STRING THEORY

Problem Set # 1

January 23, 1997

Consider the quantum field theory of a single real scalar field x, on a 2-dimensional surface Σ with Riemannian metric g and action

$$S[x,g] = \frac{1}{4\pi}\int_\Sigma d\mu_g \left\{ \frac{1}{2} g^{mn}\partial_m x \partial_n x + Q R_g x \right\}.$$

Here, $Q \in \mathbf{C}$ and R_g is the Gaussian curvature of g. Introduce local complex coordinates $z, \ \bar{z}$ in which $g = 2g_{z\bar{z}}|dz|^2$.

1.) For $Q=0$, define the renormalized stress tensor by

$$T(z) = -\frac{1}{2}\lim_{w\to z}\big(\partial_z x \partial_w x + \frac{1}{(z-w)^2}\big).$$

a) Using the OPE of $\partial_z x$ and $\partial_w x$, derive all singular terms in the OPE of $T(z)$ and $T(w)$, and confirm the form derived on general grounds

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w).$$

b) Show, by explicit conformal reparametrization $z \to z' = f(z)$, that $T(z)$ transforms as a projective connection.

c) Use the result of b) to relate the stress tensor $T(z)$ and the Virasoro generators L_m on the cylinder and on the annulus.

2.) For $Q \neq 0$, consider the flat metric $g = 2|dz|^2$.

a) Derive the OPE of $\partial_z x$ and $\partial_w x$.

b) Compute the stress tensor and show that $T = T_{zz}$ is complex analytic at the classical level. Define a suitable renormalized stress tensor.

c) Use the result of b) to obtain the central charge of the theory.

d) Show that $:\exp\{\beta x(z,\bar{z})\}:$ is a primary field for any value of $\beta \in \mathbf{C}$, and compute its conformal weight $(h, \bar{h})$.

e) Is the field $\partial_z x$ primary ? Derive its transformation law under conformal reparametrizations $z \to z' = f(z)$.