MAPPING CLASS GROUP ACTION ON SPIN STRUCTURES

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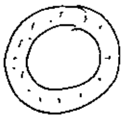
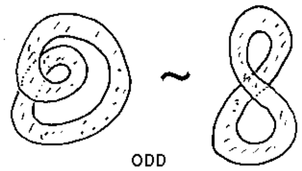
A spin structure on a (punctured, marked) orientable surface is induced by an immersion into $S^2 = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ from the unique spin structure '$\sqrt{dz}$' on S^2. In the case of an annulus, the two spin structures are:



EVEN



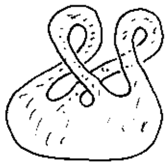
ODD

The one on the left (EVEN) extends to an immersion of the disk, whereas the one on the right (ODD) does not extend. Regularly homotopic immersions correspond to equivalent spin structures, while regularly homotopic images correspond to spin structures which are equivalent up to change of marking. The mapping class group of the surface, by definition, effects this change of marking.

The mapping class group is generated by Dehn twists. Using a "disk with bands" picture, these twists can be visualized as "band slides". The three even spin structures on a torus, and their equivalence up to change of marking, can be represented as:

ODD - EVEN　　EVEN -EVEN　　EVEN - ODD

and the odd spin structure on the torus as:



ODD - ODD

Note that the rules of (mod 2) multiplication

$$\text{ODD} \cdot \text{EVEN} = \text{EVEN} \cdot \text{EVEN} = \text{EVEN} \cdot \text{ODD} = \text{even}$$

$$\text{ODD} \cdot \text{ODD} = \text{odd}$$

correspond to the plumbing together of the annular bands (sliding an ODD band over another ODD band yields an ODD band!) For surfaces of higher genus, decompose as a connected sum of tori and use (mod 2) addition:

$$\text{odd} + \text{odd} = \text{even} = \text{even} + \text{even}$$
$$\text{even} + \text{odd} = \text{odd} = \text{odd} + \text{even}$$

(Check by using band slides that these are the only two classes of spin structure (up to change of marking), and furthermore, that the mapping class group transitively permutes all the even spin structures – and similarly, all the odd spin structures – amongst themselves!)

Algebraically, these pictures show how to associate a (mod 2) quadratic form with each spin structure on a (marked) orientable surface. The mapping class group acts on the first (mod 2) homology group (by intersection-preserving change of basis) and the (mod 2) Arf-invariant (EVEN or ODD) of this quadratic form, distinguishes the two orbits. For non-orientable surfaces, there is an analogous picture for "pin structures" using immersions to RP^2; at the algrebraic level, each pin structure corresponds to a (mod 4) quadratic form and the mapping class group orbits are determined by its (mod 8) Arf-invariant.