SOLUTION OF HOMEWORKS IN STRING THEORY

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1. Solutions to problem sets 1 and 2

1.1. Remarks on notations. In what follows Σ will denote a compact Riemann surface of genus h, usually with a fixed complex structure. We let $\Delta:\Omega^{0,0}(\Sigma)\to\Omega^{1,1}(\Sigma)$ denote the corresponding Laplace operator $\Delta=\partial\overline{\partial}$. Let us now choose a metric g on Σ compatible with the complex structure. Then we denote by $R_g\in\Omega^{1,1}(\Sigma)$ the curvature of the corresponding Levi-Civita connection and by $d\mu_g$ – the corresponding measure ($(1,1)$-form) on Σ. We set $\Omega_\Sigma=\Omega^{1,0}(\Sigma)$.

Assuming again that the metric g as above is chosen we may define an operator $\Delta_g^{-1}:\Omega^{1,1}(\Sigma)\to\mathcal{O}(\Sigma)$. By definition, $\Delta_g^{-1}(w)$ (for any $(1,1)$-form ω) is the unique function on Σ, such that

$$\int\limits_\Sigma \Delta_g^{-1}(w)d\mu_g=0 \quad\text{and}\quad \Delta(\Delta_g^{-1}(\omega))=\omega-\frac{d\mu_g}{\mathrm{vol}(\Sigma)}\cdot\int\limits_\Sigma\omega \tag{1.1}$$

1.2. Problem set 1: Problem 1b.

1.2.1. *Digression on projective connections.* Let Σ and Ω be as in 1.1. Take $c\in\mathbb{C}$. We want to associate to it a canonical torsor $\mathcal{P}_c$ over the line bundle $\Omega_\Sigma^{\otimes 2}$, called the torsor of projective c-connections. In fact, we will construct an $\mathcal{O}$-module, which is an extension of $\mathcal{O}$ by $\Omega^{\otimes 2}$ and $\mathcal{P}_c$ will be the preimage of $c\in\mathbb{C}\subset\mathcal{O}$. The basic property of E is that it is natural. In particular, vector fields on Σ act on E by Lie derivatives.

First a simple observation.

Claim. *For any $i,j,k\in\mathbb{Z}$ there exists a canonical isomorphism*

$$\Omega^{\otimes i}\boxtimes\Omega^{\otimes j}(k\Delta)/\Omega^{\otimes i}\boxtimes\Omega^{\otimes j}((k-1)\Delta)\simeq\Delta_*(\Omega^{\otimes(i+j-k)}) \tag{1.2}$$

Here Δ is the diagonal in $\Sigma\times\Sigma$.

Consider now the sheaf (it's not a sheaf of $\mathcal{O}$-modules) on $\Sigma\times\Sigma$

$$(\Omega\boxtimes\Omega(2\Delta)/\Omega\boxtimes\Omega(-\Delta))^{S_2}$$

Here S_2 denotes the corresponding symmetric group, which acts naturally on $\Omega\boxtimes\Omega(2\Delta)/\Omega\boxtimes\Omega(-\Delta)$.

Let E denote the sheaf-theoretic direct image of this sheaf to Σ with respect to either the first or the second projection (the two direct images are canonically isomorphic, since we took S_2-invariants). It is easy to see that E acquires a canonical structure of an $\mathcal{O}_\Sigma$-module (which, however, will not be used in the sequel). We have canonical exact sequence

$$0 \to \Omega^{\otimes 2} \to E \to \mathcal{O} \to 0 \tag{1.3}$$

We let p denote the above morphism $E \to \mathcal{O}$ and set $\mathcal{P}_c = p^{-1}(c)$.

If z is a local coordinate on an open subset of Σ, the torsor $\mathcal{P}_c$ acquires a canonical section P_z^c given by $(c/2)\frac{dz_1 \boxtimes dz_2}{(z_1-z_2)^2} \in \Omega \boxtimes \Omega(2\Delta)$ (one has to divide c by 2 in this formula, since the residue of $\frac{dz_1 \boxtimes dz_2}{(z_1-z_2)^2}$ on the diagonal in the natural normalization is equal to 2).

Let now $z' = f(z)$ be another local coordinate on the same open subset.

Claim. $P_z^c - P_{z'}^c = -\frac{c}{12}(dz)^{\otimes 2}\{f(z), z\}$, *where* $\{f(z), z\}$ *is the Schwartzian derivative of* f *with respect to* z

$$\{f(z), z\} = \frac{f'''f' - \frac{3}{2}(f'')^2}{(f')^2}$$

1.2.2. *Solution of (b).* Now we are ready to solve part (b) of the problem. We are working with free conformal field theory, whose classical action is

$$S[\phi, g] = \frac{1}{4\pi}\int_\Sigma \frac{1}{2}\partial\phi\overline{\partial}\phi \tag{1.4}$$

Here ϕ is a map from Σ to $\mathbb{R}^c$. We want to make sense of $:(\partial\phi(x))^2:$ as of a quantum field with values in projective c-connections (note that c is precisely the central charge of our theory). This means that for any collection of points $y_1, ..., y_n \in \Sigma$, such that $y_i \neq y_j$ for $i \neq j$ and $y_i \neq x$, and a collection of fields $\phi_1, ..., \phi_n$, the correlation function $\langle :(\partial\phi(x))^2: \phi_1(y_1)...\phi_n(y_n)\rangle$ is a section of the torsor $\mathcal{P}_c$ in the variable x.

The normal ordering $:(\partial\phi(x))^2:$ is defined as follows. Let z be a local complex coordinate on an open subset $U \subset \Sigma$ and let $y_1, ..., y_n, \phi_1, ...\phi_n$ be as above. Assume also that $y_i \notin U$ for any $i = 1, ..., n$. Then we have a section

$$\langle\partial\phi(x)\partial\phi(x')\phi_1(y_1)...\phi_n(y_n)\rangle \in \Gamma(U \times U\backslash\Delta(U), \Omega \boxtimes \Omega) \tag{1.5}$$

The correlation function $\langle :(\partial\phi(x))^2: \phi_1(y_1)...\phi_n(y_n)\rangle$ is defined by

$$\langle :(\partial\phi(x))^2: \phi_1(y_1)...\phi_n(y_n)\rangle =$$
$$\lim_{x'\to x}(\langle\partial\phi(x)\partial\phi(x')\phi_1(y_1)...\phi_n(y_n)\rangle + c\frac{dz \boxtimes dz}{(z(x_1) - z(x_2))^2} + P_z^c\langle\phi_1(y_1)...\phi_n(y_n)\rangle$$

It is now obvious that $\langle :(\partial\phi(x))^2: \phi_1(y_1)...\phi_n(y_n)\rangle$ as a section of $\mathcal{P}_c$ (when $y_1, ...y_n$ are kept fixed) does not depend on the choice of the coordinate z due to the definition of $P^c_z(x)$ and to the fact that $\langle :(\partial\phi(x)\partial\phi(x')\phi_1(y_1)...\phi_n(y_n)\rangle$ is independent of z (as a section of $\Omega\boxtimes\Omega)(2\Delta)$.

Now, in order to solve the other items of this problem we will first explain the solution of problem 1 in problem set 2 (i.e. we will explain the invariant meaning of the operator product expansion for the stress tensor).

1.3. Problem set 2: problem 1. In this problem we have to show that the right hand side of the OPE of the holomorphic stress tensor with itself is independent of the choice of coordinates. Let us say what it means using the abvoe notations.

We are working again with conformal field theory described by (1.4). Let z be a local coordinate on some open subset U of Σ. We are given the following section of E on U denoted by $T(x)$

$$\langle :(\partial\phi(x)^2):\phi_1(y_1)...\phi_n(y_n)\rangle$$

where $\phi_1...\phi_n$, $y_1,...,y_n$ are as above. The right-hand side of the OPE is the following section of $\Omega^{\otimes 2}\boxtimes\Omega^{\otimes 2}(4\Delta)/\Omega^{\otimes 2}\boxtimes\Omega^{\otimes 2}$ over $U\times U$:

$$\frac{(c/2)dz^{\otimes 2}\boxtimes dz^{\otimes 2}}{(z(x)-z(y))^4}+\frac{2dz^{\otimes 2}\boxtimes(T(y)-P^c_z(y))}{(z(x)-z(y))^2}+\frac{\mathrm{Lie}_{\partial_z}(T(y)-P^c_z(y))}{z(x)-z(y)} \tag{1.6}$$

Our task is to show that this expression is, in fact, independent of the choice of the coordinate z. We claim, that it has, in fact, the following invariant meaning.

Let T be any projective c-connection on U, i.e. a section of $(\Omega\boxtimes\Omega(2\Delta)/\Omega\boxtimes\Omega(-\Delta))^{S_2}$. Then *a priori* $T^2=T\otimes T$ is a well-defined section of $(\Omega^{\otimes 2}\boxtimes\Omega^{\otimes 2}(4\Delta)/\Omega^{\otimes 2}\boxtimes\Omega^{\otimes 2}(\Delta))^{S_2}$. However, it is easy to see, that

$$(\Omega^{\otimes 2}\boxtimes\Omega^{\otimes 2}(4\Delta)/\Omega^{\otimes 2}\boxtimes\Omega^{\otimes 2}(\Delta))^{S_2}=(\Omega^{\otimes 2}\boxtimes\Omega^{\otimes 2}(4\Delta)/\Omega^{\otimes 2}\boxtimes\Omega^{\otimes 2})^{S_2} \tag{1.7}$$

(on $U\times U$). Therefore, T^2 is a well defined section of $(\Omega^{\otimes 2}\boxtimes\Omega^{\otimes 2}(4\Delta)/\Omega^{\otimes 2}\boxtimes\Omega^{\otimes 2})^{S_2}$.

The invariance property of (1.6) follows from the next

Claim. *For any choice of a coordinate z*

(1.8)
$$T^2=\frac{c}{2}\left(\frac{(c/2)dz^{\otimes 2}\boxtimes dz^{\otimes 2}}{(z(x)-z(y))^4}+\frac{2dz^{\otimes 2}\boxtimes(T(y)-P^c_z(y))}{(z(x)-z(y))^2}\frac{dz^{\otimes 2}\boxtimes \mathrm{Lie}_{\partial_z}(T(y)-P^c_z(y)}{z(x)-z(y)}\right)$$

Proof. Let our projective connection T be written in the local coordinate z as

$$T=\frac{\frac{c}{2}dz\boxtimes dz}{(z(x)-z(y))^2}+f(x,y)dz\boxtimes dz \tag{1.9}$$

where $f(x,y)=f(y,x)$. Then T^2 as a section of $\Omega^{\otimes 2}\boxtimes\Omega^{\otimes 2}(4\Delta)/\Omega\boxtimes\Omega$ equals

$$T^2=\Big(\frac{\frac{c}{2}dz\boxtimes dz}{(z(x)-z(y))^2}+f(x,y)dz\boxtimes dz\Big)^2=$$
$$\frac{c^2}{4}\frac{dz^{\otimes 2}\boxtimes dz^{\otimes 2}}{(z(x)-z(y))^4}+cf(x,y)\frac{dz^{\otimes 2}\boxtimes dz^{\otimes 2}}{(z(x)-z(y))^2}=$$
$$\frac{c^2}{4}\frac{dz^{\otimes 2}\boxtimes dz^{\otimes 2}}{(z(x)-z(y))^4}+cf(y,y)\frac{dz^{\otimes 2}\boxtimes dz^{\otimes 2}}{(z(x)-z(y))^2}+c\frac{dz^{\otimes 2}\boxtimes dz^{\otimes 2}}{z(x)-z(y)}\Big(\frac{f(x,y)-f(y,y)}{z(x)-z(y)}=$$
$$\frac{c}{2}\Big(\frac{c}{2}\frac{dz^{\otimes 2}\boxtimes dz^{\otimes 2}}{(z(x)-z(y))^4}+\frac{2dz^{\otimes 2}\boxtimes(T(y)-P^c_z(y))}{(z(x)-z(y))^2}+\frac{dz^{\otimes 2}\boxtimes \mathrm{Lie}_{d_z}(T(y)-P^c_z(y)}{z(x)-z(y)}\Big)$$

□

1.4. Problem 1a. Here we are going to prove the OPE for the stress tensor in an arbitrary conformal field theory of central charge c. Let us remind that a conformal field theory assigns to a finite collection of labels $\phi_1,...,\phi_n$ a correlation function $\langle\phi(x_1)...\phi(x_n)\rangle$, which is an element in $\Gamma(\Sigma^n\backslash\mathrm{Diag},\mathcal{L}_1\boxtimes...\boxtimes\mathcal{L}_n)$ where $\mathcal{L}_i$ is a natural $\mathcal{O}_\Sigma$-module, assigned to the field ϕ_i (the word "natural" should be understood in the sense that vector fields on Σ act on the sections of $\mathcal{L}_i$ by Lie derivatives).

We will use also the following notation. Let $\mathcal{M}$ be an $\mathcal{O}$-module on Σ and m its section. The expression $\langle(m\otimes\phi(x))\phi_1(y_1)...\phi_n(y_n)\rangle$ is by definition a section of $\mathcal{M}\otimes\mathcal{L}\boxtimes\mathcal{L}_1...\mathcal{L}_n$, equal to the tensor product of m by $\langle\phi(x)\phi_1(y_1)...\phi_n(y_n)\rangle$.

The holomorphic stress tensor is a field, denoted by T, with the corresponding $\mathcal{O}$-module being E (cf. solution of problem 1b), with the property that

$$p_x(\langle T(x)\phi(x_1)...\phi(x_n)\rangle)=c\langle\phi(x_1)...\phi(x_n)\rangle \tag{1.10}$$

(recall that $p:E\to\mathcal{O}$ denotes the natural projection – cf. 1.1).

The basic property of the stress tensor T is its connection with Noether's currents (cf. Dan Freed's lecture). Namely, let U be an open subset of Σ with a coordinate z. The choice of z allows us to view all projective c-connections on U as quadratic differentials by means of $P\to P-P^c_z$. Let now v be a holomorphic vector field defined in a neighbourhood of a point x_0 in U.

Claim.

(1.11)
$$\mathrm{Res}_{x\to x_0}\langle(v\otimes(T(x)-P^c_z(x)))\phi_0(x_0)\phi(y_1)...\phi(y_n)\rangle=Lie_{x_0}v(\langle\phi_0(x_0)\phi(y_1)...\phi(y_n)\rangle)$$

where $Lie_{x_0}v(\langle\phi_0(x_0)\phi(y_1)...\phi(y_n)\rangle)$ denotes the Lie derivative of the correlation function $\langle\phi_0(x_0)\phi(y_1)...\phi(y_n)\rangle$ with respect to v acting on the zero-th component. Note that the right-hand side is a priori independent of the choice of the coordinate z.

This formula can be used to derive the OPE between the stress tensor and any other fields. Here we will use it in the case of the OPE for $T(x)T(y)$.

Let z be once again a coordinate on $U \subset \Sigma$. We must prove the following

Lemma.

$$(T(x) - P_z^c(x))T(x_0) - \left(\frac{(c/2)dz^{\otimes 2} \boxtimes dz^{\otimes 2}}{(z(x)-z(x_0))^4} + \frac{dz^{\otimes 2} \boxtimes (T(x_0) - P_z^c(x_0))}{(z(x)-z(x_0))^2} + \frac{Lie_{\partial_z}(T(x_0) - P_z^c(x_0))}{z(x)-z(x_0)}\right)$$

is a regular section of $\Omega^{\otimes 2} \boxtimes E$ (meaning that it becomes such, when inserted into a correlation function with any other fields).

Proof. We assume that $z(x_0) = 0$. According to the above claim, all we have to check is the following:

- $$\mathrm{Lie}_{\partial_z}(T_{x_0}) = \mathrm{Lie}_{\partial_z}(T_{x_0} - P_z^c(x_0)) \tag{1.12}$$

  This holds since $\mathrm{Lie}_{\partial_z}(P_z^c(x_0)) = 0$ (as ∂_z belongs to the Lie algebra of PGL(2) when U is embedded into $\mathbb{P}^1$ by means of z).

- $$\mathrm{Lie}_{z\partial_z}(T_{x_0}) = 2(T_{x_0} - P_z^c(x_0)) \tag{1.13}$$

  This holds because $\mathrm{Lie}_{z\partial_z}(P_z^c(x_0)) = 0$ for the same reason as above, and for any quadratic differential ω one has $\mathrm{Lie}_{z\partial_z}(\omega)(0) = 2\omega(0)$.

- $$\mathrm{Lie}_{z^3\partial_z}(T_{x_0}) = (c/2)(dz)^{\otimes 2} \tag{1.14}$$

  Indeed, $\mathrm{Lie}_{z^i\partial_z}(T_{x_0} - P_z^c(x_0)) = 0$ for $i \geq 2$ (this holds for any quadratic differential) and $\mathrm{Lie}_{z^3\partial_z}(P_z^c(x_0)) = (c/2)(dz)^{\otimes 2}$ which is easy to check by a direct calculation.

- $$\mathrm{Lie}_{z^i\partial_z}(T_{x_0}) = 0 \qquad \text{for } i = 2 \text{ or } i > 3 \tag{1.15}$$

  This holds since $\mathrm{Lie}_{z^i\partial_z}(T_{x_0} - P_z^c(x_0)) = 0$ as explained above and also one has $\mathrm{Lie}_{z^i\partial_z}(P_z^c(x_0)) = 0$, since $z^2\partial_z$ still belongs to the Lie algebra of PGL(2), and $\mathrm{Lie}_{z^i\partial_z}(P) = 0$ for $i > 3$ for any projective c-connection P (for $i > 3$ $z^i\partial_z$ is a trivial vector field on $\mathrm{Spec}(\mathbb{C}[z]/z^4)$).

□

1.5. Problem set 1: problem 2.

1.5.1. *Formulation of the problem.* Let Σ be Riemann surface of genus h, endowed with a complex structure. We are dealing with the action

$$S[\phi,g]=\frac{1}{4\pi}\int_{\Sigma}\frac{1}{2}\partial\phi\overline{\partial}\phi+Q\phi R_g \tag{1.16}$$

where ϕ is a real-valued function on Σ and g and R_g are as in 1.1. The goal of the problem is to investigate the corresponding quantum theory. Namely, we have to do the following

- to show that the quantum theory is conformal and to determine its central charge (part (c))
- to compute the OPE of $\partial\phi(x)\partial\phi(y)$ and to determine whether $\partial\phi$ is a primary field (parts (a) and (e))
- to show that $\exp(\beta\phi(x))$ is a primary field and to compute its conformal weight

1.5.2. *Computation of the partition function and the central charge.* Let us first compute the partition function. From it will be able to read off the central charge of the theory. We must compute the following functional integral:

$$Z_g^Q=\int D\phi e^{-S[\phi,g]}=\int D\phi\exp(-\frac{1}{4\pi}\int_{\Sigma}\frac{1}{2}\partial\phi\overline{\partial}\phi+Q\phi R_g) \tag{1.17}$$

where the integral is taken over all functions $\phi:\Sigma\to\mathbb{R}$ such that

$$\int_{\Sigma}\phi d\mu_g=0$$

The action is a non-homogeneous quadratic function in ϕ and therefore we may compute the functional integral by completing the square and we obtain

$$Z_g^Q=Z_g^0\exp(\frac{Q^2}{8\pi}\int_{\Sigma}\Delta_g^{-1}(R_g)\cdot R_g) \tag{1.18}$$

Let us now compute how the partition function changes when we replace g by a new metric g' of the form $e^{\sigma}g$, where σ is a real- valued function on Σ. We know that

$$Z_{g'}^0=Z_g^0\exp(\frac{1}{96\pi}(\int_{\Sigma}(\sigma,\Delta\sigma)+4\sigma R_g)) \tag{1.19}$$

(cf. Gawedzki's second lecture).

Therefore, since $R_{g'} = R_g + \frac{\Delta(\sigma)}{2}$,

$$Z^Q_{g'} = Z^Q_g e^{\frac{1}{96\pi}\int\limits_\Sigma ((\sigma,\Delta\sigma)+4\sigma R_g)} e^{\frac{Q^2}{8\pi}(\int\limits_\Sigma \Delta_g^{-1}(R_{g'})\cdot R_{g'} - \Delta_g^{-1}(R_g)\cdot R_g} = Z^Q_g \exp(\frac{1+3Q^2}{96\pi}\int\limits_\Sigma ((\sigma,\Delta\sigma)+4\sigma R_g)) \tag{1.20}$$

Therefore the central charge of the theory is equal to $1+3Q^2$ (note that we haven't defined our theory as a quantum conformal field theory yet, which is done below).

1.5.3. *Computation of the correlation functions.* We will now define the basic fields of our CFT and compute some of their correlation functions.

For $Q = 0$ we had a primary field $\partial\phi$ which took values in $\Omega^{1,0}(\Sigma)$.

For general Q we will show that $\partial\phi$ is a section of the $\Omega^{1,0}(\Sigma)$-torsor $\Omega^{1,0}_Q(\Sigma)$ of affine $2\pi Q$-connections. This torsor can be explicitly described as follows. For any metric g, compatible with the complex structure on Σ, $\Omega^{1,0}_Q(\Sigma)$ acquires a canonical trivialization (this corresponds to the fact, that given a metric, one can trivialize the torsor of affine connections by means of the $(1,0)$-part of the corresponding Levi-Civita connection) and when the metric g is replaced by $g' = e^\sigma g$, the trivialization is changed by $2\pi Q\partial\sigma$.

Claim. *$\partial\phi$ is a quantum field which takes values in $\Omega^{1,0}_Q(\Sigma)$*

Proof. We will sketch the computation of a one-point function of $\partial\phi$. So, let g be a metric on Σ, $x \in \Sigma$ – a point of Σ and ξ – a holomorphic tangent vector to Σ at x. Then the expression $\langle\partial\phi(x)\xi\rangle^Q_g$ makes sense as a complex number.

$$\langle\partial\phi(x)\xi\rangle^Q_g = \frac{1}{Z^Q_g}\int D\phi(\partial\phi(x),\xi)\cdot\exp(\frac{1}{4\pi}\int\limits_\Sigma \frac{1}{2}\partial\phi\overline{\partial}\phi + QR_g) \tag{1.21}$$

Arguing as above (by completing the square) we obtain that (1.21) is equal to

$$\mathrm{Lie}_\xi(\Delta_g^{-1}(R_g)) \tag{1.22}$$

Therefore, when g is replaced by $g' = e^\sigma g$, $\langle\partial\phi(x)\xi\rangle^Q_{g'} = \langle\partial\phi(x)\xi\rangle^Q_g + 2\pi\partial\sigma$, as claimed. □

The above claim implies, in particular, that the field $\partial\phi(x)$ is not primary, since primary fields were defined as fields whose correlation functions are sections of $\Omega^{p,q}$ (for some p,q) and $\Omega^{1,0}_Q$ is definitely not of this form for $Q \neq 0$. This solves (e).

1.5.4. *Solution of (a).* In order to establish the OPE of $\partial\phi(x)$ and $\partial\phi(y)$ we will make the computation only for the 2-point function.

Claim. *$\langle\partial\phi(x)\partial\phi(y)\rangle_g^Q - \langle\partial\phi(x)\partial\phi(y)\rangle^0$ is regular when $x \to y$.*

Proof. In fact, it is easy to compute in the same way as above, that

$$\langle\partial\phi(x)\partial\phi(y)\rangle_g^Q = \langle\partial\phi(x)\partial\phi(y)\rangle_g^0 + \langle\partial\phi(x)\xi\rangle_g^Q \otimes \langle\partial\phi(x)\xi\rangle_g^Q \tag{1.23}$$

which proves the claim □

Therefore, the OPE of $\partial\phi(x)$ and $\partial\phi(y)$ is the same as for $Q = 0$.

1.5.5. *Solution of (d).* We claim that classical field $\exp(\beta\phi(x))$ can be quantized to a primary field of conformal weight $(Q\beta + \frac{\beta^2}{2}, Q\beta + \frac{\beta^2}{2})$. We will calculate the two-point function

$$\langle e^{\beta\phi(x)} e^{-\beta\phi(y)}\rangle_g^Q = \langle e^{\beta\phi(x)} e^{-\beta\phi(y)}\rangle_g^0 \cdot \exp(Q \int_\Sigma (\Delta_g^{-1} R_g) \cdot (\delta(x) - \delta(y))) \tag{1.24}$$

When we make a Weyl transformation of the metric $g \to g' = e^\sigma g$, we see that

$$\langle e^{\beta\phi(x)} e^{-\beta\phi(y)}\rangle_{g'}^Q =$$

$$\langle e^{\beta\phi(x))} e^{-\beta\phi(y))}\rangle_g^Q \cdot \exp(\frac{\beta^2}{2}(\sigma(x) - \sigma(y))) \cdot \exp(Q\beta(\sigma(x) - \sigma(y))$$

This shows that $\exp(\beta\phi(x))$ is a field whose correlators take values in $(\Omega^{1,1})^{(\frac{\beta^2}{2}+Q\beta)}$.