Kaluza-Klein compactifications, supersymmetry, and Calabi-Yau spaces: I

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I Introduction

As we have seen in d'Hoker's lectures, study of the ten-dimensional heterotic string dynamics leads us to a low energy effective action of the form:

$$S_{\text{eff}}[g, A, \phi, \ldots] = \frac{1}{2\kappa^2} \int_{X^{10}} d^{10}x \sqrt{-\det(g)} e^{-2\phi} \left(R(g) + \frac{\alpha'}{30} \mathrm{Tr} F_A^2 + \cdots \right)$$

for a metric g of signature $(-, +, +, \cdots, +)$ on a ten-dimensional manifold X^{10}, a connection A on a principal G-bundle over X^{10}, a scalar field ϕ, called the *dilaton field*, and other fields including the supersymmetric partners of these that we have not written here. Here $R(g)$ is the scalar curvature of g, G is compact, semisimple, F_A is the curvature of A and TrF_A^2 represents a G-invariant metric on the Lie algebra of G. The study of the heterotic string shows that it must be the case that the Cartan matrix of the Dynkin diagram for G is an even unimodular lattice in dimension 16. The only two possibilities for G are $E_8 \times E_8$ or $Spin(32)/(\mathbf{Z}/2\mathbf{Z})$ for a central $\mathbf{Z}/2\mathbf{Z}$ which is not in the kernel of the projection to $SO(32)$. At the level we are working in these lectures, the delicate distinction between various groups with Lie algebra $SO(32)$ cannot be seen. For this reason, and also to simplify the

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notation, we shall consider $SO(32)$ instead of the more exotic form. Lastly, the expression $d^{10}x\sqrt{-\det(g)}$ is the volume element associated to the metric. The minus sign arises from the fact that the metric is of signature $(1,9)$ so that the sign is necessary to have a positive quantity under the square root.

Our goal in these lectures is to study and analyse this Lagranian in its own right, not considering the small coreections arising from string theory. It turns out that, because of the topological nature of the analysis, our conclusions remain largely valid in the exact string theory. In fact, this Lagrangian was written down as a Lagrangian for supergravity before string theory was invented, and so is of interest in its own right. There are two basic problems that any such analysis should resolve if this theory is to be a reasonable model of nature.

Problem 1. This action exists for metrics, connections, etc. on a ten-dimensional manifold X^{10}, yet it seems fairly apparent that any theory of nature must have to do with four-dimensional space-time of our universe. There must be an explanation of the 'collapsing' of ten space-time dimensions to four.

Problem 2. The two gauge groups that occur for this action, $E_8 \times E_8$ and $SO(32)$, are not appropriate ones for the observed matter content of the universe. Somehow by a mechanism of spontaneous symmetry-breaking, we must get down to a smaller group which is suitable for describing observed matter.

Our approach will be to first look for classical ground states (solutions) minimizing the action and study their features. At a later stage one should consider quantum corrections.

Motivated by the two problems we listed above, here are the properties that we want our ground states (or classical solutions) to have:

1. The effective theory looks four-dimensional instead of ten-dimensional.

2. The gauge group G should be broken down to a 'good' gauge group, one that fits with the GUT (grand unification theory) scheme, so that it produces matter content observed in

nature. The gauge group of the standard model is $SU(3) \times SU(2) \times U(1)$ and accounts for all the matter that we know of, in the sense that all the 'elementary' particles are grouped together into a representation of this reductive group. The GUT schemes are to embed this group in a larger group, $SU(3) \times SU(2) \times U(1) \subset G$, so that the representations that occur in the standard model occur among those arising in the restriction of an irreducible representation of G to the subgroup $SU(3) \times SU(2) \times U(1)$. This scheme (among others) allows the following set of groups for G:

$$SU(3) \times SU(2) \times U(1) \subset SU(5) \subset SO(10) \subset E_6.$$

Of course, on the level of groups on can continue the series with

$$E_6 \subset E_7 \subset E_8,$$

but the irreducible representations that we must use have to be chiral and hence not isomorphic to their duals. This rules out E_7 and E_8 where the relevant representations are real. Somehow we must break either $E_8 \times E_8$ or $SO(32)$ down to one of these groups, most likely E_6.

3. Our solution should have $N = 1$ supersymmetry in dimension 4 (the $N = 1$ super Poincaré group). We do not observe in supersymmetry in nature, so whatever supersymmetry exists in our effective four-dimensional gravity theory must be capable of being entirely broken to $N = 0$, hopefully by small quantum corrections, which we shall not discuss here. There are no mechanisms for breaking $N \geq 2$ supersymmetry down to a chiral theory. This implies that we can have at most $N = 1$ supersymmetry in our model. On the other hand, we desire nontrivial supersymmetry in our model because it is the best mechanism we know which is capable of accounting for the very small ratios of masses of the observed low energy particles, i.e., for solving the 'hierarchy problem' as it is known to physicists.

4. Our solution should yield a realistic matter content.

The story that we shall describe was worked out in the mid 1980's (see [Candelas, Horowitz, Strominger, and Witten, Nucl. Phys. B **258** (1985)] and [Green, Schwarz, Witten: Chapters 12 – 16]). More recently, with the advent of dualities, there are new wrinkles, but the basic story as it was understood ten years ago remains relevant.

II Kaluza-Klein Model

Let us begin with a much simpler and much older example, which goes back to the work of Kaluza-Klein in 1926. We consider the five-dimensional Einstein-Hilbert action

$$S[\widehat{g}] = \frac{1}{2\pi\kappa^2}\int_Y d^5z\sqrt{-\det(\widehat{g})}R(\widehat{g}).$$

(The variable z is real.) The action is for a metric $\widehat{g}$ on a five-dimensional space, and $R(\widehat{g})$ is the scalar curvature of g. Lastly, κ is Newton's constant. (The hats are simply to distinguish this from other metrics which will arise later.) The classical equations of motion for this action are Einstein's equations

$$R_{MN}(\widehat{g}) = 0,$$

the vanishing of the Ricci curvature tensor of $\widehat{g}$.

One solution $\widehat{\overline{g}}$ is on the manifold $Y = \mathbf{R}^4 \times S^1$ where the metric is

$$\widehat{\overline{g}}_{MN}(z)dz^M dz^N = \eta_{\mu\nu}dx^\mu dx^\nu + r^2 d\theta^2; \quad \text{where} \;\; z = (x,\theta).$$

Here we have introduced several conventions which we shall use consistently. The indices μ,ν range from 0 to 3 and are indices for coordinates x^μ in Minkowski four-dimensional space time. The metric tensor $\eta_{\mu\nu}$ is the standard diagonal form $(-,+,+,+)$. The indices M,N range from 0 to 4 and index the coordinates z^M on Y. Here, r is a constant which determines the radius of the internal circle.

We begin with this basic solution and study the situation near this solution. It is convenient to write metrics (or symmetric two-tensors) in a slightly different basis:

$$\widehat{g}_{MN}(z)dz^M dz^N = e^{-\rho/3}\left(e^{\rho}\left(d\theta+\kappa A_\mu dx^\mu\right)^2 + g_{\mu\nu}dx^\mu dx^\nu\right).$$

In this description the metric on the five-dimensional space has become three fields: a four-dimensional metric tensor $g_{\mu\nu}$, a one form on four-space $A_\mu dx^\mu$, and a scalar field ρ. Of course, at this point all these fields depend not only on the point in the four-space but also on the auxiliary (or *internal*) variable θ.

Now let us consider the expansion in Fourier modes with respect to the variable θ on the circle:

$$g_{\mu\nu}(z) = \sum_{n=-\infty}^{\infty} g_{\mu\nu;n}(x)e^{in\theta},$$

$$A_\mu(z) = \sum_{n=-\infty}^{\infty} A_{\mu;n}(x)e^{in\theta},$$

$$\rho(z) = \sum_{n=-\infty}^{\infty} \rho_n(x)e^{in\theta}.$$

At this point what we have done is to rewrite the five-dimensional metric as an infinite collection of purely four-dimensional fields (metrics, one-forms, scalars). As we shall see shortly, when taking the low energy effective limit we can ignore all but finitely many of these fields. (We keep only the zero modes in the above Fourier expansions.)

Now we are ready to consider excitations (or linearized fluctuations) around our solution $\widehat{\overline{g}}$. Consider a general metric $\widehat{g}_{MN}(z)dz^M dz^N$ near $\widehat{\overline{g}}$. The linearized fluctuation is

$$\tilde{h}_{MN}(z) = \widehat{g}_{MN}(z) - \widehat{\overline{g}}_{MN}.$$

It is convenient to define

$$h_{MN} = \tilde{h}_{MN} - \frac{1}{2}\widehat{\overline{g}}^{MN}\tilde{h}_{MN}\widehat{\overline{g}}_{MN}.$$

We view the symmetric two-tensor $h = h_{MN}(z)dz^M dz^N$ as parametrizing the fluctuations of the metric. We must take some care here for there are gauge symmetries, the group of diffeomorphisms of Y. Metrics lying on the same orbit of this group action must be considered as equivalent since they differ simply reparametrizing the space. The standard way around this problem is to pick a gauge (a slice perpendicular to the orbit of the gauge symmetries). In the current context the simplest gauge to pick is called the *transverse gauge*. It consists of considering metrics satisfying

$$\nabla^M h_{MN} = 0.$$

Now, as we observed above, the equations of motion are the Einstein equations

$$R_{MN}(\widehat{g}) = 0.$$

Expanding $R_{MN}(\widehat{\overline{g}} + h) = 0$ to first order in h yields

$$\Box^5_{\widehat{g}} h_{MN}(z) = \left(\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{r^2}\partial_\theta^2\right) h_{MN}(x, \theta) = 0.$$

Combining this equation with the Fourier mode expansion we see that

$$\Box^4_{\widehat{g}} h_{MN;n}(x) - \frac{n^2}{r^2} h_{MN;n}(x) = 0.$$

That is to say $h_{MN;n}(x)$ satisfies the wave equation in four-space for a field of mass $|n|/r$. Invoking a basic result from quantum mechanics we know that the lowest energy excitation of a field of mass $|n|/r$ has energy at least $|n|/r$. Now it is time to say more about the size of the circle factor in Y. So far it has been a free parameter in the theory. We shall take r to be our roughly the Planck length, which is 10^{-33} cm. At this length scale, all the excitations of $h_{MN;n}(x)$ have huge energy provided that $n \neq 0$. Since we are interested in the low energy effective theory, we can safely ignore all the modes $h_{MN;n}(x)$ except the zero modes. Thus, at energy scales below $1/r$, our effective theory is purely four-dimensional. It has three fields on four-space which are now independent of the internal variable θ. The fields are: a metric $g_0 = g_{\mu\nu;0}(x)dx^\mu dx^\nu$ on $\mathbf{R}^4$, a one-form $A_0 = A_{\mu;0}(x)dx^\mu$ on four-space and a scalar field $\rho_0(x)$ on four-space.

Next, let us plug these fields into the Kaluza-Klein action and write the resulting effective four-dimensional action. It is actually a somewhat complicated, if routine, computation to carry out the substitution; what comes out is:

$$S_{\text{eff}}[g_0, A_0, \rho_0] =$$
$$= r\int_{\mathbf{R}^4} d^4x\sqrt{-g_0}\left(\frac{1}{\kappa^2}R(g_0) - \frac{1}{4}e^{\rho_0}F^0_{\mu\nu}F_0^{\mu\nu} - \frac{1}{6\kappa^2}(\nabla\rho_0)^2 + \left((\nabla\rho_1)^2 + \frac{1}{r^2}\rho_1^2 + \cdots\right)\right),$$

where $F^0_{\mu\nu}dx^\mu dx^\nu$ is the curvature of the connection A_0. All the terms enclosed in the last pair of parentheses represent massive particles and can be ignored in our low energy effective action. Thus, we are left with

$$S_{\text{eff}}[g_0, A_0, \rho_0] = r\int_{\mathbf{R}^4} d^4x\sqrt{-g_0}\left(\frac{1}{\kappa^2}R(g_0) - \frac{1}{4}e^{\rho_0}F^0_{\mu\nu}F_0^{\mu\nu} - \frac{1}{6\kappa^2}(\nabla\rho_0)^2\right).$$

We began with a five-dimensional gravity theory and for sufficiently small radius r we have found the low energy effective four-dimensional action and written this action in terms of three four-dimensional fields: a metric, a connection which is related to the Killing vector field on the five-manifold tangent to the circle directions, and a scalar field ρ which is related to the free parameter r in the five-dimensional theory.

Let us now consider the symmetries of this effective four-dimensional action. We claim that these are simply the symmetries of the original five-manifold $Y = \mathbf{R}^4 \times S^1$ which commute with the natural S^1-action coming from this product structure of Y. This group is generated by two types of diffeomorphisms. The first are simply diffeomorphisms of $\mathbf{R}^4$. These are generated by vector fields ζ on $\mathbf{R}^4$, independent of θ. It is easy to see that the action of these on our fields are given by:

$$\delta_\zeta g_{\mu\nu} = \mathcal{L}_\zeta(g_{\mu\nu}),$$

$$\delta_\zeta A_\mu = \mathcal{L}_\zeta(A_\mu),$$

$$\delta_\zeta \rho = \mathcal{L}_\zeta(\rho).$$

Here, in all cases $\mathcal{L}_\zeta$ denotes the Lie derivative in the ζ-direction. The second type of diffeomorphisms are those generated by vector fields of the form

$$x^\mu \mapsto x^\mu; \quad \theta \mapsto \theta + \zeta^5(x).$$

Under these vector fields we have

$$\delta_\zeta(A_\mu) = -\frac{1}{\kappa}\partial_\mu \zeta^5,$$

$$\delta_\zeta g = \delta_\zeta \rho = 0.$$

Notice, in particular, that $A = A_\mu dx^\mu$ transforms like an abelian (i.e., $U(1)$) gauge field.

Of course, there are other diffeomorphisms of the five-manifold Y, which do not preserve the S^1-action. But these diffeomorphism mix the massive and low energy modes. Since we have thrown away the massive modes in forming our effective four-dimensional Lagrangian, these symmetries are broken in the limiting process, leaving the unbroken group for the four-dimensional effective action exactly the one described in the previous paragraph.

III Compactifying Einstein's Equation from ten dimensions to four dimensions

Now let us consider ten-dimensional pure gravity. The action is

$$S[\widehat{g}] = \int_{X^{10}} d^{10}x \sqrt{-\det(\widehat{g})} R(\widehat{g}),$$

where $\widehat{g}$ is a metric on X^{10} with signature $(-,+,\cdots,+)$. The classical solutions extremizing this Einstein-Hilbert action are Ricci-flat metrics

$$R_{MN}(\widehat{g}) = 0.$$

In analogy with the Kaluza-Klein model, let us study ground states (classical solutions) of the following form: The manifold X^{10} is a product $\mathbf{R}^4 \times K^6$ where K^6 is compact, and the metric is of the form $\widehat{g}_0 = (\widehat{g}_0)_{MN} dz^M dz^N$ where

$$(\widehat{g}_0)_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & (g_0)_{mn}(y) \end{pmatrix},$$

where g_0 is a positive definite metric on K^6. Our notation is similar to the five-dimensional case. The indices μ, ν run from 0 to 3 and index the coordinates x^μ on Minkowski space. The indices m, n index the internal coordinates y^m which are local coordinates on K. The indices M, N run from 0 to 9 and index local coordinates $z = (x, y)$ on the product $\mathbf{R}^4 \times K^6$. The split metric $\widehat{g}_0$ is Ricci-flat if and only if g_0 is a Ricci flat metric on K^6. The existence of such a metric, of course, imposes a topological condition on Y. We fix now a classical solution of this form.

As in the Kaluza-Klein model, let us study small fluctuations about this classical solution:

$$h_{MN}(z) = \widehat{g}_{MN}(z) - (\widehat{g}_0)_{MN}.$$

Again, there is the gauge symmetry of the diffeomorphism of X^{10} to deal with, and again we use the transverse gauge. The Ricci-flat condition to first order in h_{MN} is

$$\left(\Box^4 + \Box^6\right) h_{MN}(z) = 0,$$

where

$$\Box^4 h_{\mu\nu} = \eta^{\mu\nu} \partial_\mu \partial_\nu h_{\mu\nu}$$

and

$$\Box^6(h_{mn}) = (g_0)^{ab} \nabla_a \nabla_b h_{mn} + 2R^a{}_m{}^b{}_n h_{ab}.$$

Following the same line of reasoning as before, we expand in Fourier modes in harmonics of K^6. We write

$$h_{\mu\nu}(z) = \sum_k h_{\mu\nu;k}(x) Y_k(y)$$

where

$$\Box^6 Y_k(y) = -\lambda_k Y_k(y).$$

Here since the metric on Y is positive definite, this D'Alembertian on scalar functions on Y and is the (negative of the) usual Laplacian. It is a non-positive operator, so that $\lambda_k \geq 0$ for all k. The equation satisfied by $h_{\mu\nu;k}$ is

$$(\Box^4 - \lambda_k) h_{\mu\nu;k}(x) = 0.$$

Once again we find that $h_{\mu\nu;k}$ is a solution to the wave equation in four-space for a field of mass λ_k. If we scale the metric on K^6 by multiplying it by $1/r$, then λ_k scales by $1/r^2$. In particular, if we scale by $1/r$ for r very small, then for all k for which $\lambda_k \neq 0$, all excitations of the field $h_{\mu\nu;k}(x)$ become extremely massive, and hence can be ignored in our low energy effective action.

There is a similar analysis for the fields $h_{mn}(z)$. We impose the transverse gauge condition on the metric in the K^6-direction and we expand the equations of motion in the harmonics of K^6:

$$h_{mn}(z) = \sum_k \varphi_k(x) Y_{mn;k}(y),$$

where this time $Y_{mn;k}(y) dy^m dy^n$ are sections of the symmetric square of the cotangent bundle of Y which are eigenvectors with eigenvalue λ'_k for the D'Alembertian:

$$\Box^6 Y_{mn;k}(y) = g^{ab} \nabla_a \nabla_b Y_{mn;k}(y) + 2 R^a{}_m{}^b{}_n Y_{ab;k}(y) = -\lambda'_k Y_{mn;k}(y).$$

Thus, we obtain

$$\Box^4 \varphi_k(x) - \lambda'_k \varphi_k(x) = 0.$$

As before, this is the wave equation for a wave of mass λ'_k.

Here, though, it is not necessarily the case that $\lambda'_k \geq 0$ for all k. There can be a finite number of negative eigenvalues for the operator on Y. A negative eigenvalue means that our solution g_0 is a critical point for the Einstein-Hilbert action for positive definite metrics on Y which is not a local minimum for this action, i.e., an unstable critical point. We wish to avoid this possibility, so for now we simply assume that our operator $\Box^6$ on Y has no negative eigenvalues. (This will be automatic by supersymmetry in the Calabi-Yau case.) As long as $\lambda'_k > 0$, all excitations of this field will have mass on the order of $1/r^2$ and can be ignored in our low energy limit. Thus, we keep only those $\varphi_k(x)$ for which $\lambda'_k = 0$, i.e., we keep only those $\varphi_k(x)$ for which the corresponding $Y_{mn;k}(y)dy^m dy^n$ is a harmonic section of the symmetric square of the co-tangent bundle of K^6. This space of harmonic forms is the formal tangent space at g_0 to the moduli space of Ricci-flat metrics on K^6. So, for each parameter in the moduli space of solutions to the Einstein equations on the internal space K^6 we have a massless scalar for the effective action on four-space. These fields are the analogues of the massless field ρ we found in the Kaluza-Klein model, a field which came from the one parameter, the radius, describing the metric on the internal space in that theory.

IV Adding Matter to the Mix

We have made a fairly complete analysis of pure gravity compactified from ten dimensions to four dimensions by using a compact six-dimensional manifold K^6 with a Ricci-flat metric. It is time now to add other terms to the action so as to create a theory that has matter as well as gravity. Our goal here is to give the general nature of the terms that will add matter, both bosons and fermions, to the action that we have been studying. In the next section we will be more explicit as to the exact nature of the terms we are adding. Here the analysis is quite general and will be sketched only since much of it is similar to what we have done in great detail above for the metric.

First of all we can add boson fields by adding to the action a term of the form $S(\varphi)$

where φ is a scalar or vector-valued function on ten-dimensional space-time. Once again, we expand φ in Fourier modes using the harmonics of Y, and we can disregard all but the zero modes. In this way, after compactification φ becomes a boson field on four-space.

Let us consider fermions. We fix a spin structure on $\mathbf{R}^4 \times Y^6$ leading to spinor fields ψ which can be sections of the spin bundles tensored with other auxiliary bundles with connections. The Clifford algebra for an inner product space of type $(1, 9)$ is $\mathbf{R}[32]$, the 32×32 real matrix algebra. The even component of the Clifford algebra is the block 16×16 real matrices. This means that the spin bundle over Y has a real structure and decomposes as $S^+_{\mathbf{R}}(Y) \oplus S^-_{\mathbf{R}}(Y)$, each factor being a 16-dimensional real bundle. A section of $S^+_{\mathbf{R}}(Y)$ or $S^-_{\mathbf{R}}(Y)$ is called a Majorana-Weyl spinor. Its chirality is by definition which of these two spin bundles it is a section of. The metric $\widehat{g}_0$ leads to a spinor connection and thus to the real Dirac operator

$$\not{D}\colon \Gamma(S^{\pm}_{\mathbf{R}}(Y)) \to \Gamma(S^{\mp}_{\mathbf{R}}(Y)).$$

In terms of the gamma-matrices for Clifford multiplication we can write

$$\not{D} = \gamma^m D_m$$

where D_m is the spin covariant derivative in the direction of the m^{th} coordinate. The Dirac equation on the ten-dimensional manifold is then written as

$$i\not{D}^{(10)}\psi = 0.$$

Since the metric $\widehat{g}_0$ is split, so is the Dirac operator, and in fact the Dirac equation is

$$i(\not{D}^4 + \not{D}^6)\psi = 0,$$

where $\not{D}^4$ and $\not{D}^6$ are the Dirac operators on Minkowski four-space and K^6 respectively. The solutions to this equation can be written as a sum

$$\psi(z) = \sum_k \psi_k(x)\psi'_k(y)$$

with

$$i\not{D}^4\psi_k(x) = \lambda'_k\psi_k(x)$$

and

$$i\not{D}^6\psi'_k(y) = -\lambda'_k\psi'_k(y).$$

Again, throwing away massive fluctuations, means keeping only the zero modes of the Dirac equation on the internal space K^6.

V The Effective action from ten-dimensional heterotic string theory

The effective action that heterotic string theory in ten dimensions gives us is

$$\begin{aligned}
&S_{\text{eff}}[g,\varphi,\psi,\lambda,\chi,A,H] = \\
&= \frac{1}{2\kappa^2}\int_Y d^{10}x\sqrt{-\det(g)}e^{-2\varphi}\Big(R(g) + 4(\nabla\varphi)^2 - \frac{1}{3}H^2 \\
&+\frac{\alpha'}{30}Tr\,F_A^2 - \overline{\psi}_M\gamma^{MNP}D_N\psi_P - \overline{\lambda}\not{D}\lambda - Tr\,\overline{\chi}\not{D}\chi + (\cdots)\Big).
\end{aligned}$$

The fields in question are as follows: There is a metric g of signature $(-,+,\cdots+)$ on Y. There is the dilaton field φ which is a real scalar field. There is a connection A on a principal $E_8\times E_8$ or $SO(32)$ bundle P over Y. Its curvature is denoted F_A. The axion form H is a three-form with the property that

$$dH = \alpha'(tr\,R\wedge R - \frac{1}{30}Tr\,F_A\wedge F_A).$$

This means that we can write H at least locally in the form

$$H = dB + \alpha'\left(CS(\nabla_g) - \frac{1}{30}CS(A)\right),$$

where B is a two-form and $CS(A)$ is the Chern-Simons three-form of the connection A. The boson fields g, B, φ are grouped together in a multiplet (i.e., they are various components of an irreducible representation). This multiplet has a supersymmetric partner which is a multiplet given by a pair of spinors (ψ,λ). The spinor ψ is a Majorana-Weyl spinor with values in the cotangent bundle of Y; that is to say, it is a section of $S^+_{\mathbf{R}}(Y)\otimes T^*(Y)$. It is called the *gravitino.* The spinor λ is a Majorana-Weyl spinor of opposite chirality to that of ψ. That is to say, it is a section of $S^-_{\mathbf{R}}(Y)$. It is called the *dilatino.* The gauge field A has its supersymmetric partner χ, called the *gaugino.* This field is a Majorana-Weyl spinor with values in the adjoint bundle of P and is of the same chirality as ψ. That is to say, it is a section of $S^+_{\mathbf{R}}(Y)\otimes \mathrm{ad}P$.

Having discussed the fields that occur, let us turn to the functions of them that appear in the action. As usual, $R(g)$ is the scalar curvature of g and TrF_A^2 is the norm of F_A in the G-invariant metric in the adjoint bundle. Lastly, γ^{MNP} is obtained from the gamma-matrices by skew-symmetrizing:

$$\gamma^{MNP}=\gamma^{[M}\gamma^{N}\gamma^{P]}.$$

These are the gamma-matrices that give the representation of three-forms on spinors. In language more familiar to mathematicians, all the spinor pairings in the action arise from the two basic pairings: the nondegenerate inner product between spinors of the opposite chirality:

$$\langle\ \ \rangle\colon S^+_{\mathbf{R}}(Y)\otimes S^-_{\mathbf{R}}(Y)\to\mathbf{R}$$

and the adjoint of Clifford multiplication

$$S^+_{\mathbf{R}}(Y)\otimes S^+_{\mathbf{R}}(Y)\to T(Y). \tag{1}$$

(Here we are viewing Clifford multiplication as a pairing $T(Y)\otimes S^+_{\mathbf{R}}(Y)\to S^-_{\mathbf{R}}(Y)$. To write its adjoint as we have above, we use the isomorphisms $T(Y)\cong T^*(Y)$, given by the metric, and $S^+_{\mathbf{R}}(Y)\cong S^-_{\mathbf{R}}(Y)^*$, given by the inner product.) The term $\overline{\lambda}\not{D}\lambda$ in the action is simply the inner product $\langle\lambda,\not{D}\lambda\rangle$. Likewise, $Tr\overline{\chi}\not{D}\chi$ is $Tr\langle\chi,\not{D}\chi\rangle$ defined from the inner product on spinors and the trace in the adjoint representation. Lastly, the pairing $\overline{\psi}_M \gamma^{MNP} D_N \psi_P$ in the action means the pairing $\langle \psi, \nabla \psi \rangle$, where the pairing is between a one-form with values in the positive spin bundle and a two-form with values in the same spin bundle. The pairing is given by wedging together the forms to produce a three-form and then letting that three-form operate by Clifford multiplication on one of the spinors to produce a spinor of minus chirality which can then be paired with the plus chirality spinor to produce a real number.

In addition, there are other terms (indicated by $(\cdots)$) that we have not explicitly written down and which are not important for the analysis that we shall make. These extra terms come in two types. Some are required by supersymmetry. There are, for example, terms which are fourth power in the spinor fields such as $(\overline{\psi}\psi)^2$. There are also terms which are higher order in α', which is a dimensionful parameter. These can be ignored if we assume that, even though K^6 is on the Planck scale, it is much larger that the string length.

This action clearly has a group of symmetries: the group of orientation-preserving diffeomorphisms of the underlying manifold. But in addition, we have gone to great lengths to build it so that it has supersymmetry, $N = 1$ local supersymmetry to be more precise. For example, our boson fields form two multiplets (g, B, φ) and A, and each multiplet has its fermionic partner (ψ, λ) and χ, respectively. This of course, is only a necessary condition for the existence of supersymmetry. But in fact the $N = 1$ local supersymmetry algebra is generated by sections ϵ of the Majorana-Weyl spin bundle $S^+_{\mathbf{R}}(Y)$. These generators are odd elements in the super Lie algebra. The infinitessimal transformation laws are partially given by:

$$\begin{aligned}
\delta_\epsilon \chi &= F_{MN}\gamma^{MN}\epsilon + (\text{fermions}^2 \\
\delta_\epsilon \psi_M &= \nabla_M \epsilon - \frac{1}{4} H_{MAB}\gamma^{AB}\epsilon + (\text{fermions})^2 \\
\delta_\epsilon \lambda &= (\gamma^M \nabla_M \varphi)\epsilon + \frac{1}{24} H_{MNP}\gamma^{MNP}\epsilon + (\text{fermions})^2
\end{aligned}$$

$$\begin{aligned}
\delta_\epsilon g_{MN} &= \bar{\epsilon}\gamma_N\psi_M + \bar{\epsilon}\gamma_M\psi_N + (\text{fermions})^2 \\
\delta_\epsilon A_M &= \bar{\epsilon}\gamma_M\chi + (\text{fermions})^2 \\
\delta_\epsilon B &= \cdots \\
\delta_\epsilon \phi &= \cdots
\end{aligned}$$

The complete laws can be written down, but we shall not do that here as we will not need them.

The space of classical solutions is invariant under the supersymmetry algebra generated by these infinitessimal (odd) symmetries, and on the space of solutions the algebra generated by these infinitessimal symmetries closes to yield the $N = 1$ local supersymmetry algebra. In fact, one has the following formula for the supercommutator:

$$[\delta_\epsilon, \delta_{\epsilon'}] = \delta_v + T$$

where v is the vector field obtained by pairing ϵ and ϵ' under the pairing given in Equation 1. Written in terms of Clifford matrices we have

$$v^m = \overline{\epsilon'}\gamma^m\epsilon.$$

The term T is an expression which includes gauge transformations and terms which vanish when one imposes the equations of motion.

All of this was done without worrying about the compactification. The next step in the study is to show under what conditions we can compactify down to four-dimensions. In doing this compactification much of the $N = 1$ local supersymmetry will be broken, but in special circumstances we will be able to retain $N = 1$ super Poincaré symmetry in dimension four. This is where the next lecture will begin.