STRING THEORY

Problem Set # 4

February 13, 1997

Problem 1

In this problem, we propose to prove a certain number of intermediate steps in the calculation of the Weyl anomaly of the determinant combinations

$$Z^-_{(n)}(g) = \frac{\mathrm{Det}'\Delta^-_{(n)}}{\det(\phi_j,\phi_k)_g \det(\psi_a,\psi_b)_g}$$

Here, we use the following expressions for covariant derivatives and Laplacian operators on sections f of K^n, (K is the canonical bundle of Σ and z, $\bar z$ are local complex coordinates such that $g = 2g_{z\bar z}|dz|^2$)

$$\begin{aligned} \nabla^z_{(n)} f &= (g_{z\bar z})^{-1}\partial_{\bar z} f & \Delta^+_{(n)} f &= -2\nabla^z_{(n+1)}\nabla^{(n)}_z f \\ \nabla^{(n)}_z f &= (g_{z\bar z})^n \partial_z (g_{z\bar z})^{-n} f & \Delta^-_{(n)} f &= -2\nabla^{(n-1)}_z \nabla^z_{(n)} f \end{aligned}$$

Also, $\phi_j \in \mathrm{Ker}\nabla^{(n-1)}_z$, $\psi_a \in \mathrm{Ker}\nabla^z_{(n)}$ and these kernels have dimensions $N^+_{(n-1)}$ and $N^-_{(n)}$ respectively. Show that under infinitesimal Weyl transformations of the metric $\delta_\sigma g = 2\delta\sigma g$, we have

a) $N^\pm_{(n)}$ are invariant,

b) and the trace of the heat-kernel changes by

$$\delta_\sigma \mathrm{Tr} e^{-t\Delta^-_{(n)}} = -t\frac{\partial}{\partial t}\mathrm{Tr}\Big[2n\delta\sigma e^{-t\Delta^+_{(n-1)}} - (2n-2)\delta\sigma e^{-t\Delta^-_{(n)}}\Big].$$

c) Establish the following short time (t) expansion of the heat-kernels

$$\mathrm{Tr}\delta\sigma e^{-t\Delta^\pm_{(n)}} = \frac{1}{4\pi t}\int_\Sigma d\mu_g \delta\sigma + \frac{1\pm 3n}{12\pi}\int_\Sigma d\mu_g R_g \delta\sigma + \mathcal{O}(t),$$

d) as well as their long time limits

$$\begin{aligned} \mathrm{Tr}\delta\sigma e^{-t\Delta^-_{(n)}} &= -\frac{1}{2n-2}\delta_\sigma \ln\det(\phi_j,\phi_k)_g + \mathcal{O}(e^{-\lambda_1 t}) \\ \mathrm{Tr}\delta\sigma e^{-t\Delta^+_{(n)}} &= \frac{1}{2n}\delta_\sigma \ln\det(\psi_a,\psi_b)_g + \mathcal{O}(e^{-\lambda_1 t}), \end{aligned}$$

where $\lambda_1 > 0$.

e) Check that the result in c) reproduces the Riemann-Roch-Atiyah-Singer index theorem for the operators $\nabla^z_{(n)}$ and $\nabla_z^{(n+1)}$.

Problem 2

a) Construct the vertex operators for all physical states with mass squared $M^2 = 2$, in closed (oriented) string theory in flat space-time $M = \mathbf{R}^{26}$, by requiring that vertex operators are integrals of primary fields of conformal dimension $(1,1)$.
b) Verify that the vertex operators you find are in one to one correspondence with the physical states $|\phi\rangle$, obtained by imposing the Virasoro conditions $L_n - \delta_{n,0}$ and $\tilde{L}_n - \delta_{n,0}$ on $|\phi\rangle$, (as was carried out in Lecture 3).