Term 2

Problem Set 5

(1) Consider in two dimensions an $SO(n)$ gauge theory with scalar fields ϕ consisting of $n-1$ copies of the n dimensional representation of $SO(n)$ and a potential energy

$$V = \sum_{i=1}^{n-1} \lambda_i \left[(\phi_i, \phi_i) - a_i^2\right]^2 + \sum_{i<j} c_{ij} (\phi_i, \phi_j)^2. \tag{1}$$

The λ_i and the c_{ij} are all positive. Here $(\, , \,)$ is the invariant quadratic form on the n dimensional representation of $SO(n)$.

Because $\pi_1(SO(n)) = \mathbf{Z}/2\mathbf{Z}$, this theory has a two-valued "theta angle." Explain how you would calculate, for weak coupling, the energy difference between the vacua with the two possible values of theta.

(If you are unsure of how to think about theta for the case that $\pi_1(G)$ is a finite group, you can consult my paper on "Theta Vacua In Two Dimensional Gauge Theories" which will be available from Paula Bozzay in C building.)

(2) Consider in two dimensions the $(\psi_+\psi_-)^2$ model (introduced in Gross's last lecture in December) with the fields being two dimensional fermions ψ_i with Lagrangian

$$L = \int d^2x \left(\sum_i (\psi_{+i} \partial_- \psi_{+i} + \psi_{-i} \partial_+ \psi_{-i}) + g \sum_i (\psi_{+i} \psi_{-i})^2 \right). \tag{2}$$

(a) Consider the classical conservation law $\partial_- S = 0$ where $S = \psi_{+1}\psi_{+2}\dots\psi_{+n}$. By considering its deformation in quantum mechanics, show that the model is integrable.

In this way you obtain a conserved charge R with $[K, R] = (n/2-1)R$ where K is the generator of the Lorentz group (the subgroup $SO(1,1)$ of the two dimensional Poincaré group.) By exchanging $+$ and $-$, you get another conserved charge R' with $[K, R'] = -(n/2-1)R'$.

(b) Now set $n = 4$, so $(n/2) - 1 = 1$. Hence R and R' have the same Lorentz transformation law as the momentum components P_+ and P_-.

You should be able to sharpen the argument in (a) above to show for $n = 4$ that the theory has two stress tensors T_{mn} and T'_{mn} where T is the standard stress tensor whose appropriate integrals give the Poincaré generators, and T'_{mn} has the property that its suitable moments are R and R'. But now, by "contracting" T' with the generator of $SO(1,1)$, you get a new "Lorentz generator" K'; show that the usual Poincaré operators together with R, R', and K' generate two commuting copies of the Poincaré group. (To get the commuting copies, you have to take sums and differences of "old" and "new" operators.)

Show that no linear combination of P_+ and R (or P_- and R') vanishes so that each copy of the Poincaré group acts nontrivially. In fact, you should be able to see that the outer automorphism of $SO(4)$ (which acts in this theory by $\psi_1 \to -\psi_1$ with $\psi_i \to \psi_i$ for $i > 1$) exchanges the two copies of Poincaré.

Deduce that for $n = 4$ this theory is "reducible" as the product of two decoupled theories, in one of which one copy of Poincaré acts and in the other the second.

(c) Use the bose-fermi correspondence to verify this and identify the factors. You should be able to draw an inference about the symmetries of the quantum sine-Gordon model at a special value of the coupling. (You drew the same inference last week in a different way.)

(d) Now consider the case $n = 3$. Show that R and R' are supersymmetries, so that the model is supersymmetric for $n = 3$. Can you verify this using the bose-fermi correspondence?

A reference for this problem is my paper "Some Properties Of The $(\bar{\psi}\psi)^2$ Model in Two Dimensions" which will be available from Paula Bozzay in C building.

(3) The following problem is actually one that was assigned in the fall but not discussed (see problem set 3 from term one, which you can also consult for some additional background). It is the beginning of a series that will hopefully prepare us to do some things with supersymmetric gauge theories in two and four dimensions.

We work locally for the present; it takes some more work to make a global version.

We consider a (4,4) supermanifold W with the structure described in Bernstein's lectures – two complex conjugate integrable distributions A_+ and A_- of dimension $(0,2)$ such that $\{A_+, A_-\}$ generates $TM/(A_+ \oplus A_-)$, A_+ and A_- are endowed with volume forms. A chiral superfield is a field ϕ that is a function on W/A_-. If M is a complex manifold, a chiral map $\Phi : W \to M$ is one such that, for f a local holomorphic function on M, $\Phi^*(f)$ is a chiral function on W. In that case, $\Phi^*(\bar{f})$ is antichiral, that is, it is a function on W/A_+.

Let K be a function on W and consider the Lagrangian

$$L = \int_W d^4y d^4\theta \, \Phi^*(K). \tag{3}$$

$d^4y \, d^4\theta$ is the section of the Berezinian coming from the volume forms on $A_\pm$.

(a) Show that L is invariant under $K \to K + f + \bar{f}$, where f is a local holomorphic function on W.

(b) Recall that any K determines the two-form $\omega = -i\bar{\partial}\partial K$ which for a suitable class of K's is the Kahler class of a Kahler metric g on M. Of course, ω is invariant under the transformation considered in (a).

Show that L can be written just in terms of ω (or equivalently g) and that the "reduced" part of L is the harmonic map action, specialized to maps to a Kahler manifold:

$$L = \int_{W_{red}} d^4 y g_{i\bar{j}}(dX^i, dX^{\bar{j}}) \tag{4}$$

where X^i are local holomorphic coordinates on M.

(c) Specialize now to the flat model with $W = \mathbf{R}^{4,4}$. A_+ is generated by

$$D_A = \frac{\partial}{\partial \theta^A} - i\sigma^m_{A\dot{A}}\bar{\theta}^{\dot{A}}\frac{\partial}{\partial y^m} \tag{5}$$

where θ^A are odd coordinates transforming as "positive" spinors of $Spin(1,3) = SL(2,\mathbf{C})$, $\bar{\theta}^{\dot{A}}$ are their complex conjugates, and y^m are even coordinates. σ is the isomorphism $S_+ \otimes S_- = V$. A_- is generated by

$$D_{\dot{A}} = \frac{\partial}{\partial \theta^{\dot{A}}} - i\sigma^m_{A\dot{A}}\bar{\theta}^A\frac{\partial}{\partial y^m}. \tag{6}$$

Impose invariance under a space-like translation of W, say a translation of y^3 (the metric being $(dy^0)^2 - \sum_{i=1}^3 (dy^i)^2$ to get a model on $\mathbf{R}^{3,4}$. Show that this is equivalent to the $\mathbf{R}^{3,2}$ σ model described in problem 1(e) of term 1, problem set 3 (but specialized to the case that M is Kahler). Here is an attempt to explain the strategy: under $Spin(1,2)$, the two spinor representations of $Spin(1,3)$ become isomorphic. So after the reduction, A_+ and A_- are naturally isomorphic. One can in an $SO(1,2)$-invariant way take the "real" combination $\chi^A = \theta^A + \bar{\theta}^A$ of the θ's and $\bar{\theta}$'s. The $\mathbf{R}^{3,2}$ we want has even coordinates y^i, $i = 0, 1, 2$, and odd coordinates χ^A. By "integrating over the fibers" of a map from $\mathbf{R}^{3,4}$ to $\mathbf{R}^{3,2}$, one reduces (4) (specialized to y^3-independent maps) to the $\mathbf{R}^{3,2}$ Lagrangian of problem 1(e).