WITTEN'S PROBLEMS (Spring Term), Set One – N°. 1 (solution by D. Freed)

Problem:

(1) The strong interactions are described to good approximation by a theory which has a symmetry group $G = SU(2) \times SU(2)$ spontaneously broken down to a diagonal subgroup $H \cong SU(2)$. There are no massless particles except the Goldstone bosons, which, to the extent G is a valid symmetry, are exactly massless. The theory, to good approximation, has a G-invariant Lagrangian L.

However, G is not an exact symmetry of nature. It is broken by perturbations that transform in the (2,2) representation of G (that is, the tensor product of the two dimensional representations of the two $SU(2)$'s.) Among the operators of the G-invariant theory are two "multiplets" of operators $\mathcal{O}_i$ and $\widetilde{\mathcal{O}}_j$, $i, j = 1 \dots 4$, each transforming as (2,2). The Langrangian is, more accurately,

$$\tilde{L} = L + \sum_{i=1}^{4} \left(\varepsilon_i \mathcal{O}_i + \tilde{\varepsilon}_i \widetilde{\mathcal{O}}_i \right). \tag{1}$$

The ε_i and $\tilde{\varepsilon}_j$ are, of course, all real. (That there are precisely two possible multiplets $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ of perturbations is explained by the underlying gauge theory, but for the present exercise we simply assume it.)

For a generic choice of the ε_i and $\tilde{\varepsilon}_i$ the exact symmetry group of the model is only $H' = U(1)$, and this is, in fact, the situation in nature.

Treat the effective theory to first order in ε and $\tilde{\varepsilon}$ (which are, in fact, in nature small and comparable to each other).

(a) Show that, in first order, you would expect the degeneracy of the vacuum to be lifted, and there to appear a unique vacuum with mass gap. In particular, you expect the H' symmetry to be unbroken. (Abstractly, H' can be embedded in H; the point that is slightly non-trivial is that the perturbation picks a vacuum, of the unperturbed theory, that is invariant under an $H \cong SU(2)$ that actually does contain H'. H could of course have been conjugated to a group that does not contain H'.) In particular, there is no exact symmetry that is spontaneously broken, and no exact Goldstone boson; the former Goldstone bosons all get mass.

(b) More surprisingly, show that to first order in ε and $\tilde{\varepsilon}$, the three Goldstone bosons – the one that is invariant under H' and the two that transform in a non-trivial representation of H' – all get *equal* masses. (This is also observed in nature, to high precision, the particles being respectively the neutral and charged pions.)

(c) The three Goldstone bosons will not have *exactly* equal masses. In what order in ε and $\tilde{\varepsilon}$ will the degeneracy be lifted and how would you describe this lifting in the effective Lagrangian?

Solution:

Since the only massless particles in the unbroken ($\varepsilon = \tilde{\varepsilon} = 0$) theory are the Goldstone bosons, in the infrared it flows to a σ-model into the homogeneous space G/H. Topologically G/H is a 3-sphere. Now with $\varepsilon, \tilde{\varepsilon}$ nonzero we expect a perturbation of the σ-model Langrangian, and we are asked to contemplate first a perturbation *linear* in $\varepsilon, \tilde{\varepsilon}$. The most relevant perturbation involves no derivative of the field in the σ-model, so is simply a potential function V on G/H. Now G double covers SO_4, and we are given that the operators $\mathcal{O}$, $\widetilde{\mathcal{O}}$ in the problem transform as vectors under SO_4. Thus if we transform $\varepsilon, \tilde{\varepsilon}$ as vectors then the perturbed theory retains the full G invariance. For fixed $\varepsilon, \tilde{\varepsilon}$ the group G is broken to the subgroup which fixes $\varepsilon, \tilde{\varepsilon}$, which for generic vectors $\varepsilon, \tilde{\varepsilon}$ is isomorphic to SO_2. (This is called "H'" in the problem.) Now the potential V must be SO_4-invariant if we transform $\varepsilon, \tilde{\varepsilon}$, and it follows easily that it has the form

$$V_{\varepsilon,\tilde{\varepsilon}}(x) = (\lambda\varepsilon + \tilde{\lambda}\tilde{\varepsilon}) \cdot x.$$

Here we write $x \in G/H \simeq S^3 \subset \mathbb{R}^4$, the '$\cdot$' denotes the dot product in $\mathbb{R}^4$, and $\lambda, \tilde{\lambda}$ are real constants. For fixed $\varepsilon, \tilde{\varepsilon}$ the potential $V_{\varepsilon,\tilde{\varepsilon}}$ is the restriction to S^3 of a linear function in $\mathbb{R}^4$. Thus it has a unique minimum on S^3. This is the unique vacuum of the perturbed theory (question (a)). Furthermore, it is clear that the Hessian of $V_{\varepsilon,\tilde{\varepsilon}}$ at the minimum is scalar; that is, its eigenvalues – the masses of the pseudo Goldstone bosons – are all equal (question (b)). Observe that the potential is invariant under an $S0_3$ subgroup of SO_4 (which fixes the vector $\lambda\varepsilon + \tilde{\lambda}\tilde{\varepsilon}$), an accidental symmetry of the infrared theory. (We only expected SO_2 invariance.) Finally, a perturbation quadratic in $\varepsilon, \tilde{\varepsilon}$ — for example $V_{\varepsilon,\tilde{\varepsilon}}(x) = (\varepsilon \cdot x)(\tilde{\varepsilon} \cdot x)$ — can break this SO_3 invariance and so destroy the equality of the Goldstone boson masses (question (c)).

Spring Term QFT Problems, Set One – N°. 2

(a) In this problem we work classically. Then the minimum occurs when the gauge fields are zero and $\phi = \phi_0$ is a constant with $|\phi_0|^2 = v^2$. Locally we can identify the $SU_2 \times U_1$ in the problem with U_2, which then acts on ϕ via the two dimensional representation. Fixing the gauge fields and ϕ breaks the gauge symmetry down to the global U_1 gauge transformations which fix ϕ. (A nonzero vector in $\mathbb{C}^2$ is fixed by $U_1 \subset U_2$.)

(b), (c) We expand the action around the minimum up to quadratic terms. Write A, iB for the SU_2 and U_1 connections and set

$$A = \begin{pmatrix} iA_3 & A_1 + iA_2 \\ -A_1 + iA_2 & -iA_3 \end{pmatrix}.$$

Here A_1, A_2, A_3, B are all real. Choose $\phi_0 = \binom{0}{v}$ and write

$$\phi = \begin{pmatrix} 0 \\ v \end{pmatrix} + \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Then up to quadratic terms we have

$$\begin{aligned}
|F_B|^2 &\sim |dB|^2, \\
|F_A|^2 &\sim |dA_1|^2 + |dA_2|^2 + |dA_3|^2, \\
|D\phi|^2 &\sim \left| \begin{pmatrix} d\phi_1 + (A_1 + iA_2)v \\ d\phi_2 + iBv - iA_3 v \end{pmatrix} \right|^2, \\
&\sim |d\phi_1|^2 + |d\phi_2|^2 + v^2 A_1^2 + v_2 A_2^2 + v^2 (B - A_3)^2, \\
\frac{\lambda}{8}\left(|\phi|^2 - v^2\right) &\sim \frac{\lambda}{8}|\phi_1|^2 + \frac{\lambda}{8}|\phi_2|^2 + 2v \;\mathrm{Re}\,\phi_2.
\end{aligned}$$

Now rescale $A \to e_2 A,\ B \to e_1 B$ so that after introducing a gauge fixing term (as in Problem Set Six, N°. 2 in the fall), the quadratic Langrangian is (putting a coefficient $1/2$ in front of $|D\phi|^2$):

$$\begin{aligned}
L_{\text{quad}} = \int d_x^4 \quad & \left(\frac{1}{2}|d\phi_1|^2 + \frac{\lambda}{8}|\phi_1|^2\right) + \left(\frac{1}{2}|d\phi_2|^2 + 2v\,\mathrm{Re}\,\phi_2 + \frac{\lambda}{8}|\phi_2|^2\right) \\
&+ \left(\frac{1}{2}(\Delta A_1, A_1) + \frac{1}{2}v^2 e_2^2 A_1^2\right) + \left(\frac{1}{2}(\Delta A_2, A_2) + \frac{1}{2}v^2 e_2^2 A_2^2\right) \\
&+ \left(\frac{1}{2}(\Delta A_3, A_3) + \frac{1}{2}(\Delta B, B) + \frac{1}{2}v^2 (e_1 B - e_2 A_3)^2\right).
\end{aligned}$$

From this we see that there are two massive complex scalars (ϕ_1, ϕ_2); two massive vectors (A_1, A_2) with mass ve_2; a massless vector and a massive vector with mass $v\sqrt{e_1^2 + e_2^2}$. To prove the last statement observe that in L_{quad} the gauge fields A_3, B have a mass matrix with eigenvalues 0 and $v\sqrt{e_1^2 + e_2^2}$.

The Lie algebra of the unbroken U_1 is the subalgebra where $A_1 = A_2 = 0$ and $e_2 A_3 = e_1 B$. Thus ϕ_1 has charge 2, ϕ_2 is uncharged, A_3 and B are uncharged, and A_1, A_2 have charge 2.

Note: If we had followed the normalization indicated in the problem, we would have replaced A, B by $\frac{A}{2}$, $\frac{B}{2}$, and then the charge of ϕ_1, A_1, A_2 would be 1 rather than 2. However, we wanted to avoid the denominators in the computation.