Homework, May 9

These exercises are concerned with twisted versions of $N=2$ supersymmetric theories in two dimensions. Recall that in such a twisted theory one considers only operators that commute with a Q such that $Q = Q_+ + Q_-$ or $Q = \overline{Q}_+ + Q_-$ (or the hermitian conjugate of one of these). If one works on a flat $\mathbf{R}^2$, Q commutes with $S+S'$ where S is the generator of rotations of $\mathbf{R}^2$ and S' is a sum or difference of left and right moving R-symmetries.

(1) Consider a theory of chiral superfields Φ_i with some polynomial superpotential $W(\Phi_i)$. We assume that W has only finitely many critical points (perhaps degenerate).

The Lagrangian is

$$L = \int d^2x d^4\theta \overline{\Phi}_i \Phi_i + \left(\int d^2x d^2\theta W(\Phi) + c.c. \right). \tag{1}$$

We recall that this theory can be obtained by dimensional reduction of a four-dimensional theory with coordinates $x^1, x^2, \ldots, x^4$; the reduction to two dimensions is made by taking fields to be independent of x^3, x^4. Let S be the generator of rotations of the $1-2$ plane and S' the generator of rotations of the $3-4$ plane. S' is a symmetry for any W. Twist by S' to get a topological field theory in which the Q-invariant operators are chiral superfields (plus their q-form descendants for $q=1,2$); this theory can be defined for any W as S' is a symmetry for any W.

Let $\mathcal{O}_i$ be chiral superfields (note that one can take these to be polynomial functions of the Φ_i only, without derivatives) and compute in genus zero the path integral

$$\langle \prod_{i=1}^{s} \mathcal{O}_i(x_i) \rangle. \tag{2}$$

As a preliminary reduction, note that by using the chiral ring you can reduce to the case $s=1$.

To compute the path integral in genus zero, the main step is to reduce to a finite dimensional integral by throwing away the non-zero modes of all fields. If you don't know how to do or interpret the resulting finite dimensional integral, practice with the case of only one chiral superfield.

(2) Consider first of all an $N=2$ gauge theory in two dimensions with gauge group G. The curvature is the twisted chiral superfield $\Sigma = \sigma + \ldots$. This theory also can come by dimensional reduction from four dimensions. In four dimensions there is an R symmetry S''. Upon reduction to two dimensions, we made a twisted topological theory in which

rotations were generated by $S + S''$ and the topological symmetry is $\overline{Q}_+ + Q_-$; observables in this theory are twisted chiral superfields (and not chiral superfields, as in the previous problem).

We showed the following:

(a) Invariant polynomials in σ (and their q-form descendants) gave the topological observables.

(b) Such a polynomial P corresponds to a cohomology class w_P on the moduli space of flat connections.

(c) Correlation functions are given by integrals over moduli space of a product of such cohomology classes. (It is important to consider the q-form descendants since otherwise everything vanishes, a statement first conjectured by Newstead for $G = SU(2)$.)

Now modify this by including chiral superfields Φ_i in some representation of G. Note that (in contrast to problem (1) where we made a twist that allows an arbitrary superpotential) here we are making the twist by $S + S''$ that forbids a generic superpotential.[1] Because of this, take the superpotential to be zero.

There is no change in (a) and (b). However, (c) will be modified. The correlation functions of the operators $P^{(q)}$ (integrated over q-cycles in Σ) in the twisted theory with the chiral superfields will be the integral over the moduli space of flat connections of the same product of cohomology classes that we had before times a new class that depends on the chiral superfields. Can you identify this class?

[1] If one wants to include a superpotential, (i) it must have weight 2 under some $\mathbf{C}^*$ action on the Φ_i, (ii) one must add to S'' the generator of this $\mathbf{C}^*$, thus modifying how the geometrical R symmetry acts on the chiral superfields.