\bigg|_{\xi'=0} \end{aligned} \tag{6.F.4}$$

Here, $\bar S$ admits a Taylor series expansion in ℓ. Notice that S and $S^{(1)}_\mu$ are each the sum of a term of order ℓ^{-2} and a term of order ℓ^0 arising from the dilaton action S_Φ. The effective action $\Gamma[x_0,g]$ is now obtained just as in (6.E.8), except with the field φ replaced by x_0 and ϕ replaced by ξ.

$$e^{-\Gamma[x_0,g]} = \int D\xi^\mu \exp\left\{ -S[e^{\ell\xi}x_0,g] + \ell \int_\Sigma d\mu_g \xi^\mu \Gamma^{(1)}_\mu[x,g] \right\} \tag{6.F.5}$$

It will be understood throughout that S stands for the renormalized action, denoted by S_R previously. We have also used the definition

$$\Gamma^{(1)}_\mu[x_0,g] \equiv \frac{1}{\sqrt{\det\, g}} \frac{\delta \Gamma[e^{\xi'}x_0,g]}{\delta \xi'^\mu}\bigg|_{\xi'=0} . \tag{6.F.5'}$$

The functional measure $D\xi^\mu$ is defined through the L^2 norm on TM, which is given explicitly by

$$\|\xi^\mu\|^2 \equiv \int_\Sigma d\mu_g \, G_{\mu\nu}(x_0)\xi^\mu\xi^\nu . \tag{6.F.6}$$

Using again the property that $\Gamma[x_0,g]$ admits a Laurent expansion in terms of ℓ^2 rather than ℓ, we have

$$\Gamma[x_0,g] = S[x_0,g] + \bar\Gamma[x_0,g] \tag{6.F.7}$$

where $\bar\Gamma$ admits a Taylor series expansion in ℓ^2, and is given by

$$e^{-\bar\Gamma[x_0,g]} = \int D\xi^\mu \, e^{-\bar S[x,g] + \ell\int_\Sigma d\mu_g \xi^\mu \, \bar\Gamma^{(1)}_\mu[x_0,g]} . \tag{6.F.8}$$

Here, $\bar S$ is defined as in (6.F.4), and $\bar\Gamma^{(1)}_\mu$ is defined as in (6.F.5$'$) by replacing Γ by $\bar\Gamma$.

It remains to list the explicit expansion terms that we shall need to evaluate β^G, β^B and β^Φ, up to order ℓ^0 in β^G and β^B and ℓ^0, ℓ^2 in β^Φ. They are given by

$$\bar S = \bar S_0 + \ell\, \bar S_1 + \ell^2 \bar S_2 + O(\ell^3) \tag{6.F.9}$$

where

$$\begin{aligned}
\bar{S}_0 &= \frac{1}{8\pi}\int_\Sigma d\mu_g \Big\{ \mathcal{D}^*_m\xi^\mu \mathcal{D}^*_n\xi^\nu G_{\mu\nu}(x_0) g^{mn} + \mathcal{R}_{\mu\nu\rho\sigma}(x_0)\partial_m x_0^\mu \partial_n x_0^\rho \xi^\nu \xi^\sigma (g^{mn} - \varepsilon^{mn}) \Big\} \\
\bar{S}_1 &= \frac{1}{8\pi}\int_\Sigma d\mu_g \Big\{ \frac{1}{3} H_{\mu\nu\rho} \xi^\mu \mathcal{D}^*_m \xi^\nu \mathcal{D}^*_n \xi^\rho \varepsilon^{mn} \Big\} \qquad\qquad (6.F.10) \\
\bar{S}_2 &= \frac{1}{8\pi}\int_\Sigma d\mu_g \Big\{ \frac{1}{3} R_{\mu\nu\rho\sigma} \xi^\nu \xi^\rho \mathcal{D}^*_m \xi^\mu \mathcal{D}^*_n \xi^\sigma g^{mn} - \frac{1}{2} \mathcal{R}_{\mu\nu\rho\sigma} \xi^\nu \xi^\rho \mathcal{D}^*_m \xi^\mu \mathcal{D}^*_n \xi^\sigma \varepsilon^{mn} \\
&\qquad\qquad + 2 D_\mu D_\nu \Phi(x_0) R_g \xi^\mu \xi^\nu \Big\} \ .
\end{aligned}$$

Here, the covariant derivative with torsion on TM and pulled back to Σ, is defined by

$$\mathcal{D}^*_m \xi^\mu \equiv D^*_m \xi^\mu + \frac{1}{2} H^\sigma{}_{\nu\rho} g_{mp} \varepsilon^{pq} \partial_q x^\nu \xi^\rho \ . \qquad (6.F.11)$$

(This definition agrees with that of (6.C.9).) Furthermore, we have the following covariant derivatives with torsion on TM

$$\mathcal{D}_\mu \xi^\nu \equiv D_\mu \xi^\nu + \frac{1}{2} H^\sigma{}_{\mu\rho} \xi^\rho \qquad (6.F.12)$$

whose structure relations yield the torsion H and the curvature tensor $\mathcal{R}$ of the connection with torsion $\Gamma + \frac{1}{2} H$:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\, \xi^\rho = H^\sigma{}_{\mu\nu} \mathcal{D}_\sigma \xi^\rho + \mathcal{R}^\rho_{\sigma\mu\nu} \xi^\sigma \ . \qquad (6.F.13)$$

It is easy to express $\mathcal{R}$ in terms of the Riemann tensor R and the torsion tensor H:

$$\begin{aligned}
\mathcal{R}_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} + \frac{1}{2} D_\rho H_{\sigma\mu\nu} - \frac{1}{2} D_\sigma H_{\rho\mu\nu} \\
&\qquad + \frac{1}{4} H_{\rho\mu\alpha} H_{\sigma\nu}{}^\alpha - \frac{1}{4} H_{\sigma\mu\alpha} H_{\rho\nu}{}^\alpha
\end{aligned} \qquad (6.F.14)$$

G) Reformulation as an $\mathrm{SO}(1, D-1)$ gauge theory

The difficulty encountered in the above formulation is that the quadratic form representing the kinetic term for the field ξ^μ in (6.F.10) depends in a non-trivial way on the metric $G_{\mu\nu}$ and on the background field x_0^μ. As a result, perturbation theory will involve determining the ξ-propagator in arbitrary x_0 background, which is an impossible task. One may get around this complication by changing variables and absorbing the $G_{\mu\nu}$ and x_0^μ dependence into the new field.

One introduces an orthonormal frame, which we denote by ${e_\mu}^a$, with $a = 0, 1, \ldots, D-1$ representing the tangent space direction of TM. We have, by definition

$$G_{\mu\nu} = {e_\mu}^a\, {e_\nu}^b\, \eta_{ab} \tag{6.G.1}$$

where $\eta_{ab} = \mathrm{diag}(- \ + \cdots +)$ is the $\mathrm{SO}(1, D-1)$ invariant flat metric. The inverse frame is denoted by ${e_a}^\mu$, so that ${e_a}^\mu\, {e_\mu}^b = {\delta_a}^b$ and ${e_\mu}^a\, {e_a}^\nu = {\delta_\mu}^\nu$. We also introduce the $\mathrm{SO}(1, D-1)$-valued connection ω_μ, with components ${\omega_\mu}^a{}_b$, such hat under the covariant derivative with respect to the affine connection Γ, and the spin connection ω_μ, the frame ${e_\mu}^a$ is covariantly constant

$$D_\mu {e_\nu}^a \equiv \partial_\mu {e_\nu}^a - \Gamma^\kappa_{\mu\nu} {e_\kappa}^a + {\omega_\mu}^a{}_b\, {e_\nu}^b = 0 \tag{6.G.2}$$

The covariant derivative $\mathcal{D}$, including the torsion connection $\frac{1}{2} H$, is similarly defined. For example, on ${e_\nu}^a$, we have

$$\mathcal{D}_\mu {e_\nu}^a = D_\mu {e_\nu}^a + \frac{1}{2} H_{\mu\nu}{}^\rho {e_\rho}^a \tag{6.G.3}$$

We now rewrite all tensors with respect to the orthonormal frame basis. Thus, we have

$$\begin{aligned} H_{\mu\nu\rho} &= {e_\mu}^a {e_\nu}^b {e_\rho}^c H_{abc} \\ \mathcal{R}_{\mu\nu\rho\sigma} &= {e_\mu}^a {e_\nu}^b {e_\rho}^c {e_\sigma}^d \mathcal{R}_{abcd} \\ \xi^\mu &= \xi^a {e_a}^\mu \\ \mathcal{D}_a &= {e_a}^\mu \mathcal{D}_\mu \end{aligned} \tag{6.G.4}$$

where all tensors are evaluated at the same point on M, which in our background calculation will be ${x_0}^\mu$. We also introduce the pullback of the tangent vector $\partial_m x_0^\mu$, expressed in orthonormal frame basis:

$$e^{*\ a}_m \equiv \partial_m {x_0}^\mu {e_\mu}^a(x_0) \tag{6.G.5}$$

The expression for the quantum contributions to the effective action $\bar\Gamma[x_0, g]$ of (6.F.8) is now

$$e^{-\Gamma[x_0,g]} = \int D\xi^a \exp\left\{ -\bar S[e^{\ell\xi} x_0, g] + \ell \int_\Sigma d\mu_g \xi^a \Gamma^{(1)}_a \right\} \tag{6.G.6}$$

The functional measure $D\xi^a$ is again defined with respect to the L^2 norm

$$\|\xi^a\|^2 = \int_\Sigma d\mu_g \xi^a \xi^b \eta_{ab} \tag{6.G.7}$$

which does not depend upon x_0 or $G_{\mu\nu}$ anymore. The action $\bar{S}$ may be expanded, just as in (6.F.10), and we obtain

$$\bar{S}_0 = \frac{1}{8\pi} \int_\Sigma d\mu_g \{\mathcal{D}^*_m \xi^a \mathcal{D}^*_n \xi^b \eta_{ab} g^{mn} + \mathcal{R}_{abcd}\, e^{*\ a}_m e^{*\ c}_n \xi^b \xi^d (g^{mn} - \varepsilon^{mn})\} \tag{6.G.8a}$$

$$\bar{S}_1 = \frac{1}{8\pi} \int_\Sigma d\mu_g \{\frac{1}{3} H_{abc} \xi^a \mathcal{D}^*_m \xi^b \mathcal{D}^*_n \xi^c \varepsilon^{mn}\} \tag{6.G.8b}$$

$$\begin{aligned} \bar{S}_2 = \frac{1}{8\pi} \int_\Sigma d\mu_g \{ & \frac{1}{3}(\mathcal{R}_{abcd} - \frac{1}{4} H_{caf} H_{db}{}^f) \xi^b \xi^c \mathcal{D}^*_m \xi^a \mathcal{D}^*_n \xi^d g^{mn} \\ & - \frac{1}{2} \mathcal{R}_{abcd} \xi^b \xi^c \mathcal{D}^*_m \xi^a \mathcal{D}^*_n \xi^d \varepsilon^{mn} + 2\mathcal{D}_a \mathcal{D}_b \Phi\, R_g\, \xi^a \xi^b \} \; . \end{aligned} \tag{6.G.8c}$$

The pullback covariant derivative is now very simple:

$$\mathcal{D}^*_m \xi^a = \nabla_m \xi^a + A_m{}^{ab}\, \xi^b \tag{6.G.9}$$

The effective SO$(1, D-1)$ gauge field A_m is defined by the pullback of the spin connection ω_μ:

$$A^a_{mb}(\xi) \equiv \partial_m x^\mu_0(\xi) \omega_\mu{}^a{}_b \tag{6.G.10}$$

In particular, the kinetic term for the ξ^a-field is now also independent of the metric $G_{\mu\nu}$ and the background field $x_0{}^\mu$. As a result, standard perturbation methods in ℓ may be used to evaluate the effective action $\Gamma[x_0, g]$, with a standard propagation for ξ^a.

H) Weyl Variation of the Effective Action

The definition of the trace of the stress tensor for Green functions introduced in (6.C.1) may be translated into an equation for the Weyl variation of the effective action, $\Gamma[x_0, g]$. One finds

$$\delta_\sigma \Gamma[x_0, g] = -\frac{1}{2\pi} \int_\Sigma d\mu_g\, \delta\sigma\, T_m{}^m \; . \tag{6.H.1}$$

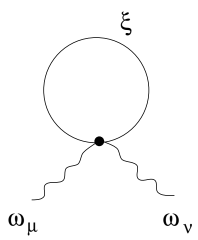
where $T_m{}^m$ denotes the expectation value of the operator $T_m{}^m$ in the bakcground field x_0, and metric g. The derivation of (6.H.1) from (6.C.1) is given in Appendix A. It is understood that x_0 is also allowed to transform under Weyl(Σ), as explained in §C. The arguments developed in §C for what the most general structure of $T_m{}^m$ is, carry over here, and we find, for a suitable Weyl(Σ) action on x_0:

$$T_m{}^m = \partial_m x_0^\mu \partial_n x_0^\nu (\beta^G_{\mu\nu} g^{mn} + \beta^B_{\mu\nu} \varepsilon^{mn}) + \beta^\Phi R_g \qquad (6.H.2)$$

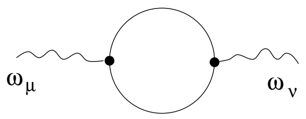
where β^G, β^B and β^Φ are evaluated at x_0. This definition of $T_m{}^m$ and β^G, β^B, and β^Φ is now completely precise, and these quantities may be directly evaluated from $\Gamma[x_0, g]$. Notice that the variation $\delta_\sigma \Gamma$ obtained by substituting (6.H.2) into (6.H.1) must satisfy an integrability condition, known as the Wess-Zumino consistency condition. This condition provides a relation between β^G, β^B and β^Φ, which we shall discuss in §I.

The effective action $\Gamma[x_0, g]$ is the sum of the classical action $S[x_0, g]$ and the quantum corrections $\bar\Gamma[x_0, g]$. The contribution of S to β^G, β^B and β^Φ was already evaluated previously (Problem Set #7). It remains to evaluate the Weyl transformation of $\bar\Gamma$ to the order we are computing. We begin by evaluating the $O(\ell^0)$ contribution, which arises solely from $\bar S_0$. We shall treat the contributions to $T_m{}^m$ perturbatively in the number of M-derivatives applied to the fields G, B and Φ. This expansion is reliable since we know from general considerations that we should only retain contributions to $T_m{}^m$ with two derivatives. For example, the $\mathcal{R}$-term in $\bar S_0$ can contribute at most to first order.

The form of the first term in $\bar S_0$ is that of a gauge field — the spin connection ω_μ — minimally coupled to ξ^a. By power counting its one-loop graphs with one insertion of $\omega_\mu \omega_\nu$ and with two insertions of $\omega_\mu \partial_\nu$ are the only ones that could potentially contribute to the Weyl anomaly.



graph $\omega_\mu\omega_\nu$



graph $\omega_\mu\partial_\nu$

All other graphs are absolutely convergent and obey anomaly-free Ward identities. By gauge invariance, however, the degree of divergence is lowered by 2 and the sum of the two graphs does not contribute to the Weyl anomaly either (see also Gawedzki's lectures). Thus, the only contribution from the first term in $\bar{S}_0$ is to the central charge: $+D/6$ contribution to β^Φ, which we have already discussed previously by fixing $c_0 = +D/6$.

It remains to evaluate the contributions from the first order expansion of the $\mathcal{R}$-term in $\bar{S}_0$ to β^G and β^B. We directly use the Ansatz for these functions in terms of G, B, Φ: we denote their contribution to this order by $\tilde{\beta}^G$, $\tilde{\beta}^B$, $\tilde{\beta}^\Phi$. Clearly $\tilde{\beta}^\Phi = 0$, and the Weyl transformation of this part is given by

$$\begin{aligned} &-\frac{1}{2\pi}\int_\Sigma d\mu_g\,\delta\sigma\,\partial_m x_0^\mu \partial_n x_0^\nu \left(\tilde{\beta}^G_{\mu\nu}(x_0) g^{mn} + \tilde{\beta}^B_{\mu\nu}(x_0)\varepsilon^{mn}\right) \\ &= \delta_\sigma \Big\langle -\frac{1}{8\pi}\int_\Sigma d\mu_g\,\mathcal{R}_{\mu a \rho b}(x_0)\partial_m x_0^\mu \partial_n x_0^\rho \xi^a \xi^b (g^{mn} - \varepsilon^{mn})\Big\rangle\ , \end{aligned} \tag{6.H.3}$$

where $\langle\quad\rangle$ stands for the normalized correlation function taken with respect to the action given by only the first term in $\bar{S}_0$. But, since x_0 was arbitrary throughout, we may identify the symmetric and antisymmetric tensor parts:

$$\begin{aligned} \tilde{\beta}^G_{\mu\nu}(x_0)\delta\sigma &= \delta_\sigma \left\langle \xi^a \xi^b \right\rangle \frac{1}{8}(\mathcal{R}_{\mu a \nu b} + \mathcal{R}_{\nu a \mu b}) \\ \tilde{\beta}^B_{\mu\nu}(x_0)\delta\sigma &= -\delta_\sigma \left\langle \xi^a \xi^b \right\rangle \frac{1}{8}(\mathcal{R}_{\mu a \nu b} - \mathcal{R}_{\nu a \mu b}) \end{aligned} \tag{6.H.4}$$

The correlation function of ξ's to this order is independent of x_0, and we we find

$$\delta_\sigma \left\langle \xi^a \xi^b \right\rangle = 2\delta\sigma\eta^{ab}\ . \tag{6.H.5}$$

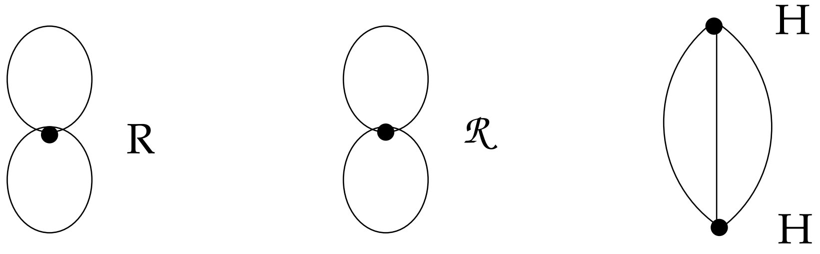
Making use of the expression for $\mathcal{R}$ in terms of R^G and H of (6.F.14), we find

$$\begin{aligned} \tilde{\beta}^G_{\mu\nu} &= \tfrac{1}{2} R^G_{\mu\nu} - \tfrac{1}{8} H_{\mu\alpha\beta} H_\nu{}^{\alpha\beta} \\ \tilde{\beta}^B_{\mu\nu} &= -\tfrac{1}{4} D^\alpha H_{\alpha\mu\nu}\ . \end{aligned} \tag{6.H.6}$$

We now combine this result with the one obtained to tree level: this yields the full β-functions β^G, β^B to order ℓ^0; we also collect the results obtained thus far for β^Φ:

$$\begin{aligned} \beta^G_{\mu\nu} &= \tfrac{1}{2} R^G_{\mu\nu} - \tfrac{1}{8} H_{\mu\alpha\beta} H_\nu{}^{\alpha\beta} + D_\mu D_\nu \Phi \\ \beta^B_{\mu\nu} &= -\tfrac{1}{4} D^\alpha H_{\alpha\mu\nu} + \tfrac{1}{2} D^\alpha \Phi\, H_{\alpha\mu\nu} \\ \beta^\Phi &= D/6 + \ell^2\{2D_\alpha \Phi D^\alpha \Phi - 2D_\alpha D^\alpha \Phi + c_1 R^G + c_4 H^2\}\ . \end{aligned} \tag{6.H.7}$$

Here, we have included the coefficient of $D_\alpha D^\alpha \Phi$, which is obtained from a simple 1-loop calculation, from the last term in $\bar{S}_2$. The calculation of c_1, c_4 requires a full 2-loop computation, with the following Feynman diagram contributions:



The result is

$$c_1 = -\frac{1}{2}; \qquad c_4 = \frac{1}{24} \ . \tag{6.H.8}$$

I) Low Energy String Field Equations and String Effective Action

The string field equations are obtained by including also the effects of the Faddeev-Popov ghosts b c; this effect is limited to an addition of $-13/3$ to β^Φ. We then obtain the following equations for string dynamics:

$$\begin{cases} \beta^G_{\mu\nu} = \frac{1}{2} R^G_{\mu\nu} - \frac{1}{8} H_{\mu\alpha\beta} H_\nu{}^{\alpha\beta} + D_\mu D_\nu \Phi = 0 \\[2ex] \beta^B_{\mu\nu} = -\frac{1}{4} D^\alpha H_{\alpha\mu\nu} + \frac{1}{2} D^\alpha \Phi \, H_{\alpha\mu\nu} = 0 \\[2ex] \beta^\Phi = (D-26)/6 + \ell^2 \{ 2 D_\alpha \Phi \, D^\alpha \Phi - 2 D_\alpha D^\alpha \Phi - \frac{1}{2} R^G + \frac{1}{24} H^2 \} = 0 \ . \end{cases} \tag{6.I.1}$$

to leading order in the expansion in powers of derivatives on G, B and Φ.

There is an interesting consistency condition, whose origin may be understood as follows. On a (locally) flat worldsheet, conformal invariance requires only that $\beta^G = \beta^B = 0$. One expects that these conditions should suffice to make the model conformal on non-flat worldsheets as well. This in turn suggests that the equation $\beta^\Phi = 0$ is not independent from the equations $\beta^G = \beta^B = 0$. In fact, this interdependence may indeed be established with the help of the Bianchi identities:

$$\begin{aligned} D^\mu R^G_{\mu\nu} &= \frac{1}{2} D_\nu R^G \\ D^\mu (H_{\mu\alpha\beta} H_\nu{}^{\alpha\beta}) &= D^\mu H_{\mu\alpha\beta} H_\nu{}^{\alpha\beta} + \frac{1}{6} D_\nu H^2 \ , \end{aligned} \tag{6.I.2}$$

where $H^2 \equiv H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}$. The following relation is then found to hold to this order in powers of ℓ:

$$D^\mu \beta^G_{\mu\nu} = -\frac{1}{2\ell^2}\, D_\nu \beta^\Phi + \tfrac{1}{2}\beta^B_{\alpha\beta} H_\nu{}^{\alpha\beta} + 2\beta^G_{\mu\nu} D^\mu \Phi \ , \tag{6.I.3}$$

so that the equations $\beta^G = \beta^B = 0$ imply that β^Φ is constant. We see that it suffices indeed to render the quantum-field theory of the x-field conformal on a (locally) flat worldsheet. The only quantity left undetermined is then the central charge. There are arguments in the physics literature that such an equation for β^Φ holds to all orders in ℓ.

The only way to satisfy $\beta^\Phi = 0$, given that D must be an integer, and that ℓ is an expansion parameter, is to set $D = 26$.

The above equations can be put in a more standard form, which reflects the fact that $B_{\mu\nu}$ and Φ may be viewed as matter fields, coupling to the metric $G_{\mu\nu}$ by their energy-momentum tensor. Alternatively, the above field equations are seen to derive from an action

$$I(G,B,\Phi) = \frac{1}{2\kappa^2}\int d^{26}x\sqrt{\det\, G}\ e^{-2\Phi}\left\{R^G + 4D_\mu\Phi D^\mu\Phi - \frac{1}{12}\, H^2\right\} , \tag{6.I.4}$$

provided the dimension is critical $D = 26$. The way the dilaton coupling enters exposes the fact that $e^{2\Phi_0}$ is the string loop expansion parameter, or string coupling constant, when the dilaton vacuum expection value is Φ_0. The presence of the factor $e^{-2\Phi_0}$ reveals that this effective action arises from string tree level effects. (One loop contributions would have no exponential Φ dependence.) It is possible to put the action in a more standard form, by performing a Weyl transformation on G. (For later use, we consider arbitrary dimension D_i on results will then have application also to superstring theory, where $D = 10$.)

$$G_{\mu\nu} \to G'_{\mu\nu} = e^{-4\Phi/(D-2)} G_{\mu\nu} \ . \tag{6.I.5}$$

The action in this metric assumes the form

$$I(G',B,\Phi) = \frac{1}{2\kappa^2}\int d^D x\sqrt{\det\, G'}\ \left\{R^{G'} - \frac{4}{D-2}\, D_\mu\Phi D^\mu\Phi - \frac{1}{12}\, e^{-8\Phi/(D-2)} H^2\right\} . \tag{6.I.6}$$

The metrics G and G' are usually called the *string metric* and the *Einstein metric*, respectively.

The action $I(G', B, \Phi)$ is in fact well-known in field theory. If we ignore the peculiar value of the $D = 26$ dimension, and arbitrarily set $D = 10$ instead, we recover the action for the bosonic part of the $N = 1$ supergravity multiplet in $D = 10$ which is part of the so-called Chapline-Manton action (see §X). When we deal will with superstrings, we shall see that the superstring calculation to this order yields the same as the bosonic string, except for the dimension!

Finally, we point out that the field equations (6.I.1) are consistent with the low energy limit of transition amplitudes of strings, scattering in flat Minkowski space-time. The simplest example of such a low energy limit was encountered in §II, where we argued that the transversality conditions on the graviton polarization tensor required conditions that arise as the low energy limit of the linearized limit of $R^G_{\mu\nu} = 0$. In fact, string scattering amplitudes may be used directly to compute the string field equations. Unfortunately, it is difficult to compute the effective action directly from string transition amplitudes, since the latter are always evaluated on-shell only.

J) A first Look at Compactification

One of the most remarkable results of string theory is that the dimension of space-time (at long distances) is dynamically determined: $D = 26$ for bosonic strings and $D = 10$ for superstrings in flat (or nearly flat) space-time. The nagging aspect of this result, however, is that the number of dimensions we observe is only 4. Well, at least 10 is *larger* than 4!

In attempts at unifying general relativity and Maxwell's electromagnetism, Kaluza and Klein proposed in the 1920's that space-time could really be of higher dimension. And that as long as space-time effectively appears to be 4-dimensional at all length scales where present day experiments can probe it, the extra dimensions would not be directly observable. This scheme is usually termed Kaluza-Klein theory.

The most straightforward way to realize such unobservable extra dimensions is to consider space-times M of the form $M = \mathbb{R}^4 \times K$. Here, $\dim K = \dim M - 4$, and K is a compact manifold, whose (largest) size will be denoted by R_K. As long as $R_K \ll L_{\mathrm{expt}}$, the K manifold should not be directly observable. More generally, we may consider M to be of the form $M = M_4 \times K$, where M_4 is approximately Minkowskian (such as would be the case for our universe, where M_4 is a manifold curved by the presence of matter).

The low energy approximation to string theory yields information on the possible solutions for the compact manifold K, provided the size R_K of K is much larger than the Planck scale:

$$R_K \gg \ell = \sqrt{\alpha'/2} \tag{6.J.1}$$

In this case the approxiamtion in which higher powers of ℓ are neglected is reliable. The string field equations (6.I.1) restrict the fields G, B and Φ that are allowed on K and thus restrict K.

If it is assumed that space-time sypersymmetry remains partially unbroken at distance scales of order R_K, then K must be a Kähler manifold, and Φ must be constant. A particular solution corresponds to $H = 0$, so that K must be Kähler and Ricci flat. For superstrings, where the critical dimension is $D = 10$, the combination of these conditions amounts to demanding that K be a Calabi Yau manifold of complex dimension 3.

Appendix A

The correlation functions in (6.C.1) are defined by

$$\begin{aligned}\langle \phi_1 \ldots \phi_n \rangle_g &= \int D\phi\, \phi_1 \ldots \phi_n\, e^{-S[\phi,g]} \\ &= \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n}\, e^{W[J,g]} \Big|_{J=0}\end{aligned}$$

Here, we consider correlation functions of a canonical field ϕ, which in our case is just the map x or the field ξ and J is some function, just as in (6.E.1). We now perform an infinitesimal Weyl rescaling δ_σ, whose support does not contain the points where the operators $\phi_1 \ldots \phi_n$ are applied. Thus the Weyl variation becomes:

$$\delta_\sigma \langle \phi_1 \ldots \phi_n \rangle_g = \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n}\, \delta_\sigma W[J,g]\, e^{-W[J,g]} \Big|_{J=0}$$

and using the definition of the effective action by Legendre transform

$$\delta_\sigma W[J,g] \Big|_{J=0} = -\delta_\sigma \Gamma[\varphi, g] \Big|_{\varphi}$$

where φ is such that $J = 0$. Equation (6.H.1) follows immediately.