Spinors: tables

P. Deligne

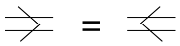
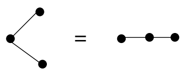
1. Complex story.

The Dynkin diagram of the group $\mathrm{Spin}(n)$ is:

$n = 2r$ $\qquad\qquad$ $n = 2r + 1$

where r is the rank. The spin or semi-spin representations are the fundamental representations corresponding to the vertex, or vertices, at the right of the diagram. If one restricts the spin representation of $\mathrm{Spin}(2r+1)$ to $\mathrm{Spin}(2r)$, one obtains the sum of the two semi-spin representations of $\mathrm{Spin}(2r)$. Each of the two semi-spin representations of $\mathrm{Spin}(2r)$ restricts as the spin representation of $\mathrm{Spin}(2r-1)$.

The Dynkin diagram makes clear the low rank exceptional isomorphisms. We list in the next table $n = 2, 3, 4, 5$ or 6, the Dynkin diagram, a group G to which $\mathrm{Spin}(n)$ is isomorphic, the description of spin or semi-spin representations as representations of G, and the description of the defining representation of $\mathrm{SO}(n) = \mathrm{Spin}(n)/(\mathbb{Z}/(2))$ as representation of G.

n	diagram	G	(semi)spin	defining orthogonal
2	none	$\mathbb{G}_m$	characters z, z^{-1}	$z^2 \oplus z^{-2}$
3	•	$\mathrm{SL}(2)$	defining V	adjoint $= \mathrm{Sym}^2(V)$
4	:	$\mathrm{SL}(2) \times \mathrm{SL}(2)$	defining V_1, V_2	$V_1 \otimes V_2$
5		$\mathrm{Sp}(4)$	defining V	$\overset{2}{\wedge} V$
6		$\mathrm{SL}(4)$	defining V, and V^*	$\overset{2}{\wedge} V$

The symmetric bilinear form on the defining orthogonal representation is given by

$n=3$: Killing form $\mathrm{Tr}(adx\, ady)$, or

$\left\langle v^2, w^2\right\rangle = \psi(v,w)^2$, for ψ symplectic (= volume) form of V.

$n=4$: $\langle v_1 \otimes v_2,\, w_1 \otimes w_2\rangle = \psi(v_1,w_1)\psi_2(v_2,w_2)$.

$n=5$: $\langle v_1 \wedge v_2,\, w_1 \wedge w_2\rangle = \psi(v_1,w_1)\psi(v_2,w_2) - \psi(v_1,w_2)\psi(v_2,w_1)$.

$n=6$: $\langle v_1 \wedge v_2,\, w_1 \wedge w_2\rangle = v_1 \wedge v_2 \wedge w_1 \wedge w_2 \in \overset{4}{\wedge}\, V = \mathbb{C}$.

For n odd, the spin representation S of $\mathrm{Spin}(n)$ is orhtogonal or symplectic. There is a morphism of representations from $S \otimes S$ to the defining representation V; it is symmetric or antisymmetric.

For n even, the semi-spin representations S^+ and S^- are orthogonal, symplectic, or dual of each other. There is either a morphism of representations $S^+ \otimes S^- \to V$, or morphisms $S^+ \otimes S^+ \to V$ and $S^- \otimes S^- \to V$, symmetric or antisymmetric. Which is the case depends only on n mod 8. The next table gives n mod 8, the kind of bilinear form on spinors, and the kind of bilinear forms (spinors) $\otimes$ (spinors) $\to V$.

n mod 8	forms on spinors	symmetry of spinors, spinors $\to V$
1	orthogonal	symmetric
2	S^+ dual to S^-	symmetric (on S^+, and on S^-)
3	symplectic	symmetric
4	S^+ and S^- sympletic	$S^+ \otimes S^- \to V$
5	symplectic	antisymmetric
6	S^+ dual to S^-	antisymmetric (on S^+, and on S^-)
7	orthogonal	antisymmetric
8	S^+ and S^- orthogonal	$S^+ \otimes S^- \to V$

2. Real story.

An irreducible representation (over $\mathbb{R}$) of $\mathrm{Spin}(p,q)$ is said to be spinorial if, after extensioin of scalars to $\mathbb{C}$, it becomes a sum of spinorial or semi-spinorial representations.

If $p-q$ is odd (equivalent to $p+q$ odd), there is a unique spinorial irreducible representation. It is either real or quaternionic. if $p+q$ is even, either there is only one such representation, in which case it is complex, giving over $\mathbb{C}$ the sum of the two semi-spinorial representations, or there are two of them, both real or both quaternionic. If real, they remain irreducible over $\mathbb{C}$: S^+ and S^- have a real form. If they are quaternionic, after extension of scalars to $\mathbb{C}$, one becomes $2S^+$, the other becomes $2S^-$. In which case we are depends only on the signature $p-q$ mod 8. Table:

$p-q$	real, complex or quaternionic
1 or 7	$\mathbb{R}$
2 or 6	$\mathbb{C}$
3 or 5	$\mathbb{H}$
4	$\mathbb{H}$, $\mathbb{H}$
8	$\mathbb{R}$, $\mathbb{R}$

3. Minkowski signature.

In Minkowski signature, $(+,-,-,\dots)$, and with a positive light cone chosen on V, if S is an irreducible spinorial representation, there is up to a real factor a unique symmetric bilinear maps B: $S\otimes S\to V$. It can be normalized so that $Q(s):=\frac{1}{2}B(s,s)$ is with values in the closed positive cone. Such a Q is now unique up to a positive real factor.

If S is complex, B is the real part of an hermitian bilinear form with values in $V_{\mathbb{C}}$: if J is the complex structure, Q is invariant by the J, as well as by the circle group of $\exp(\theta J)$. After extension of scalars to $\mathbb{C}$, S becomes the sum of the two semi-spinorial representations, and B corresponds to a pairing $S^+\otimes S^-\to V$.

If S is quaternionic, the quaternions with absolute value one preserve Q. After extension of scalors to $\mathbb{C}$, S becomes the sum $S_0\otimes T$ of two copies of a spinorial or semi-spinorial representation S_0, and B the tensor product of an antisymmetric pairing $S_0\otimes S_0\to V_{\mathbb{C}}$ with a scalar alternating form $T\otimes T\to\mathbb{C}$. The quaternions become $\mathrm{End}(T)$.

The next table gives the nature of irreducible spinorial representations, as a function of n mod 8, for $\mathrm{Spin}(1,n-1)$. If $\mathrm{Spin}(1,n-1)$ has two irreducible spinorial representations, they have isomorphic restrictions to $\mathrm{Spin}(1, n-2)$. The last column of the table describes the restriction of an irreducible representation of $\mathrm{Spin}(1, n-1)$ to $\mathrm{Spin}(1, n-2)$. It is either an irreducible spinorial representation, or twice it, or the sum of two distinct spinorial representations. Noted: S, $2S$, $S^+ + S^-$.

n mod 8	nature	restriction to $\mathrm{SO}(1, n-2)$
1	$\mathbb{R}$	S
2	$\mathbb{R}$, $\mathbb{R}$ (S^+ and S^-, in duality)	S
3	$\mathbb{R}$	$S^+ + S^-$
4	$\mathbb{C}$	$2S$
5	$\mathbb{H}$	$2S$
6	$\mathbb{H}$, $\mathbb{H}$ (S^+ and S^-, in duality)	S
7	$\mathbb{H}$	$S^+ + S^-$
8	$\mathbb{C}$	S