Real versus complex

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We want to explain how to go back and forth between two modes of writing:

real: one considers real varieties, functions on them (possibly complex valued: $F = f + ig$), maps between them, ...

complex: one considers complex varieties, and *only holomorphic functions* on them, complex analytic maps between them, ...

A real analytic variety X can be complexified to a complex variety $X_{\mathbb{C}}$. In fact, only the germ of $X_{\mathbb{C}}$ around X is defined. One can think of X as being given by $X_{\mathbb{C}}$, plus some auxiliary data. For some purposes, this auxiliary data is not needed, and computations are better done in "complex mode", on $X_{\mathbb{C}}$.

To recover X from $X_{\mathbb{C}}$, one could give

(A) The antiholomorphic involution "complex conjugation" on $X_{\mathbb{C}}$, with fixed point $X \subset X_{\mathbb{C}}$ — or

(B) the data of which (holomorphic) functions are real (on X).

There is not much difference between complex valued functions on X, and (holomorphic) functions on $X_{\mathbb{C}}$, and similarly for forms. A local coordinate system on X gives a (holomorphic) local coordinate system on $X_{\mathbb{C}}$.

If α is a form of maximal degree on X, viewed as an (holomorphic) form on $X_{\mathbb{C}}$, integration of α becomes, on $X_{\mathbb{C}}$, integration on a cycle, and we need to know this cycle only up to suitable homologies.

Example. To compute $\int e^{-x^2/2+iax}dx$, it is better to go to the holomorphic picture, where the integration cycle $\mathbb{R} \subset \mathbb{C}$ is needed up to homologies given by 2-chains on which the integrand decays fast. In this setting, the completing the square change of variables $y = (x - ia)$ makes good sense, giving the answer $\sqrt{2\pi}\, e^{-a^2/2}$.

Similarly, a $(0|n)$ dimensional real variety$X$ can be complexified. Here, (holomorphic) functions on $X_{\mathbb{C}}$ are the same as complex valued functions on X. The real X is recovered from $X_{\mathbb{C}}$ by giving (A') an antilinear involutuion σ of $X_{\mathbb{C}}$: $\overline{X_{\mathbb{C}}} \to X_{\mathbb{C}}$ or (B') which functions are real. Here, to integrate a density is a purely algebraic story: only the density on $X_{\mathbb{C}}$ matter, the real form X is irrelevant.

That integration is a purely algebraic story, making the real structure irrelevant, can happen also for ordinary varieties. Example: let V be a real vector space, Q a quadratic form, P a polynomial and dv a volume form. Then, up to an annoying square root, $\int e^{-Q}P.dv$ depends only on $e^{-Q}Pdv$ on the complexification $V_{\mathbb{C}}$.

In dimension $(n|m)$, Bernstein introduced the notion of cs-spaces. I like to think of a cs space as being a complex super space $X_{\mathbb{C}}$, plus the data of an "integration" cycle C on the reduced space. In many local computations only $X_{\mathbb{C}}$ matters. If one wants to integrate densities (holomorphic sections of $\mathrm{Ber}(\Omega^1)$), the cycle C matters only up to suitable homologies. Sometimes one wants to germify along C (which is the real locus for a real structure on the reduced space). This gives Bernstein definition, where the cs space is C, with the sheaf of rings $\mathcal{O}_{X_{\mathbb{C}}}$ restricted to C.

If X has complex structure J, $X_{\mathbb{C}}$ is $X \times \bar{X}$, with X sitting in $X \times \bar{X}$ as the diagonal. As before, this is not much difference between a (complex valued) function on X and an (holomorphic) function on $X_{\mathbb{C}} = X \times \bar{X}$, defined near the diagonal. Holomorphic functions on X become functions $f(x, y)$ on $X_{\mathbb{C}} = X \times \bar{X}$ depending only on x. In local coordinates: an holomorphic coordinate system on X, (z_i), gives one on $X_{\mathbb{C}} = X \times \bar{X}$: $(z_i, \bar{z}_i)$. A function on X is then viewed as an (holomorphic) function of z_i and $\bar{z}_i$. It is an holomorphic function on X if it is an (holomorphic) function not depending on the $\bar{z}_i$.

For functions spaces also it can be convenient to work in complexified spaces, with reality conditions coming as an afterthought, if needed. The complexification of $\underline{\mathrm{Hom}}(X, Y)$ is the space $\underline{\mathrm{Hom}}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ of (holomorphic) maps f from $X_{\mathbb{C}}$ to $Y_{\mathbb{C}}$. One requires f to be defined only in a suitable neighborhood of $X \subset X_{\mathbb{C}}$, and that $f(X)$ is close to $Y \subset Y_{\mathbb{C}}$.

Example. (i) If v is a section of the complexified tangent bundle of X, with X compact and X and v real analytic, $\exp(tv)$ (t small) belongs to $\mathrm{Diff}(X)_{\mathbb{C}}$.

(ii) (After G. Segal), Let S^1 be the unit circle in $\mathbb{C}$. A point of $\mathrm{Diff}(S^1)_{\mathbb{C}}$ is an holomorphic map from an annulus $1-\eta < r < 1+\eta$ to an annulus $1-\varepsilon < r < 1+\varepsilon$. It should have a restriction to S^1 close to a diffeomorphism of S^1.

Composition is only partially defined. If f maps S^1 inside S^1, one can attach to f the "annulus" $[f]$ between $f(S^1)$ and S^1 in $\mathbb{C}$. Both boundary components of $[f]$ are parametrized by S^1, using the identity, resp. f. To composition corresponds gluing of annuli, and this is how a good substitute to $\mathrm{Diff}(S^1)_{\mathbb{C}}$ is the monoid of isomorphism classes of annuli with parametrized boundaries. In conformal field theory, one meets complex (projective) representations of real semi-group underlying the complex semi-group $\mathrm{Diff}(S^1)_{\mathbb{C}}$. At the Lie algebra level, if W is the Lie algebra of vector fields on S^1, with complexification $W_{\mathbb{C}}$, a complex (projective) representation of the real Lie algebra $r(W_{\mathbb{C}})$ underlying $W_{\mathbb{C}}$ is the same as a complex (projective) representation of $r(W_{\mathbb{C}}) \otimes \mathbb{C} = W_{\mathbb{C}} \otimes_{\mathbb{R}} \mathbb{C} = W_{\mathbb{C}} \times W_{\mathbb{C}}$.

(iii) Take $X = \mathbb{R}^{0|1}$, hence $X_{\mathbb{C}} = \mathbb{C}^{0|1}$. Here $\mathrm{Hom}(X,Y) = Y$, with the sheaf of ring Ω^*, while $\mathrm{Hom}(X_{\mathbb{C}}, Y_{\mathbb{C}})$, taken in the holomorphic world is $Y_{\mathbb{C}}$, with the holomorphic Ω^*, and $(Y,\Omega^*)_{\mathbb{C}} = (Y_{\mathbb{C}}, \Omega^*)$.

In the complex setting, if X is of dimension $(0|1)$, and Y of dimension $(n|0)$, a real structure on Y induces a *cs* structure on $\underline{\mathrm{Hom}}(X,Y)$. Indeed, $\underline{\mathrm{Hom}}(X,Y)$ has a reduced space Y.

The same is not true for X of dimension $(0,2)$. Indeed, take $Y = \mathbb{C}$ and coordinates θ_1, θ_2 on X. A map to Y, i.e. an (holomorphic) function, is written

$$a + \alpha_1\theta_1 + \alpha_2\theta_2 + b\theta_1\theta_2 \ :$$

the space $\underline{\mathrm{Hom}}(X,Y)$ is a $\mathbb{C}^{2|2}$ with coordinates $(a, b, \alpha_1, \alpha_2)$. If we give a real structure on Y, we know what it means for a to be real. To know what it means for b to be real, we need in addition to give a real structure on X.

In practice, this will not matter, for two reasons:

(A) The spaces of externals $\mathcal{S} \subset \mathrm{Hom}(X, Y)$ we will consider will be such that $\mathcal{S}_{\mathrm{red}}$ embeds into $\mathrm{Hom}(X_{\mathrm{red}}, Y)$ (and is a space of extremal there), so that a real structure on Y gives one on $\mathcal{S}_{\mathrm{red}}$: we get a cs structure on $\mathcal{S}$.

(B) In path integrals, integrals on fields playing the role of "c" above can be handled algebraically.

In the real setting, if we are to consider $\mathrm{Hom}(X, Y)$ with Y having a complex structure, it is usually convenient to work on $\mathrm{Hom}(X, Y)_{\mathbb{C}} = \mathrm{Hom}(X_{\mathbb{C}}, Y_{\mathbb{C}})$.

The fact that Y was complex gives us two transversal foliations on $Y_{\mathbb{C}} = Y \times \bar{Y}$. In local oordinates: for (z_i) local coordinates on Y, we get local coordinates z_i, $\bar{z}_i$ on $Y_{\mathbb{C}}$. It follows that $f \in \mathrm{Hom}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ is given by the $f_i = z_i(f)$ and $\bar{f}_i = \bar{z}_i(f)$, where the f_i and $\bar{f}_i$ are (independent) holomorphic functions on $X_{\mathbb{C}}$ (or complex valued functions on X). If on $X_{\mathbb{C}}$ we have a real structure X, we get the real structure $\mathrm{Hom}(X, Y)$ on $\mathrm{Hom}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ by telling that f is real in f_i and $\bar{f}_i$ are complex conjugate (on X).