Lectures on quantization of gauge theories.

Lecture 1.

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In gauge theory the space of field is the space of connections (with additional data) considered modulo gauge transformation. Geometry of this space is extremely nontrivial. The idea of BRST approach to construction of quantum gauge theory is rougly as follows. One first quantizes a larger but simpler linear space (this linear space contains the space of all connections), and then takes invariants of the gauge group in the resulting Hilbert space. In functional integral approach to quantization one also replaces the integral over the space of connections modulo gauge transformation by a certain integral over this huge linear space. Both procedures are not quite straightforward: the first one leads to the notion of semiinfinite cohomology; the second one involves equivariant cohomology of the gauge group.

We start with a simple example which does not belong to the gauge theory but is in certain aspects analogous to some gauge invariant problems.

1.Theory of free vector boson. A field is a map $A : V \to V$. We can and will consider A as a 1-form on V. The Lagrangian is then given by the formula:

$$\mathcal{L}(A) = \frac{1}{2} dA \wedge *dA - \frac{m^2}{2} A \wedge *A$$

The second term is not invariant under the gauge transformations $A \to A + df$ thus it is not a gauge theory if $m \neq 0$.

Let us write A in components, $A = \sum_{i=0}^{d-1} A_i dx_i$. The classical equations of motion are:

$$\Box A_i + \frac{\partial}{\partial x_i} * d * A = m^2 A_i$$

As usual it is convenient to describe the space of solutions in Fourier coordinates. Let $\tilde{A}$ be the Fourier transform of A. Then the equations of motion read as:

$$(p^2 + m^2)\tilde{A}(p) = (\tilde{A}(p), p)p$$

For $m \neq 0$ this is equivalent to saying that $\mathrm{supp}(\tilde{A}) \subset \mathcal{O}_m = \{p \,|\, p^2 + m^2 = 0\}$ and $(\tilde{A}(p), p) = 0$. If $m = 0$ then the space of solutions is invariant under the gauge tranformations $\tilde{A} \to \tilde{A} + f(p)p$, and the space of solutions is generated by $\tilde{A}$ such that $\mathrm{supp}\tilde{A} \subset \mathcal{O}_0$, $(\tilde{A}(p), p) = 0$ under the gauge transformations. Thus solutions modulo gauge are identified with the space of sections of a bundle on $\mathcal{O}_0$ whose fiber is $p^\perp/\mathbb{R}p$ (see the lecture 3 from the fall semester).

The closed 2-form on the space of solutions is given by

$$\langle A, B \rangle = \int_{\mathcal{O}_m^+} (\tilde{A}(p)\tilde{B}(-p) - \tilde{A}(-p)\tilde{B}(p)) d\mu$$

where μ is the invariant measure on $\mathcal{O}_m^+$. This form is non-degenerate if $m \neq 0$; it is pull-back of a non-degenerate 2-form on the space of solutions modulo gauge transformations if $m = 0$.

We next recall the decomposition $V = \mathbb{R} \times \mathbb{R}^{d-1}$, and view $\mathrm{Maps}(V, V)$ as the space of paths on the manifold $\mathrm{Maps}(\mathbb{R}^{d-1}, V)$.

The value of $\mathcal{L}(A)$ at a point $x \in \{t\} \times \mathbb{R}^{d-1}$ depends only on $A_{space} = A|_{\{t\}\times\mathbb{R}^{d-1}}$ and $\dot{A}_{space} = \dot{A}|_{\{t\}\times\mathbb{R}^{d-1}}$ (where $\dot{A} = \frac{\partial A}{\partial x_0}$).

Let $\mathcal{L}_{space}$ be a functional on the tangent bundle $T(\mathrm{Maps}(\mathbb{R}^{d-1}, V))$ with values in functions on $\mathbb{R}^{d-1}$ such that $\mathcal{L}(A)|_{\{t\}\times\mathbb{R}^{d-1}} = \mathcal{L}_{space}(A_{space}, \dot{A}_{space})$.

We put $L = \int\limits_{\mathbb{R}^{d-1}} \mathcal{L}_{space}$.

Then L is quadratic-linear in fibers of the tangent bundle, but the corresponding quadratic form is degenerate: the Lagrangian does not depend on $\dot{A}_0$.

To exploit the analogy with classical mechanics further let us briefly recall the corresponding classical formalism.

1.1. Digression in classical mechanics. Assume that M is a manifold, and $\mathcal{T}$ is a quotient of the tangent bundle to M; let $p : TM \to \mathcal{T}$ stand for the projection. Let L be a function on $\mathcal{T}$ which is assumed to be nondegenerate quadratic-linear in fibers. We consider the Euler-Lagrange equation with Lagrangian $p^*(L)$.

As usual Lagrangian defines a map $P : TM \to T^*M$. The only difference with the standard case is that P is no longer an isomorphism; its image is the subbundle $\mathcal{T}\check{} = Ker(p)^\perp \subset T^*M$.

One can define the Hamiltonian function H on $\mathcal{T}\check{}$ as the Legendre transform of the function L on $\mathcal{T}$.

For a path $\gamma : \mathbb{R} \to M$ let $T\gamma : \mathbb{R} \to TM$ be its canonical lifting to the tangent bundle.

A map $\tilde{\gamma} : \mathbb{R} \to \mathcal{T}\check{}$ will be called a classical trajectory if $\tilde{\gamma} = P \circ T\gamma$ where γ is a solution of the Euler-Lagrange equations.

Note that the symplectic form restricted to $\mathcal{T}\check{}$ is degenerate; its kernel is an integrable distribution of dimension $\dim(M) - \mathrm{rank}(\mathcal{T})$ which we denote by $\mathcal{K}$. Let us define the submanifold $X = \{x \in \mathcal{T}\check{} : d_x H|_{\mathcal{K}_x} = 0\}$.

We have a correctly defined section of $T(\mathcal{T}\check{})/\mathcal{K}|_X$ which we denote by ξ_H. (It is characterized by the standard formula $\omega(\xi_H, v) = \langle dH, v\rangle$ for all $v \in T(\mathcal{T}\check{})$.)

Now it is not hard to see that a path $\tilde{\gamma} : \mathbb{R} \to \mathcal{T}\check{}$ is a classical trajectory iff $\tilde{\gamma}(t) \in X$ for all t and $\partial\tilde{\gamma}/\partial t = \xi_H \mod \mathcal{K}$.

Assume for simplicity that X is smooth and $\dim(TX_x \cap \mathcal{K}_x)$ is constant. One can check that ξ_H lies actually in the subbundle $TX/(TX \cap \mathcal{K}) \subset T(\mathcal{T}\check{})/\mathcal{K}|_X$.

To avoid irrelevant notational complications let us suppose also that the foliation is split, i.e. we have a decomposition $X = Y \times Z$ such that $\{y\} \times Z$ for $y \in Y$ is a leaf of the foliation. The section ξ_H defines a vector field on Y, and a general classical trajectory has the form $\tilde{\gamma} = \tilde{\gamma}_1 \times \tilde{\gamma}_2$ where $\tilde{\gamma}_1$ is a trajectory of the vector field (defined by) ξ_H and $\tilde{\gamma}_2 : \mathbb{R} \to Z$ is arbitrary.

Thus the space of solutions of the Euler-Lagrange equation is identified with $Y \times \mathrm{Maps}(\mathbb{R}, Z)$.

1.2. Let us return to our situation. So $M = \text{Maps}(\mathbb{R}^{d-1}, V)$; the cotangent bundle $T^*(M)$ is identified with $M \times \Gamma(\Omega^1(V)|_{\mathbb{R}^{d-1}})$.

The kernel of p is generated by $\dot{A}_0 \in T(\text{Maps}(\mathbb{R}^{d-1}, V))$; thus $\mathcal{T}\check{} \cong M \times \Omega^1(\mathbb{R}^{d-1})$.

The Hamiltonian h is a functional on $T^*(M)$ with values in functions on $\mathbb{R}^{d-1}$. It is defined by

$$h(A_{space}, E) = E_k \dot{A}_k - \mathcal{L}_{space}(A_{space}, \dot{A}_{space})$$

where $E_k = \frac{\partial \mathcal{L}}{\partial \dot{A}_k} = \dot{A}_k - \frac{\partial A_0}{\partial x_k}$.

Then we put $H = \int\limits_{\mathbb{R}^{d-1}} h$.

We have

$$h(A_{space}, E) = \frac{1}{2}(E_k \cdot (\dot{A}_k + \frac{\partial A_0}{\partial x_k}) - (d\vec{A})^2 - m^2(A)^2) = \frac{1}{2}((E)^2 - (d\vec{A})^2 - m^2(\vec{A})^2 - A_0^2) + E \wedge * dA_0$$

where $E = \sum\limits_{k=1}^{d-1} E_k dx_k$ and $\vec{A} = \sum\limits_{k=1}^{d-1} A_k dx_k$ are viewed as 1-forms on $\mathbb{R}^{d-1}$.

The distribution $\mathcal{K}$ is generated by $A_0 \in TM \subset T(M \times \Omega^1(\mathbb{R}^{d-1}))$. Thus X is given by the equation $\frac{\partial H}{\partial A_0} = 0$, i.e. $\frac{\delta h}{\delta A_0} = 0$ where δ is the variational derivative. Integrating by parts the last summand in the formula for h we get:

$$-\frac{\delta h}{\delta A_0} = d * E + m^2 A_0$$

So X is a subvariety in $\text{Maps}(\mathbb{R}^{d-1}, V) \times \Omega^1(\mathbb{R}^{d-1})$ defined by the equation

$$d * E + m^2 A_0 = 0$$

If $m \neq 0$ then X is symplectic; the space of classical solutions is identified with X. Since A_0 is determined from the equation we see that $X \cong T^*(\Omega^1(\mathbb{R}^{d-1}))$ in this case.

If $m = 0$ then X is no longer symplectic; we get a constraint on $(\vec{A}, E)$ but A_0 can be arbitrary. In terms of 1.1 we have $X = Y \times Z$ where $Y \subset T^*(\Omega^1(\mathbb{R}^{d-1}))$ is defined by $d * E = 0$, and $Z = \Omega^0(\mathbb{R}^{d-1})$. The space of classical solutions is $Y \times \text{Maps}(\mathbb{R}, Z)$.

We finish the discussion of the free vector QFT with the following remark.

Suppose we want to add a "small" term to the above Lagrangian and study the resulting theory perturbatively. The corresponding propagator in momentum space is an $\text{End}(V)$ valued function on V given by $G(p) = (D_0(p) + (p^2 + m^2)Id)^{-1}$ where $D_0(p)v = -(p, v)p$. We see that $G(p)$ does not tend to 0 when $p \to \infty$ (for example $G(p)p = m^{-2}p$). Thus the methods of perturbation theory do not work: all Feynman integrals diverge, and we have no way to renormalize them. This is an indication of a well-known fact that a non-free non-gauge vector QFT does not exist.

This also illustrates one of the difficulties in quantizing gauge theory. Let us consider a $U(1)$-gauge theory as an example. Thus fields are pairs (A, ϕ) where A is a connection on the (trivial) $U(1)$-bundle, and ϕ is a section of the corresponding complex line bundle. The Lagrangian is:

$$\mathcal{L}(A, \phi) = F_A \wedge * F_A + D_A \phi \wedge * D_A \phi + W(\phi)$$

where $W(\phi) = (|\phi|^2 - a^2)^2$. For $L = \int \mathcal{L}$ to make sense we need the condition $|\phi|^2 \to a^2$ for $v \to \infty$.

Let us choose a gauge (i.e. a trivialization of the $U(1)$-bundle) in such a way that ϕ is real and $\phi \to a$ on infinity. We can write $\phi = a + \sigma$ where σ is a real-valued rapidly decreasing function on V. We have:

$$\mathcal{L}(A, \sigma) = dA \wedge *dA + a^2 A \wedge *A + d\sigma \wedge *d\sigma + 4a^2 \sigma \cdot *\sigma + \cdots$$

where $\cdots$ stand for terms of order ≥ 3 in A and σ.

Thus the sum of quadratic terms is Lagrangian for the sum of 2 noninteracting free massive vector bosons; a naive approach to the quantization problem would be to study gauge theory as a perturbation of that free QFT. As was explained this method does not have a chance to work.

2. Pure gauge theory. Let G be a compact Lie group, $\mathcal{T} \to V$ be a principle G-bundle on V. Let $\mathfrak{g}_{\mathcal{T}}$ be the adjoint vector bundle of $\mathcal{T}$. We fix a positive invariant scalar product on the Lie algebra of G, and will not distinguish between the Lie algebra and its dual.

The Lagrangian of the theory is $\mathcal{L}(A) = \frac{1}{2} F_A \wedge *F_A$.

The classical field equation is then $d_A * F_A = 0$.

Let us rewrite the theory in Hamiltonian form. Thus we fix a decomposition $V = \mathbb{R} \times \overline{V}$. Let $\mathcal{T}_{space}(t)$ be the restriction of $\mathcal{T}$ to $\{t\} \times \overline{V}$, and $A_{space}(t) = A|_{\{t\} \times \overline{V}}$ be the connection on $\mathcal{T}_{space}(t)$. Introduce notations: $B = B(t) = F_A|_{\{t\} \times \overline{V}} = F_{A_{space}} \in \Omega^2(\{t\} \times \overline{V}, \mathfrak{g}_{\overline{\mathcal{T}}})$ (the magnetic field), and $E = i_{\partial_t}(F_A)|_{\{t\} \times \overline{V}} \in \Omega^1(\{t\} \times \overline{V}, \mathfrak{g}_{\overline{\mathcal{T}}})$ (the electric field).

Let $\varkappa$ be the projection $\varkappa : V \to \overline{V}$. Let us fix an isomorphism $\mathcal{T} \cong \varkappa^*(\overline{\mathcal{T}})$ for a G-bundle $\overline{\mathcal{T}}$ on $\overline{V}$. Then $\mathcal{T}|_{\mathbb{R} \times \{x\}}$ is trivialized for all $x \in \overline{V}$. So we can write any connection on $\mathcal{T}$ as $A = A_{space}(t) + A_0 dx_0$ where $A_0 \in \Omega^0(\mathfrak{g}_{\mathcal{T}})$; thus a connection on $\mathcal{T}$ is the same as a path in M, where M is the space of pairs: a connection on $\overline{\mathcal{T}}$, and a section of the adjoint bundle of $\overline{\mathcal{T}}$.

We have:

$$E(t) = d_{A_{space}(t)} A_0 + \dot{A}_{space}(t)$$

$$\mathcal{L}(A) = \frac{1}{2}(F_A)^2 = \frac{1}{2}((B)^2 + (E)^2)$$

Exactly as above we see that Lagrangian is nondegenerate in $\dot{A}_{space}$, but does not depend on $\dot{A}_0$. Thus we can define the Hamiltonian functional h on $M \times \Omega^1(\overline{V}, \mathfrak{g}_{\mathcal{T}})$ by:

$$\mathcal{L}(A) = E \wedge *\dot{A}_{space} - h(A_{space}, E)$$

(Notice that $E = \frac{\partial \mathcal{L}}{\partial \dot{A}_{space}}$). We have:

$$h(A_{space}, E) = \frac{1}{2}(E^2 - B^2) - E \wedge *d_{A_{space}} A_0$$

In terms of 1.1 we have: X is the subspace in $M \times \Omega^1(\overline{V}, \mathfrak{g}_{\mathcal{T}})$ given by

$$\frac{\delta h}{\delta A_0} = 0$$

where δ is the variational derivative. In the formula for h we can integrate by parts to get:

$$\frac{\delta h}{\delta A_0} = d_{A_{space}}(*E)$$

So we see that $Y \subset T^*(Conn)$ where $Conn$ stands for the space of connections on $\overline{\mathfrak{T}}$, and Y is defined by

$$d_{A_{space}}(*E) = 0$$

we also have $Z = \Omega^1(\mathfrak{g}_{\overline{\mathfrak{T}}})$. So the space of classical solutions is $Y \times \mathrm{Maps}(\mathbb{R}, Z)$.

Now we notice that any connection is gauge-eqiuvalent to a one with $A_0 = 0$, i.e. a one which is constant on every line $\mathbb{R} \times \{x\}$ in the above trivialization. (Such a connection is called a connection with temporal gauge).

A connection with a temporal gauge is the same as a map from $\mathbb{R}$ to $Conn$.

Let us restrict our Lagrangian to the space of connections with temporal gauge, which is identified with $\mathrm{Maps}(\mathbb{R}, Conn)$. Then we obtain a functional on the tangent bundle $T(Conn)$ which is readily seen to be quadratic nondegenerate in fibers. Hence the space Y of critical points of Lagrangian on the space of temporaly gauged connections is the (co)tangent bundle to $Conn$, which is identified with the space of pairs (A_{space}, E) where A_{space} is a connection on $\overline{\mathfrak{T}}$ and $E \in \Omega^1(\overline{\mathfrak{T}})$.

Obviously the space X of temporaly gauged connections which are classical solutions of the original problem (i.e. critical points of $\mathfrak{L}$ as a function on the space of all connections) is a subspace of Y (we have to impose the condition that variation in A_0 vanishes). Moreover we have shown that X is a subspace of Y defined by the equation $d_A * E = 0$.

Remark. A naive analogy with the finite dimensional case may be misleading, as is illustrated by the following example.

Let U be a finite dimensional manifold with an action of a Lie group G; let $Z \subset U$ be a submanifold such that $G \cdot Z = U$. Assume that L is a G-invariant function on U. Then a point $y \in Z$ is a critical point of $L|_Z$ iff it is a critical point of L as a function on U.

However if we take U to be the space of connections, take Z to be the space of connections with temporal gauge, and $L = \int\limits_V \mathfrak{L}$ to be the action then this statement is no longer true, as we saw few lines above.

This has the following reason.

A classical field (a connection in our case) is a solution of the classical equations of motion iff it is a stationary point of the action with respect to variations with *compact support.*

Suppose that A is a connection with temporal gauge which is stationary with respect to such variations in the class of connections with temporal gauge. Consider a compactly supported variation in A_0-direction. We can apply to it an infinitesimal gauge transformation so that the resulting variation will stay in the class of connections with temporal gauge. However we can not guarantee that this new variation has compact support, so the action can actually vary infinitesimaly.

Proposition. *Let $Y = \{(A_{space}, E)\} \cong T^*(Conn)$ and $X \subset Y$ be as above. Then the map $m : Y \to \Omega^{d-1}(\mathfrak{g}_{\overline{\mathfrak{T}}})$ given by $(A_{space}, E) \to d_{A_{space}} * E$ is the moment map for the action of* *the gauge group $\overline{\mathcal{G}} = Aut(\overline{\mathcal{T}})$ on Y. Thus X is the 0-set of the moment map, and the space of extremals of $\mathcal{L}$ modulo gauge transformations is symplectic reduction of Y with respect to the action of the gauge group.*

Proof. It remains only to check that m is the moment map.

Let us take $x \in \Omega^0(\mathfrak{g}_{\overline{\mathcal{T}}}) = \mathrm{Lie}(\overline{\mathcal{G}})$ and a constant vector field $w = (A', E')$ on Y for $A', E' \in \Omega^1(\mathfrak{g}_{\overline{\mathcal{T}}})$. Let j denote the action of $\mathrm{Lie}(\overline{\mathcal{G}})$ on Y. Then we have to check the equality:

$$d \langle x, d_A * E \rangle (w) = \omega(j(x), w)$$

where first d in the left-hand side is external derivative of a function on Y, and ω is the standard symplectic form on Y.

Now j is given by $j(x)|_{(A,E)} = (d_A x, [x, E])$. So the equality reads as:

$$\langle x, d_A * E' + [A', *E] \rangle = \langle d_A x, *E' \rangle + \langle [x, E], *A' \rangle$$

which is true by integration by parts.