Homework, April 10

In class we considered in two dimensions a theory with n chiral superfields A_i, $i = 1, \ldots, n$, and superpotential $F(A_1, \ldots, A_n)$ a homogeneous function of degree n. Then we constructed a $\mathbf{Z_n}$ orbifold of this theory in which one divides by the $\mathbf{Z_n}$ group that acts as $A_i \to e^{2\pi i/n} A_i$.

The theory has a left and right moving R symmetry group $U(1) \times U(1)$, with which you should be familiar. So the Hilbert space is bigraded.

Formulate this theory on a circle $\mathbf{S}^1$ (with supersymmetric, periodic boundary conditions for all fields). There is a finite-dimensional space of supersymmetric ground states of the theory, in fact for reasons that should be clear from the lecture this space is closely related to the cohomology of the projective hypersurface defined by the equation $F = 0$ in $\mathbf{P}^{n-1}$. Both are bigraded, the bigrading on one side coming from the R symmetries and the bigrading on the other side coming from the Hodge decomposition of the cohomology of the hypersurface; the two bigradings will match (after a shift by $n/2$, that is what is usually called $H^{p,q}$ maps to states whose R charges are $p', q' = p - n/2, q - n/2$).

Compute the space of supersymmetric ground states of the orbifold theory. There are different pieces:

(1) There are $n - 1$ twisted sectors. How many zero energy states come from each twisted sector? To find the p', q' values of the ground states in the twisted sectors, you can note that (a) perturbation theory is straightforward as there are no zero modes, (b) you can go to the weak coupling limit in computing p' and q', (c) therefore you can use free field theory, (d) the computation is hence extremely similar to computations that appear in superstring theory in comparing Ramond and Neveu-Schwarz sectors, (e) you essentially just have to compute an eta invariant of a rather trivial one-dimensional operator.

(2) In the untwisted sector, perturbation theory is not straightforward. However, in this case the non-zero modes do not make any important contribution (there is no eta invariant as in the previous case) and hence you lose nothing essential if you think of the $\mathbf{S}^1$ as very small, drop the non-zero modes, and reduce to $0 + 1$ dimensions. We are hence in the realm of classical mathematics. Using whatever knowledge you have of quantum mechanics, $\mathbf{L}^2$ cohomology, singularity theory, or whatever, determine the spectrum of zero energy states of this $0 + 1$ dimensional problem.

Now combine the answers in (1) and (2) and compare to the cohomology of the projective hypersurface.