Term 2, Problem Set Four

(1) In the two-dimensional sigma model with target space a sphere S^n, for large n, we introduced a Lagrange multiplier field σ in order to quantize the theory.

(a) By using the equations of motion, identify σ as an operator in terms of the original variables.

(b) What is the dimension of the σ field and therefore what would you expect to be the large q behavior (with Euclidean q) of $\langle\sigma(q)\sigma(-q)\rangle$. Compare to the actual behavior.

(2) Consider in two dimensions a $U(1)$ gauge theory with *two* species of fermion, ψ_i, $i = 1, 2$, in the same representation of $U(1)$, of bare masses m_i.

The Lagrangian is thus

$$L_\psi(\psi_1, m_1) + L_\psi(\psi_2, m_2) + \int \left(\frac{F^2}{4e^2} + i\frac{F}{2\pi}\right) \tag{1}$$

where L_ψ is the Lagrangian that we considered in class:

$$L_\psi(\psi, m) = \int \left(\overline{\psi}_+ D_- \psi_+ + \overline{\psi}_- D_+ \psi_- + m\overline{\psi}_- \psi_+ + \overline{m}\overline{\psi}_+ \psi_-\right) d^2x. \tag{2}$$

(a) Show that the θ dependence disappears if either of the m_i vanishes.

(b) Show that in the limit that m_1 becomes very large keeping m_2 fixed (or letting m_2 go to zero while m_1 becomes large) the theory reduces to the one that we studied in class, but with a possibly shifted effective value of θ.

(c) Describe qualitatively the behavior when both of the m_i are large.

(d) Describe the behavior when both of the m_i are small.

(e) Show that if the m_i are equal, the theory has an $SU(2)$ symmetry. Now, by considering the case that the m_i are equal and small, deduce that the two-dimensional sine-Gordon model, that is the model of a scalar field with potential $V(\phi) = \mu^2 \cdot \cos(\beta\phi)$, has an $SU(2)$ symmetry for a certain value of β which you should identify.

(f) Taking the limit as μ goes to zero, where the sine-Gordon theory mentioned in (e) becomes a massless conformally invariant free field theory, deduce that the two-dimensional theory of a free massless scalar of the correct radius (or equivalently of fixed radius 2π and the correct coefficient k of the kinetic energy in $L = k \cdot \int d^2x\, |d\phi|^2$) has an $SU(2)$ symmetry. Because of the conformal invariance this theory actually represents the affinization of $SU(2)$ (more exactly, two copies of it, one for left-movers and one for right-movers) and not only the finite-dimensional group $SU(2)$. This is called the vertex construction of the affine Lie group of $SU(2)$.

To obtain it, we didn't really need to go via the gauge theory. One could, more generally, simply consider several massless free fermions and, by applying the bose-fermi correspondence to them, get the vertex construction for $SO(2n)$ and (with a slight modification) for $SU(n)$.

Hint: a reference for most of Problem 2 (except the last paragraph) is Coleman's "More On The Massive Schwinger Model."