STRING THEORY

Problem Set # 9

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In this problem set, we investigate the so-called lightcone gauge quantization method. The key ingredient of this method is to eliminate at the classical level the constraints resulting from Diff(Σ) invariance (and also other symmetries for superstring theories), by solving for the constraints explicitly. Quantization is then carried out on the remaining "transverse degrees of freedom" only, and automatically results in a Hilbert space, with positive definite norm. Solving the constraints is achieved by choosing a null vector ξ in Minkowski space-time (i.e. a vector on the lightcone, whence the name of the method). Poincare symmetry will be realized in the quantum theory provided Poincare generators can be constructed that obey the Poincare algebra, and this is possible again only in the critical dimension.

Let $M = \mathbf{R}^D$ be flat Minkowski space-time of dimension D, with flat metric $\eta = \mathrm{diag}(-,+,\cdots,+,+)$. We choose the null vector $\xi = (1,0,\cdots,0,1)$, introduce associated lightcone coordinates

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{D-1})$$

and denote the remaining "transverse coordinates" by x^i, $i = 1,\cdots,D-2$. The scalar product in M reads

$$v \cdot w = v^\mu w^\nu \eta_{\mu\nu} = -v^+ w^- - v^- w^+ + v^i w^i.$$

The string worldsheet Σ is taken to be an annulus with global complex coordinates $z, \bar z$ and worldsheet metric $g = 2|dz|^2$.

Closed Bosonic String

We specialize here to the closed oriented bosonic string; the open and unoriented strings may be constructed by truncation. The classical field equations are $\partial_z \partial_{\bar z} x^\mu(z,\bar z) = 0$, and are solved by

$$x^\mu(z,\bar z) = q^\mu + ip^\mu \ln|z|^2 + i\sum_{n\neq 0}\frac{1}{n}(x_n^\mu z^{-n} + \tilde x_n^\mu \bar z^{-n}) \qquad (1)$$

The classical constraints are

$$\partial_z x \cdot \partial_z x = \partial_{\bar z} x \cdot \partial_{\bar z} x = 0 \tag{2}$$

The generators of the Poincare algebra are given by p^μ and by the Lorentz generators

$$J^{\mu\nu} = q^\mu p^\nu - q^\nu p^\mu + S^{\mu\nu} + \tilde S^{\mu\nu} \tag{3}$$

where the spin operators of the massive states are defined by

$$S^{\mu\nu} = -i \sum_{n=1}^\infty \{x^\mu_{-n} x^\nu_n - x^\nu_{-n} x^\mu_n\} \qquad \tilde S^{\mu\nu} = -i \sum_{n=1}^\infty \{\tilde x^\mu_{-n} \tilde x^\nu_n - \tilde x^\nu_{-n} \tilde x^\mu_n\} \tag{4}$$

a) Show that it is consistent with the field equations for x^μ to make a choice of coordinates $z, \bar z$ such that $x_n^+ = \tilde x_n^+ = 0$ for all $n \in \mathbf{Z},\ n \neq 0$.

b) In these coordinates, solve for the classical constraint equations (2), by solving explicitly for $x^-(z,\bar z)$ in terms of $x^i(z,\bar z)$, for x_n^- in terms of x_n^i and for $\tilde x_n^-$ in terms of $\tilde x_n^-$.

c) Quantize the independent degrees of freedom x_n^i and $\tilde x_n^i$ by requiring for $m, n \in \mathbf{Z}$ and $i, j = 1, \cdots, D-2$ that

$$[x^i_m, \tilde x^j_n] = 0 \qquad\qquad [x^i_m, x^j_n] = [\tilde x^i_m, \tilde x^j_n] = \delta^{ij} \delta_{m+n,0}. \tag{4}$$

It is convenient to work on a sector of Fock space with definite eigenvalue of the operator p^+, so that p^+ may effectively be regarded as a c-number. Define a suitable renormalization for the composite field x^-. This renormalization will be unique up to a constant, which we denote by a for later reference. Show that the oscillators $p^+ x_n^-$ (and similarly $p^+ \tilde x_n^-$) obey a Virasoro algebra with central charge $D-2$.

d) Show that the algebra of the transverse Lorentz group, SO(24), generated by J^{ij} closes for any D and any value of a. Show that the commutators $[J^{ij}, J^{k-}]$ obey the Lorentz algebra of SO(1,D-1), for all values of a and D.

e) From the general structure of the Lorentz generators, show that the commutators $[J^{i-}, J^{j-}]$ must be of the form

$$[J^{i-}, J^{j-}] = \frac{1}{(p^+)^2} \sum_{n=1}^\infty \Delta_n (x^i_{-n} x^j_n - x^j_{-n} x^i_n) + \{x \to \tilde x\}. \tag{5}$$

Determine Δ_n, and find the conditions under which Δ_n vanishes for all $n \in \mathbf{Z},\ n \neq 0$. (This is a necessary condition for the closure of the Lorentz and Poincare algebras.)

f) Show that the remaining Poincare structure relations are obeyed.

g) We define the Del Giudice Di Vecchia Fubini (DDF) operators (for simplicity only in the left moving sector) by

$$V^i(k) = \oint dz \partial_z x_+^i e^{ik \cdot x_+}$$

where

$$x^i(z, \bar{z}) = x_+^i(z) + x_-^i(\bar{z}).$$

Show that, for a suitable choice of momenta k, the operators map positive norm states into positive norm states, and physical states into physical states.

Ramond-Neveu-Schwarz strings

Generalize the set-up and the analysis carried out above to the case of the Ramond-Neveu-Schwarz string.