Spring Term Problem Set 1

(1) The strong interactions are described to good approximation by a theory which has a symmetry group $G = SU(2) \times SU(2)$ spontaneously broken down to a diagonal subgroup $H \cong SU(2)$. There are no massless particles except the Goldstone bosons, which, to the extent G is a valid symmetry, are exactly massless. The theory, to good approximation, has a G-invariant Lagrangian L.

However, G is not an exact symmetry of nature. It is broken by perturbations that transform in the $(2,2)$ representation of G (that is, the tensor product of the two dimensional representations of the two $SU(2)$'s.) Among the operators of the G-invariant theory are two "multiplets" of operators $\mathcal{O}_i$ and $\widetilde{\mathcal{O}}_j$, $i, j = 1 \dots 4$, each transforming as $(2,2)$. The Lagrangian is, more accurately,

$$\widetilde{L} = L + \sum_{i=1}^{4} \left(\epsilon_i \mathcal{O}_i + \widetilde{\epsilon}_i \widetilde{\mathcal{O}}_i \right). \tag{1}$$

The ϵ_i and $\widetilde{\epsilon}_j$ are, of course, all real. (That there are precisely two possible multiplets $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ of perturbations is explained by the underlying gauge theory, but for the present exercise we simply assume it.)

For a generic choice of the ϵ_i and $\widetilde{\epsilon}_i$ the exact symmetry group of the model is only $H' = U(1)$, and this is, in fact, the situation in nature.

Treat the effective theory to first order in ϵ and $\widetilde{\epsilon}$ (which are in fact in nature small and comparable to each other).

(a) Show that, in first order, you would expect the degeneracy of the vacuum to be lifted, and there to appear a unique vacuum with mass gap. In particular, you expect the H' symmetry to be unbroken. (Abstractly, H' can be embedded in H; the point that is slightly non-trivial is that the perturbation picks a vacuum, of the unperturbed theory, that is invariant under an $H \cong SU(2)$ that actually does contain H'. H could of course have been conjugated to a group that does not contain H'.) In particular, there is no exact symmetry that is spontaneously broken, and no exact Goldstone boson; the former Goldstone bosons all get mass.

(b) More surprisingly, show that to first order in ϵ and $\widetilde{\epsilon}$, the three Goldstone bosons – the one that is invariant under H' and the two that transform in a non-trivial representation of H' – all get *equal* masses. (This is also observed in nature, to high precision, the particles being respectively the neutral and charged pions.)

(c) The three Goldstone bosons will not have *exactly* equal masses. In what order in ϵ and $\widetilde{\epsilon}$ will the degeneracy be lifted and how would you describe this lifting in the effective Lagrangian?

(2) Here we consider (as in the standard model of weak interactions) an $SU(2)\times U(1)$ gauge theory in four dimensions. The $SU(2)$ and $U(1)$ gauge fields will be called A_2 and A_1, the curvatures F_2 and F_1, and the gauge couplings e_2 and e_1. The Lagrangian of the pure gauge theory would be

$$L=\int d^4x\left(\frac{1}{4e_1^2}|F_1|^2+\frac{1}{4e_2^2}\mathrm{Tr}F_2^2\right) \tag{2}$$

where Tr is the trace in the two-dimensional representation of $SU(2)$.

In addition there are complex scalar fields ϕ transforming in the two-dimensional representation of $SU(2)$ and with "charge 1/2" for the $U(1)$. Their Lagrangian is

$$L_\phi=\int d^4x\left(|D\phi|^2+\frac{\lambda}{8}\left(|\phi|^2-v^2\right)^2\right). \tag{3}$$

v^2 is positive.

(a) Show that $SU(2)\times U(1)$ is spontaneously broken to a diagonal $U(1)$.

(b) Determine the spectrum of the model: what are the massless and massive scalars and vectors, and how do they transform under the unbroken $U(1)$?

(c) The quantities of the model that are observed to the highest precision are the two gauge couplings and the masses of the charged and neutral massive vectors. This makes four observables. As the model has four parameters (e_1, e_2, λ, and v), one might not expect any relation among these four observables. However, the four observables are independent of λ in the classical approximation which (as the model is weakly coupled) is a good approximation. So in practice the model does predict a relation between the two vector masses (given a measurement of e_1 and e_2). This relation was one of the early successes of the electroweak standard model.

Of course, this relation will not be exact. It will be modified by loops, and the loops will depend on additional fields that are present in nature but were not mentioned so far (and do not affect the classical discussion carried out here). Present experimental accuracy is just about good enough to see the loop effects, and thereby to get an estimate of the parameter λ that does not enter the classical relation but does enter the loops.