Witten's Problems, Set Four N°. 1 (solution written by D. Freed)

Problem:

(1) In the two-dimensional sigma model with target space a sphere S^n, for large n, we introduced a Lagrange multiplier field σ in order to quantize the theory.

(a) By using the equations of motion, identify σ as an operator in terms of the original variables.

(b) What is the dimension of the σ field and therefore what would you expect to be the large q behavior (with Euclidean q) of $\langle\sigma(q)\sigma(-q)\rangle$. Compare to the actual behavior.

Solution:

(a) This model has two bosonic fields: a map $\phi : \mathbb{R}^2 \to \mathbb{R}^{n+1}$ and a "Lagrange multiplier" $\sigma : \mathbb{R}^2 \to \mathbb{R}$. The action is

$$\mathcal{L} = \frac{1}{2\lambda}\,|d\phi|^2 - \frac{i\sigma}{2}(|\phi|^2 - 1).$$

It is easy to compute the classical equations of motion

$$\begin{aligned} |\phi|^2 &= 1 \\ \frac{\Delta}{\lambda}\phi &= i\sigma\phi. \end{aligned}$$

Our Laplacian 'Δ' is nonnegative. The first says that ϕ maps into the n-sphere S^n and the second that the Laplacian of ϕ is perpendicular to S^n. This means that ϕ is a harmonic map into S^n. We express σ in terms of ϕ as

$$\sigma = \frac{1}{i\lambda}(\Delta\phi, \phi). \tag{1}$$

Up to a total derivative $\sigma = \frac{1}{i\lambda}|d\phi|^2$.

(b) According to equation (1) the σ field has dimension 2, since ϕ is a scalar field in two dimensions and so has dimension 0. We expect the behaviour of the 2-point function in position space to be

$$\langle\sigma(x)\sigma(0)\rangle \sim \frac{1}{|x|^4} \text{ as } x \to 0,$$

or equivalently in momentum space to be

$$\langle\sigma(q)\sigma(-q)\rangle \sim q^2 \quad \text{on} \quad q \to \infty, \tag{2}$$

with possible logarithmic connections due to anomalous dimensions. We could integrate out the σ field first, thereby obtaining the σ-model into S^n, and since this theory is asymptotically free the 2-point function of $|d\phi|^2$ does exhibit the behaviour (2) (up to logarithms). Instead, we integrate out ϕ first and compute the effective action for σ, at least in the large n limit.

The computation of the effective action appears in Witten's lecture II-3:

$$\Gamma(\tilde{\sigma}) = \frac{n}{2}\{Tr \ln(\Delta - i\tilde{\sigma}\tilde{\lambda}) + i\tilde{\sigma}\}. \tag{3}$$

where $\tilde{\sigma} = \sigma/n$ and $\tilde{\lambda} = \lambda n$. We know that we should expand around the value $-i\tilde{\sigma}_0\tilde{\lambda} = M^2$ for a positive constant M^2. The $\tilde{\sigma}$ propagator is the inverse of the second derivative of the action (3), which we evaluate at the constant $\tilde{\sigma}_0 = iM^2/\tilde{\lambda}$:

$$\begin{aligned}\left.\frac{\delta^2\Gamma}{\delta\tilde{\sigma}^2}\right|_{\tilde{\sigma}_0} &= \frac{n\tilde{\lambda}^2}{2} Tr\big((\Delta+M^2)^{-1}\delta\tilde{\sigma}(\Delta+M^2)^{-1}\delta\tilde{\sigma}\big)\\ &= \frac{n\tilde{\lambda}^2}{2}\int dxdy\, G(x,y)\,\delta\tilde{\sigma}(y)\,G(y,x)\,\delta\tilde{\sigma}(x)\\ &= \frac{n\tilde{\lambda}^2}{2}\int dxdy\, G(x,y)^2\,\delta\tilde{\sigma}(x)\,\delta\tilde{\sigma}(y),\end{aligned} \tag{4}$$

where G is the kernel of the Greens function $(\Delta + M^2)^{-1}$. So the kernel of the inverse $\tilde{\sigma}$-propagator is $\frac{n\tilde{\lambda}^2}{2}G(x,y)^2$. In momentum space the square goes over to convolution, and so the kernel of the inverse $\tilde{\sigma}$-propagator is

$$\begin{aligned}K(q,-q) &= \frac{n\tilde{\lambda}^2}{2}\int \frac{d^2p}{(2\pi)^2}\,\frac{1}{p^2+M^2}\,\frac{1}{(q-p)^2+M^2}.\\ &= \frac{n\tilde{\lambda}^2}{8\pi^2}\int_0^1 d\alpha \int d^2p\, \frac{1}{[p^2+M^2+\alpha(1-\alpha)q^2]^2}\\ &= \frac{n\tilde{\lambda}^2}{8\pi}\int_0^1 \frac{d\alpha}{M^2+\alpha(1-\alpha)q^2}\\ &= \frac{n\tilde{\lambda}^2}{4\pi}\,\frac{\ln\left(\frac{\sqrt{1+4M^2/q^2}+1}{\sqrt{1+4M^2/q^2}-1}\right)}{q^2\sqrt{1+4M^2/q^2}}\\ &\sim \frac{n\tilde{\lambda}^2}{2\pi}\,\frac{\ln(q^2/M^2)}{q^2} \quad \text{as} \quad q\to\infty.\end{aligned} \tag{5}$$

So the $\tilde{\sigma}$-propagator behaves as

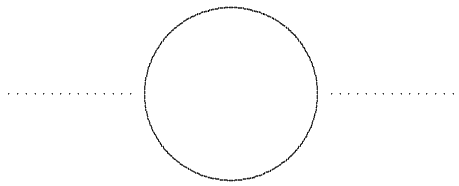
$$\langle\tilde{\sigma}(q)\tilde{\sigma}(-q)\rangle \sim \frac{2\pi}{n\tilde{\lambda}^2}\,\frac{q^2}{\ln(q^2/M^2)} \quad \text{as} \quad q\to\infty.$$

Since $\tilde{\sigma} = \sigma/n$, we have finally

$$\langle\sigma(q)\sigma(-q)\rangle \sim \frac{2\pi n}{\tilde{\lambda}^2}\,\frac{q^2}{\ln(q^2/M^2)} \quad \text{as} \quad q\to\infty. \tag{6}$$

Notice that the behaviour in n could have been predicted from (1), since the 2-point function of $(\Delta\phi,\,\phi)$ is $n+1$ times the 2-point function of a single scalar field. The large q behavior is as predicted in (2), but corrected by a logarithm.

Notice that the inverse propagator (5) can be computed by diagrams. The effective action obtained by integrating our ϕ is computed by diagrams with external σ lines and internal ϕ lines. We should expand around $\sigma = iM^2/\lambda$, of course. For the quadratic term there is a single diagram, as shown, where the solid line



represents ϕ and the dotted line represents σ. Note the ϕ propagator is massive due to the shift in σ. This diagram is evaluated by (5).