# Kaluza-Klein compactifications, supersymmetry, and Calabi-Yau spaces: II

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## V Review of material from the first lecture

Let us recall the fields and $N = 1$ local supersymmetries that we developed last time for our ten-dimensional supergravity coupled to super Yang-Mills theory. We have bosonic fields: a metric $g_{MN}$ on space-time $X^{10}$ which is a ten-dimensional manifold, a two-form potential $B$, the dilaton $\phi$ which is a scalar field, and a gauge field in the adjoint bundle of a principal $E_8 \times E_8$ or $SO(32)$ bundle $P$ over space-time. The fermion fields are the gravitino $\psi_M$ which is a Majorana-Weyl spinor of plus chirality with values in the tangent bundle, the dilation $\lambda$, which is a Majorana-Weyl spinor of minus chirality, and the gaugino $\chi$ which is a Majorana-Weyl spinor of plus chirality with values in the adjoint bundle of $P$. The fields $(g_{MN}, B, \phi, \psi_M, \lambda)$ form an $N = 1$ supermultiplet, called the gravity supermultiplet, and $(A, \chi)$ form the $N = 1$ Yang-Mills supermultiplet, which is a matter supermultiplet. Because of anomalies we are not free to have arbitrary numbers of these supermultiplets: there must be exactly one ($E_8 \times E_8$ or $SO(32)$ matter supermultiplet to go with the supergravity supermultiplet.

Recall from last time that the supersymmetric (odd) generators $\epsilon$ are Majorana-Weyl

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spinors on ten-dimensional space-time, i.e., sections of $S^+_{\mathbf{R}}(X^{10})$. Their action on the fields is given by;

$$\begin{aligned}
\delta_\epsilon \chi &= F_{MN}\gamma^{MN}\epsilon + (\text{fermions})^2 \\
\delta_\epsilon \psi_M &= \nabla_M \epsilon - \frac{1}{4} H_{MAB}\gamma^{AB}\epsilon + (\text{fermions})^2 \\
\delta_\epsilon \lambda &= (\gamma^M \nabla_M \varphi)\epsilon + \frac{1}{24} H_{MNP}\gamma^{MNP}\epsilon + (\text{fermions})^2 \\
\delta_\epsilon g_{MN} &= \bar{\epsilon}\gamma_N \psi_M + \bar{\epsilon}\gamma_M \psi_N + (\text{fermions})^2 \\
\delta_\epsilon A_M &= \bar{\epsilon}\gamma_M \chi + (\text{fermions})^2 \\
\delta_\epsilon B &= \cdots \\
\delta_\epsilon \phi &= \cdots
\end{aligned}$$

## VI Partially breaking the Supersymmetry by compactifying down to four-dimensions

Now we are ready to study what happens to these $N = 1$ local supersymmetries as we compactify from ten dimensions to four dimensions in the way we discussed last time. Let us recall what we hope to achieve. We wish to find a four-dimensional low energy effective theory which has $N = 1$ super Poincaré symmetry. The odd generators of the $N = 1$ super Poincaré group in dimension four are constant four-dimensional Majorana spinors over four-dimensional Minkowski space-time. (Recall, that the spin group from Minkowski space time is $SL_2(\mathbf{C})$ which has an obvious real four-dimensional representation which is the underlying real representation of the defining 2-dimenisonal complex representation. A section of this spin bundle is a Majorana spinor.) We begin with a supersymmetry group generated by the sections of a 16-dimensional real spin bundle over ten-dimensional space-time, and we must somehow break this down so that after compactification the remaining supersymmetry generators are the constant Majorana spinors over Minkowski four-space. We need to break the symmetries in three-quarters of the components as well as for the non-constant sections.

As a basic model of how this works, recall what happens in the Kaluza-Klein model, where we compactify along a circle from five dimensions to four dimensions. We began with a five-dimensional theory with the full five-dimensional diffeomorphism group as symmetries. But the only diffeomorphisms which remained after compactifying and taking the low energy effective action for small circle radius were the diffeomorphisms which did not mix up the massive and massless modes. Recall that the five-dimensional vacuum we consider has a metric of the form

$$\eta_{\mu\nu} dx^{\mu} dx^{\nu} + \frac{1}{r^2} d\theta^2.$$

The group of global symmetries which remain unbroken (i.e., leave the vacuum invariant) are exactly the Poincaré group of Minkowski four-space times the group of constant $U(1)$ rotations. The infinitessimal generators of the original group of symmetries are the vector fields on the five-dimensional space-times. The action of a vector field $v$ on a metric $g$ on space-time is given by $\delta_v g = \mathcal{L}_v(g)$, the Lie derivative of $g$ in the $v$-direction. The condition on a vector field that it preserve a given vacuum $g_0$ is that $\delta_v g_0 = \mathcal{L}_v(g_0) = 0$, in other words, that $v$ be a Killing vector field for $g_0$.

Now let us return to our ten-dimensional theory. The situation is analogous to the Kaluza-Klein model. The main difference is that this time our generators are supersymmetric (odd) generators instead of (even) vector fields. Nevertheless, the condition on a generator $\epsilon$ that it be unbroken, i.e., preserve a given vacuum is that the first variation of all the fields with respect to $\epsilon$ must be trivial. As we develop the argument we will list the properties our solutions have. Some come by assumption: others are consequences of these assumptions. The first property comes from the basic set-up of the reduction as discussed last time:

**Assumption (i): The ten-dimensional space-time manifold of the solution is of the form $\mathbf{R}^4 \times K^6$, where $K^6$ is compact, and the metric is of the form**

$$\begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & (g_0)_{mn} \end{pmatrix}$$

**where $g_0$ is a Ricci-flat metric on $K^6$ and $\eta_{mu\nu} dx^\mu dx^\nu$ is the standard metric on Minkowski four-space. Furthermore, the obvious isometric action of the four-dimensional Poincaré group on this riemannian manifold is covered by an $N = 1$ four-dimensional super Poincaré symmetry group of the solution.**

Since the metric is split as a four-dimensional metric times a six-dimensional metric, the same is true of the spin bundles. They decompose

$$S^+(\mathbf{R}^4 \times K^6) = \left(S^+(\mathbf{R}^4) \otimes S^+(K^6)\right) \oplus \left(S^-(\mathbf{R}^4) \otimes S^-(K^6)\right).$$

Since there are no nonzero spinors in $S^\pm(\mathbf{R}^4)$ which are invariant under the Poincaré group, there are no nonzero spinors on the ten-dimensional manifold which are invariant under the Poincaré group.

**Consequence (ii): All the fermionic fields $(\psi_M, \lambda, \chi)$ of our solution must be trivial.**

Now, since all the fermionic fields vanish, it follows that all the boson fields are automatically invariant under any odd supersymmetry, simply by parity considerations. Thus, to understand which of the $N = 1$ local supersymmetry generators are unbroken in this vacuum (i.e., leave it invariant), we have only three equations to study:

$$\begin{aligned}
\delta_\epsilon \chi &= F_{MN}\gamma^{MN}\epsilon + (\text{fermions})^2 = 0 \\
\delta_\epsilon \psi_M &= \nabla_M \epsilon - \frac{1}{4} H_{MAB}\gamma^{AB}\epsilon + (\text{fermions})^2 = 0 \\
\delta_\epsilon \lambda &= (\gamma^m \nabla_m \varphi)\epsilon + \frac{1}{24} H_{MNP}\gamma^{MNP}\epsilon + (\text{ferminos})^2 = 0.
\end{aligned}$$

Furthermore, in these equations we can now drop the terms (fermion)$^2$ as well, leaving

$$\begin{aligned}
\delta_\epsilon \chi &= F_{MN}\gamma^{MN}\epsilon = 0 && (1)\\
\delta_\epsilon \psi_M &= \nabla_M \epsilon - \frac{1}{4} H_{MAB}\gamma^{AB}\epsilon = 0 && (2)\\
\delta_\epsilon \lambda &= \gamma^M \nabla_M(\phi)\cdot\epsilon + \frac{1}{24} H_{MNP}\gamma^{MNP}\epsilon = 0. && (3)
\end{aligned}$$

Our goal is, given the metric $g_0$ on $K^6$, to find $A, H, \phi$ so that there are exactly four spinor fields $\epsilon$ (Majorana spinors on Minkowski four-space) for which these equations are satisfied. One can then ask, if these equations are satisfied then are the equations of motion satisfied as well, or is that an extra condition? In this case, and in fact almost always, the (first-order) supersymmetry equations imply the equations of motion. As a simple example of this principle, let us suppose we are dealing with a case of four-dimensional supergravity. Thus, we have $H = F = 0$. Then the only equation for a Majorana spinor $\epsilon$ to be an unbroken supersymmetry is $\nabla_\mu \epsilon = 0$, i.e., $\epsilon$ is a supersymmetry if it is a covariantly constant spinor over four-dimensional space-time. But the existence of a non-zero convariantly constant spinor field implies that the metric on space-time must be Ricci-flat. The can be seen as follows: $\nabla_\mu \epsilon = 0$ implies that $\gamma^\nu[\nabla_\nu, \nabla_\mu]\epsilon = 0$ and hence that $R_{\mu\nu}\gamma^\mu\epsilon = 0$. Since $\epsilon$ is non-zero, this implies that the Ricci curvature $R_{\mu\nu} = 0$, i.e., that space-time is Ricci-flat.

## VII Geometric consequence of the unbroken supersymmetry

Our assumption is that we have a split metric on $\mathbf{R}^4 \times K^6$

$$g = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & (g_0)_{mn} \end{pmatrix}$$

with $g_0$ being a Ricci-flat metric on $K^6$. We are also assuming that we have nontrivial supersymmetry generator by Majorana spinor fields $\epsilon$ on space-time, and that there is an unbroken $N=1$ supersymmetry of our vacuum. We shall make further assumptions in order to keep the analysis simple. Here, we are following the analysis in [Candelas, Horowitz, Strominger, and Witten, Nucl. Phys. B **258** (1985), p.46]. (To see what happens in the more general case without these assumptions consult [Witten, Nucl. Phys. B **268** (1986) p. 95] and [Strominger, Nucl. Phys. B **274** (1986), p. 253].)

**Assumption (iii): the scalar field $\phi = \phi_0$ is a constant function and the three-form $H$ is zero.** Recall from last time that the axion field $H$ and the $B$ field are related by

$$H = \alpha'\left(CS(\nabla_g) - \frac{1}{30}CS(A)\right) + dB,$$

where $CS$ is the Chern-Simons functional. Thus, this assumption is equivalent to assuming that the field $B$ is a closed two-form.

Of course, as we have already argued, all the fermion fields must be zero. With all these assumptions, Equation 1 is automatically satisfied and Equations 2 and 3 simplify to

$$\delta_\epsilon \psi_M = \nabla_M \epsilon = 0 \tag{4}$$

$$\delta_\epsilon \chi = F_{MN}\gamma^{MN}\epsilon = 0 \tag{5}$$

In more invariant language, $\epsilon$ must be covariantly constant Majorana Weyl spinor of positive chirality and the effect of Clifford multiplying this spinor by the curvature two-form of the connection on the principal bundle is zero.

## VII.1 Consequences of the first equation

The $N=1$ supersymmetry that we are searching for is to be global super Poincaré symmetry. In particular, the four (odd) spinor generators of the superalgebra must make the standard four-dimensional representation of the four-dimensional Poincaré group acting in its usual way on the first factor of space-time. Since the spinor bundle $S^+(\mathbf{R}^4 \times K^6)$ decomposes as $S^+(\mathbf{R}^4) \otimes S^+(K^6) \oplus S^-(\mathbf{R}^4) \oplus S^-(K^6)$, the only Weyl-Majorana spinors of plus chirality on the ten-dimensional manifold which transform as required under the Poincaré group of Minkowski four-space are of the form

$$\epsilon = \sum_i \eta_+^i(y) \cdot (\epsilon_+^i)^{(4)} + \text{c.c.}$$

where $\eta_+^i(y)$ is a 6-dimensional Weyl spinor of positive chirality on $K$ and $(\epsilon_+^i)^{(4)}$ is a constant Weyl spinor of positive chirality on Minkowski four-space. (Here, c.c. means the complex conjugate. The complex conjugate is a product of Weyl spinors of negative chirality.)

Equation 4 becomes two equations: one in the $\mathbf{R}^4$-direction and one in the $K^6$-direction:

$$\delta_\epsilon \psi_\mu = \sum_i \eta_+^i \partial_\mu (\epsilon_+^i)^{(4)} + \text{c.c.} = 0$$

$$\delta_\epsilon \psi_m = \sum_i \nabla_m (\eta_+^i)(\epsilon_+^i)^{(4)} + \text{c.c.} = 0.$$

The first of these equations is automatic since $(\epsilon_+^i)^{(4)}$ is constant. (We could also use it to prove that $(\epsilon_+^i)^{(4)}$ is constant.) As to the second, since the complex conjugate of $\eta_+^i$ is a spinor of opposite chirality, and since covariant differentiation preserves chirality, it implies that $\nabla_m \eta_+^i(y) = 0$. This means that $\eta_+^i$ must be a covariantly constant non-zero Weyl spinor of plus chirality on $K^6$. It is clear that then that this spinor has constant norm, which we can renormalize to be 1. In particular, $\eta_+^i$ is nowhere zero.

In particular, we have seen that there must be at least one nonzero, covariantly constant spinor $\eta_+$ on $K$. Next, we study the consequences of this fact for the geometry of $(K^6, g_0)$.

**Claim VII.1.1** *The fact that $\eta_+$ is a covariantly constant spinor field of norm one implies that the endomorphism of the tangent bundle of $K$ given by*

$$J_m{}^n = i(\eta_+)^\dagger \gamma_m{}^n \eta_+$$

*is a real endomorphism whose square is minus the identity. Consequently, it defines an almost complex structure on $K$. This endomorphism an isometry with respect to $g_0$ and is* *parallel under the metric. This means that it defines an integrable almost complex structure (i.e., a complex structure) for which the metric $g_0$ is a Kähler metric.*

Here we are using the notation ${\gamma_m}^n = \frac{1}{2}(\gamma_m\gamma^n - \gamma^n\gamma_m)$ and $\gamma_n = \gamma^p g_{np}$.

**Proof.** In more invariant language, the element $J$ is determined by the spinor field $\eta_+$ is simply the image under the natural maps

$$S(K) \otimes S^*(K) \cong \mathrm{Cl}(K) \cong \wedge^* TK \to \wedge^2 TK \subset T^*K \otimes TK$$

of the element $\eta_+ \otimes \eta_+^\dagger$. It is clear from the defining formula that $J$ is a real tensor since ${\gamma_n}^m = -({\gamma_n}^m)^\dagger$. It follows from the Fierz identity among Clifford matrices that

$${J_m}^n {J_n}^p = -{\delta_m}^p,$$

i.e., that $J$ is an endomorphism whose square is minus the identity. Since $\eta_+$ and ${\gamma_m}^n$ are covariantly constant, it is clear that $J$ is covariantly constant. It is now a well-known fact that this implies that $J$ is integrable and that the metric is Kähler, but let us briefly go through the derivation of these facts. The obstruction to integrability is the Nejenhuis tensor

$${N_{mn}}^p = {J_m}^q(\nabla_q {J_n}^p - \nabla_n {J_q}^p) - {J_n}^q(\nabla_q {J_m}^p - \nabla_m {J_q}^p).$$

Since $J$ is covariantly constant, it follows that $N = 0$ and that $J$ defines a complex structure on $K$. Next, let us see that $J$ is an orthogonal transformation with respect to the metric $g$ or equivalently that $g$ defines a hermitian metric on the complex manifold determined by $J$. This is simply the equation ${J_m}^q {J_n}^p g_{pq} = g_{mn}$, which follows easily from the definitions, the fact that $|\eta_+|^2 = 1$, and the relations among the $\gamma$-matrices. Last, since $J$ is covariantly constant, so is the two-form associated to the hermitian metric determined by $g$. This means in particular, that the two-form is closed, and consequently that the metric is Kähler. □

Note that when it is not assumed that $H = 0$, the almost complex structure $J$ is not covariantly constant. Nevertheless, it can still be shown that the Nejenhuis tensor vanishes (see [Strominger, ibid.]), so that $J$ defines the structure of a complex manifold. Of course, it is not Kähler.

The complex structure on $K^6$ which is parallel under the metric reduces the holonomy group of parallel translation for the metric from $SO(6)$ to $U(3)$. But in fact, the existence of the self-parallel spinor $\eta_+$ implies that the holonomy reduces further. Decomposing the eight-dimensional Weyl spin bundle $S^+(Y)$ under $U(3)$ we get

$$8 \mapsto 3_1 \oplus 1_1 \oplus \overline{3}_{-1} \oplus \overline{1}_{-1}$$

where the first integer indexes the representation of $SU(3)$ and the subscript indexes the action of $U(1)$. (Thus, $3_1$ is the standard three-dimensional complex representation of $U(3)$ and $\overline{3}_{-1}$ is its dual, whereas $1_1$ is the determinant representation of $U(3)$.) The existence of a parallel non-zero spinor means that there must be a non-zero vector in the representation which is invariant under the image of the holonomy representation. Clearly, the only vector in this space invariant under the $U(1)$ diagonal subgroup is the trivial vector. This means then that the image of holonomy lies in $SU(3) \subset U(3)$. There are several possible subgroups with this property, $SU(3), SU(2)$, etc, but $SU(3)$ is the only one which has exactly a one-dimensional space of invariant spinors of each chirality. Thus, if we want to have exactly $N = 1$ supersymmetry and no more, then the holonomy group for the metric must be $SU(3)$. In this case there is a one-dimensional space of covariantly constant spinor fields, generated by say $\eta_+$.

There is another way to think about all of this. Consider the action of $Spin(6) = SU(4)$ on the complex four-dimensional space of Weyl spinors $S^+$ of positive chirality. The stabilizer of an point in the unit sphere in $S^+$ is conjugate to $SU(3) \subset SU(4) = Spin(6)$. The image in $SO(6)$ is the usual embedding of $SU(3) \subset SO(6)$. If there is a parallel non-zero spinor, then the holonomy of the metric must lie in this $SU(3)$. This reduces the structure group of $K^6$ to $SU(3)$ and the reduction is parallel under the metric. That is to say we have produced a parallel almost complex structure $J$. The arguments above show that this structure is integrable and that the metric is Kähler with respect to this complex structure. Since the holonomy is reduced to $SU(3)$, the metric is Ricci-flat.

We have shown:

**Consequence (iv): The odd generators of the super Poincaré algebra are covariantly constant spinors $\epsilon$ of the form $(\eta_+\epsilon_+^{(4)} + \mathrm{c.c})$ where $\eta_+$ is the unique (up to constant multiples) covariantly constant Weyl spinor of plus chirality on $K^6$ and $\epsilon_+^{(4)}$ is a constant Weyl spinor of plus chirality on Minkowski four-space.**

In all cases, since the holonomy is $SU(3)$, the Ricci tensor of the metric is trivial, and in particular the first Chern class of the metric is trivial. According to Yau's theorem on the existence of Ricci-flat metrics on Kähler manifolds, $c_1 = 0$ is sufficient for the existence of such a metric (if the manifold is non-simply connected then the condition must be interpreted as saying that the canonical bundle of the Kähler manifold is trivial as a holomorphic line bundle.) According to the uniqueness part of Yau's theorem, the moduli of Ricci flat Kähler metrics is the moduli space of complex structures and Kähler classes. As we remarked in the last lecture, and as we shall explain in more detail below, these moduli are associated with massless fields in the low energy effective four-dimensional action.

This completes the study of Equations 2 and 3 for the existence of an unbroken supersymmetry of the solution equations under the assumption that $H = 0$ and $\phi = \phi_0$ is constant. We have found the necessary and sufficient conditions for the existence of non-zero supersymmetries satisfying these equations:

**Consequence (v): The riemannian manifold $(K, g_0)$ be a Calabi-Yau three-fold with holonomy $SU(3)$.**

We have two more things to deal with: we must examine the consequences of assuming

$dH = 0$, and we must show that Equation 1 is automatically satisfied under the assumptions that we are making.

### VII.2 The condition $H = 0$.

So far we have not mentioned the gauge field $A$ of the solution. This field is constrained by our assumption that $H = 0$. We have

$$0 = dH = tr(R \wedge R) - \frac{1}{30} Tr(F_A \wedge F_A),$$

where $tr$ refers to the trace in the standard representation, that is to say the trace of the composite endomorphism of the tangent bundle, and $Tr$ denotes the trace in the adjoint representation of the gauge group. This means that

$$tr R \wedge R = \frac{1}{30} Tr F_A \wedge F_A,$$

There are two possible gauge groups $SO(32)$ and $E_8 \times E_8$. First, let us consider $SO(32)$. The trace in the adjoint representation is equal to 30 times the trace in the fundamental representation of $SO(32)$. Thus, we are looking for an $SO(32)$ connection $A$ with $tr F_A \wedge F_A = tr R \wedge R$, where $tr$ on the left-hand-side refers to the trace in the defining $SO(32)$ representation. One way to arrange this is to 'embed the spin connection in the gauge connections'. That means that there is a natural embedding $SU(3) \subset SO(32)$ with commutant being $SO(26) \times U(1)$. Using this embedding, we can view the connection on the tangent bundle (which has had its structure group reduced to $SU(3)$) as being a principal $SO(32)$ connection. Clearly, it satisfies the trace condition above. In this way, we can construct the last of the fields $A$ necessary to specify a solution.

We can perform a similar construction with $E_8 \times E_8$, using only one of the factors. There is an embedding $SU(3) \subset E_8$ with commutant being $E_6$. Once again we can use this to embed the spin connection as a gauge connection for the group $E_8 \times E_8$, with symmetry group $E_6 \times E_8$.

**Assumption (vii): We obtain the gauge field by embedding the spin connection.**

With this assumption, $H = 0$ is consistent with the defining equation for $H$.

### VII.3 The remaining supersymmetry equation

Now let us examine the remaining equation for the existence of an unbroken supersymmetry $\epsilon$:

$$\delta_\epsilon \chi = F_{MN}\gamma^{MN}\epsilon = 0.$$

The two-form $F$ with values in the adjoint bundle to $P$ is acting on the spinor field $\epsilon$ by Clifford multiplication of the two-form to produce a section of the spin bundle tensored with the adjoint bundle. We claim that with our hypotheses this equation is always satisfied. Since the gauge connection is, by assumption, obtained by embedding the spin connection, this equation will follow immediately from the analogous equation for the spin connection. Thus, the result we need to establish is

$$\frac{1}{2}R_{MNA}{}^{B}\gamma^{MN}\epsilon = 0, \qquad (6)$$

where $R_{MNA}{}^{B}dx^M \wedge dx^N$ is the riemann curvature tensor. The condition that $\epsilon$ is covariantly constant implies that the curvature two-form evaluates trivially on $\epsilon$, i.e., that

$$R_{MNA}{}^{B}\gamma^{A}{}_{B}\epsilon = 0.$$

Using the metric to raise and lower indices, we can rewrite this as

$$R_{MNAB}\gamma^{AB}\epsilon = 0.$$

Now using the fact that our connection is torsion-free, we can switch the roles of $AB$ and $MN$ and rewrite

$$R_{MNAB}\gamma^{MN}(\epsilon) = 0,$$

which is equivalent to Equation 6.

Let us say a few words about what happens in the more general situation where do not require $H = 0$, following [Strominger, ibid]. The equation $F_{MN}\gamma^{MN}\epsilon = 0$ is related to the hermitian Yang-Mills condition for the connection. That is to say the connection must be a holomorphic connection and the curvature must be a primitive two-form, i.e., $\omega \wedge \omega \wedge F = 0$ where $\omega$ is the fundamental (Kähler) two-form. Then using this, the relationship between $F, R$, and $H$ and supersymmetry we can then determine the value of $H$.

Let us return to the case of present interest where $H = 0$. In this case we have completed the descriptions of all the fields and a study of these supersymmetry equations. We have shown:

**Theorem VII.3.1** *Let $(K^6, J, g_0)$ is a Ricci-flat compact Kähler manifold with holonomy $SU(3)$. Then there is a classical solution to ten-dimensional supergravity coupled with super Yang-Mills with gauge group either $E_8 \times E_8$ or $SO(32)$. Space-time for this solution is $\mathbf{R}^4 \times K^6$, the metric on space-time is ,*

$$\begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & (g_0)_{mn} \end{pmatrix}.$$

*The axion field $H$ is zero and the dilaton field $\phi$ a constant. The fermion fields (the dilatino, the gravitino, and the gaugino) of this solution vanish and the gauge field $A$ is obtained by embedding the spin connection on $K^6$ as a gauge connection. If the gauge group is $E_8 \times E_8$, then the commutant of the connection, (i.e., the automorphism group of $P$ which leaves the connection invariant) is $E_6 \times E_8$. If the gauge group is $SO(32)$, then the commutant is $SO(26) \times U(1)$. Furthermore, this solution has $N = 1$ four-dimensional super Poincaré symmetry lifting the natural action of the four-dimensional Poincaré group space-time.*

# VIII Massless fields in the low energy effective Lagrangian

As we remarked when discussing the Kaluza-Klein theory, parameters for the internal space of the high-dimensional theory (ten-dimensional in this case) yield massless fields in the effective, low energy four-dimensional Lagrangian. By Yau's existence and uniqueness theorem for Ricci-flat Kähler metrics, the moduli space of Calabi-Yau structures on $K$ is the moduli space of isomorphism classes of pairs $(J, \omega)$ consisting of a holomorphic structure $J$ on $K$ with $c_1 = 0$ (or more precisely with trivial canonical bundle) and a Kähler class $\omega$. Let us consider the space of holomorphic structures; a similar discussion can be made for the variation of the Kähler class. The massless fields in the low energy effective Lagrangian associated to the variations of the Calabi-Yau metric come from expanding the metric $g_{MN}$ on ten-space in terms of the decomposition of space-time and keeping only the harmonic modes of eigenvalue zero on the internal space:

$$g_{MN}(z) = M^{\alpha}(x) h^{\alpha}_{mn}(y) + \cdots.$$

The other terms in this expansion are not relevant for the massless fields coming from the variations of the Calabi-Yau structure: They lead to massless fileds of other types. Since we are working at a critical point, there will be no terms in the effective low energy action involving only first derivatives of $g_{MN}$. As we take second derivatives we get three types of terms from $M^{\alpha}(x) h^{\alpha}_{mn}(y)$. We can differentiate $M^{\alpha}(x)$ twice, we can differentiate $h^{\alpha}_{mn}(y)$ twice, or we can differentiate each of $M^{\alpha}(x)$ and $h^{\alpha}_{mn}(y)$ once. Terms of the last type cannot contribute to the action because we have to pair the two derivatives using the metric of the solution. Since this metric is split, the pairing between the cross terms vanishes. Applying two derivatives to $h^{\alpha}_{mn}(y)$ would lead to a term in the Lagrangian which involves $M(x)^2$. Such a term would give $M(x)$ a mass. We know that there are no such terms because $h^{\alpha}_{mn}(y)$ is a harmonic variation of the linearized equations of motion. Thus, the only terms that can contribute to the action are where both derivatives are applied to $M(x)$. This leads to a term in the low energy effective action of the form

$$\int_{\mathbf{R}^4} d^4x \sqrt{-\det(g)} g^{\mu\nu} \partial_{\mu} M^{\alpha}(x) \partial_{\nu} M^{\beta}(x) G_{\alpha\beta}(M(x))$$

for some symmetric two-tensor $G = G_{\alpha\beta} dM^{\alpha} dM^{\beta}$ on moduli space.

In fact, it is possible to explicitly identity $G$. In general, the tangent space to the moduli space of holomorphic structures on a complex manifold $K$ is $H^1(TK)$. In the case that $K$ is a Calabi-Yau three-fold, we can use the non-vanishing holomorphic three-form $\Omega$ on $K$ (normalized to norm 1) to give an identification between $H^1(TK)$ and $H^{2,1}(K)$. Let us suppose that we are at a smooth point of the moduli space of holomorphic structures and let $M^\alpha, \alpha = 1, \ldots, h^{2,1}$ be local coordinates on the moduli space of holomorphic structures centered at the given point. A direct computation shows that under the identification of the tangent space of moduli space at $M$ with $H^{2,1}(M)$ the pairing $G(M)$ is the usual pairing of $(2,1)$-forms on $M$, namely if $a, b \in H^{2,1}(M)$ then

$$\langle a, b \rangle = \int_M a \wedge \overline{b}.$$

If our manifold is simply connected, or in fact if $h^{1,0} = 0$, then this pairing is positive definite. (It is always non-degenerate.) But, because it arises from an $N = 1$ supersymmetric Lagrangian, we can deduce highly non-trivial facts about this pairing. It follows from a general $N = 1$ supersymmetric computation that this metric is in fact a Kähler metric. Indeed, it is a Hodge metric. Much more can be obtained using the fact that not only can we work with the heterotic string in ten dimensions, we can also work with the type II string. Putting these together, we are able to extend this Lagrangian to a four-dimensional $N = 2$ supersymmetric Lagrangian. This imposes even more stringent restrictions, leading to what is referred to as 'special geometry'. For instance, the Kähler potential is locally determined by a certain holomorphic 'potential'. In this way supergravity makes contact with the theory of filtrations of Hodge structures.

## IX Relation to Grand Unification Theory.

As we remarked in the first lecture, $E_6$ is a possible group for GUT's. In fact, the representation that we need to use of $E_6$ in order to do GUT is the 27-dimensional defining representation. In our context we begin with the adjoint representation of $E_8$ and break it down to $SU(3) \times E_6$. Under this subgroup the 248-dimensional adjoint representation of $E_8$ decomposes

$$(8,1) \oplus (3,27) \oplus (\overline{3},\overline{27}) \oplus (1,78).$$

Here, the first term is the adjoint representation of $SU(3)$ and the last is the adjoint representation of $E_6$. The second and third terms are tensor products of three-dimensional representations of $SU(3)$ with 27-dimensional representations of $E_6$. Thus, among the representations that we are dealing with is the 27-dimensional representation of $E_6$, which fits with what GUT's requires.

Now the number of families massless modes in $(3,27)$ is given by an index theorem. Actually, the difference of massless modes of distinct (left and right) chiralities is computed by the equation:

$$n^L_{27} - n^R_{27} = \mathrm{index}(i\not{D}^{(6)}_3) = \frac{1}{48}(2\pi)^3 \int tr_3 F \wedge F \wedge F = \frac{-\chi(K)}{2},$$

where the Dirac operator and the trace are taken in the vector bundle associated to the fundamental three-dimensional representation of $SU(3)$. The fact that it is the difference rather than the actual number of families is not too serious a problem. It turns out to be relatively easy for massless modes of opposite chirality to combine and become massive. Thus, one can believe that the absolute value of this index is the correct number for the massless families. Now experiment shows us that in the universe there are exactly three families (quarks and leptons) of these light modes. This means that we should be looking for Calabi-Yau's with Euler characteristic $\pm 6$. A few of these have been found.

From the physics point of view, it is very exciting that the simplest compactifications lead naturally to realistic GUT gauge groups, realistic chiral representations of matter, and the structure of family replication. This remains a significant source of optimism for the eventual physical relevance of string theory. But important and difficult problems – for example, supersymmetry breaking – remain to be solved.