STRING THEORY

Problem Set # 2

January 30, 1997

The purpose of this problem set is to establish that the singular parts of the OPE of two stress tensors

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w). \tag{1}$$

and of a stress tensor and a primary field, are valid for any worldsheet metric g in any complex coordinate system in which $g = 2g_{z\bar z}|dz|^2$, and are thus covariant under analytic diffeomorphisms of the complex coordinates. (The issue of covariance of the OPE was raised by Pierre Deligne.)

1) Using the fact that $T(z)$ transforms as a projective connection, establish directly that the OPE in (1) transforms covariantly under analytic diffeomorphisms.

2) An alternative derivation of the covariance of (1), and in addition of the validity of (1) on any surface with any metric, may be obtained by retracing the derivation of the OPE using Diff(Σ) and Weyl(Σ) Ward identities. The derivation was carried out in Gawedski's lectures, but the actual OPE seems to have been derived only for locally flat metrics. Here, we propose to complete the analysis for general metric. For the sake of clarity, I shall go through all the steps of the derivation.

Let ϕ_i be a set of primary fields, with conformal weights $(h_i, \bar h_i)$, $i = 1, \cdots, n$, and consider their (unnormalized) correlation functions $\langle \phi_1 \cdots \phi_n \rangle_g$ for worldsheet metric g. We define the (covariant) quantum stress tensor T^{cov}_{mn} by

$$\delta_g \langle \phi_1 \cdots \phi_n \rangle_g = \frac{1}{4\pi}\int_\Sigma d\mu_g \delta g^{mn}(w,\bar w)\langle T^{cov}_{mn}(w.\bar w)\phi_1 \cdots \phi_n\rangle_g$$

and the variation of the correlation function on the r.h.s is given by

$$\delta_g \langle T^{cov}_{mn}(w,\bar w)\phi_1 \cdots \phi_n \rangle_g = \frac{1}{4\pi}\int_\Sigma d\mu_g \delta g^{pq}(z,\bar z)\langle T^{cov}_{pq}(z,\bar z) T^{cov}_{mn}(w,\bar w)\phi_1 \cdots \phi_n\rangle_g$$

We define a conformal field theory on any surface Σ, and for any metric g by the following requirement on its full (covariant) quantum stress tensor T^{cov}_{mn}

$$\nabla^m T^{cov}_{mn} = 0 \qquad\qquad T^{cov}{}_m{}^m = -\frac{c}{3}R_g$$

where ∇ is the covariant derivative with respect to the affine connection. This definition is identical to the axiomatic definition used in Gawedski's lectures.

a) Use the covariance of the correlation function $\langle\phi_1 \cdots \phi_n\rangle_g$ under general infinitesimal diffeomorphisms (not just analytic ones) to derive a first order differential equation relating $\langle T^{cov}_{ww}(w)\phi_1 \cdots \phi_n\rangle_g$ to $\langle\phi_1 \cdots \phi_n\rangle_g$. Deduce the OPE of T^{cov}_{ww} with $\phi_i(z)$ from this equation by solving for its singular part as $z \to w$.

b) By evaluating the variation of $\langle T^{cov}_{mn}(w, \bar{w})\phi_1 \cdots \phi_n\rangle_g$ under a change δg^{zz}, derive an equation for $\langle T^{cov}_{zz}(z, \bar{z})T^{cov}_{ww}(w, \bar{w})\phi_1 \cdots \phi_n\rangle_g$. Deduce the OPE $T^{cov}_{zz}T^{cov}_{ww}$ from this equation by solving for its singular part as $z \to w$.

c) Show that there exists a quantity Q_{zz} such that $T(z)$ defined by

$$T(z) = T^{cov}_{zz}(z, \bar{z}) + Q_{zz}(z, \bar{z})$$

is analytic and obeys the standard OPE in (1). Determine Q, and compute it explicitly when $g_{z\bar{z}} = 1$.

d) Deduce from the results of c) for general metric that $T(z)$ transforms as a projective connection under analytic diffeomorphisms.