STRING THEORY

Problem Set # 8

March 20, 1997

Let G be a semi-simple compact Lie group, with Lie algebra $\mathcal{G}$, and structure constants f_{abc}, $a,b,c = 1, \cdots, \dim G$. Let Σ be a surface with the topology of the plane, with a flat metric. The Wess-Zumino-Witten model is defined in terms of maps $g: \Sigma \rightarrow G$ and classical action

$$S[g] = \frac{k}{16\pi}\int_\Sigma \mathrm{tr}\theta \wedge *\theta + \frac{k}{24\pi}\int_B \mathrm{tr}\tilde\theta \wedge \tilde\theta \wedge \tilde\theta.$$

Here, θ denotes the $\mathcal{G}$-valued left-invariant one form $\theta = g^{-1}dg$, B is a 3-dimensional manifold whose boundary is Σ. Also, we have $\tilde\theta = \tilde g^{-1}d\tilde g$ where $\tilde g: B \rightarrow G$ is any differentiable extension of g to B such that $\tilde g|_\Sigma = g$.

Classical Part

a) Show that the action $S[g]$ is of the same form as the generalized non-linear sigma models studied in the lectures, with action (for definitions, see problem set 7)

$$S_{G,B}[x] = \frac{1}{8\pi l^2}\int_\Sigma d\mu_\delta \left[\delta^{mn}G_{\mu\nu} + \epsilon^{mn}B_{\mu\nu}\right]\partial_m x^\mu \partial_n x^\nu$$

b) Show that the following multiplication formula holds

$$S[gh] = S[g] + S[h] + \frac{k}{16\pi}\int_\Sigma \mathrm{tr} g^{-1}\bar\partial g h^{-1}\partial h$$

and use it to show that S is well-defined, up to shifts by integer multiples of $2\pi k$.

c) Using the formula of b), show that $S[g]$ is invariant under the infinite dimensional current (Kac-Moody) algebra $\hat{\mathcal{G}} \times \hat{\mathcal{G}}$:

$$g(z,\bar z) \rightarrow \Omega(z) g(z,\bar z)\bar\Omega(\bar z)^{-1}$$

where $\Omega(z)$ and $\bar\Omega(\bar z)$ are holomorphic and anti-holomorphic maps from Σ into G (actually into the complexification of G) respectively. Derive the $\mathcal{G}$-valued conserved current $J = J^a t^a$ resulting from the transformations Ω. (t^a are generators of $\mathcal{G}$, satsifying the structure relations $[t_a, t_b] = f_{abc}t_c$.) Derive the transformation law of J under Ω.

d) Derive the stress tensor T, show that it is holomorphic and express T as a quadratic form in terms of J.

Quantum Part

We now quantize the theory defined by $S[g]$, and assume that the quantum current J and stress tensor T continue to be holomorphic. We also assume in part f) that the relation between the quantum current J and the quantum stress tensor T is given by the relation derived in d), except for an unknown multiplicative constant κ :

$$2\kappa T(z) =: J^a(z)J^a(z):$$

e) Show that the Ward identities for Diff(Σ) and $\hat{\mathcal{G}}$ yield the following OPE's on T and J

$$\begin{aligned}
T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) \\
T(z)J^a(w) &\sim \frac{1}{(z-w)^2}J^a(w) + \frac{1}{z-w}\partial_w J^a(w) \\
J^a(z)J^b(w) &\sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{f^{abc}}{z-w}J^c(w)
\end{aligned}$$

f) By considering the current correlation functions

$$\langle J^{a_1}(z_1)\cdots J^{a_n}(z_n)\rangle$$

and limits thereof when different z_i approach one another, show that the OPE's for J and T are consistent only if

$$c = \frac{k \dim G}{k + C_2(G)}$$

where $C_2(G)$ is the quadratic Casimir value of the adjoint representation of $\mathcal{G}$, defined by $f^{acd}f^{bcd} = C_2(G)\delta^{ab}$.

g) Show that, in the large k approximation, the metric G and the anti-symmetric tensor field B of the Wess-Zumino-Witten model, satisfy the equations for conformal invariance and for the central charge that were derived (in the small l^2 limit) in the lectures.