Superhomework, Term 2

The purpose of the following is to describe facts that we will need in order to study supersymmetric quantum field theories.

In doing so, we will mainly be interested in flat models (for the supermanifolds on which the quantum field theory is formulated), and in any event our supermanifolds will always be "split" (because we will not include supergravity) which generally means that one won't lose much that is essential by considering the flat model.

(Even without doing quantum field theory, one of the important things one learns from these exercises is that the interesting relativistic field theories have supersymmetric generalizations, with some restriction on the dimension of spacetime.)

We make one small change in notation. A manifold with p even and q odd coordinates will be said to be of dimension $(p|q)$. This frees up the comma for other purposes; for instance, if the reduced space has a metric of signature (n, m) (with $n + m = p$) we might call this a manifold of dimension $(n, m|q)$.

I haven't tried to explain everything below. You will have to consult Wess and Bagger, or other references, especially for the four-dimensional gauge theory. However, if you understand these problems, you are probably familiar with the classical facts that are most important background for beginning to discuss supersymmetric quantum theories.

(1) The first part is review from last fall. We consider a $(3|2)$ supermanifold $\mathbf{R}^{(3|2)}$ (or one can dimensionally reduce to $\mathbf{R}^{(n|2)}$ with $n < 3$ by imposing invariances under some translations). We call the fermionic coordinates θ^A, $A = 1, 2$ and the bosonic coordinates y^{AB} (symmetric in A and B). The supersymmetry generators are

$$Q_A = \frac{\partial}{\partial\theta^A} + i\theta^B \frac{\partial}{\partial y^{AB}} \tag{1}$$

and commute with

$$D_A = \frac{\partial}{\partial\theta^A} - i\theta^B \frac{\partial}{\partial y^{AB}} \tag{2}$$

which is used in writing Lagrangians.

The symbol ϵ^{AB} $(= -\epsilon^{BA})$ will denote a translation-invariant volume form on what Bernstein called the odd distribution – the odd subspace of the tangent bundle generated by the D's.

The super-Poincaré group is generated by the Q_A, their anticommutators $\partial/\partial y^{AB}$, and an $SU(2) \cong SO(3)$ for which the θ^A and the y^{AB} are in the two- and three- dimensional representations, respectively.

A superfield is just a function Φ on $\mathbf{R}^{(3|2)}$. Let X be a fixed Riemannian manifold with metric g. Let Φ be a map from $\mathbf{R}^{(3|2)}$ to X. Picking local coordinates on X, we describe Φ via functions Φ^I and write simply g_{IJ} for the pullback via Φ.

(a) Then as we hopefully remember from last fall we can make the supersymmetric Lagrangian

$$L_0 = \int d^3y\, d^2\theta\, \epsilon^{AB} g_{IJ} D_A \Phi^I D_B \Phi^J. \tag{3}$$

Refresh your recollections; show that this is equivalent to an ordinary sigma model with target space X plus fermions and describe the fermions.

Determine the conserved supercurrent.

(b) Now let h be a smooth function on X and let $L = L_0 + L_1$ where

$$L_1 = \int d^3y\, d^2\theta\, \Phi^*(h). \tag{4}$$

Identify the reduced model. In particular, what is the potential energy and what is the mass term for the fermions?

Suppose that X is compact and that h has only isolated critical points. Can you state a lower bound for the number of classical minima of the energy? (After dimensional reduction to $\mathbf{R}^{(1|2)}$ this observation was of course the starting point for my article on Morse theory in 1982. Even if h does not have isolated critical points, there is a sense in which the same lower bound on the number of vacua holds.)

(c) If X is noncompact it may be the case that there is no critical point of h and moreover that the set with $|dh| < c$ is compact for any constant c. In this case the energy is bounded strictly above zero. This is a theory in which supersymmetry is spontaneously broken. As you can read in Wess and Bagger, unbroken supersymmetry means that the energy of the vacuum is zero.

Consider an arbitrary local minimum of the classical energy function at which the energy is not zero. Show that there is a massless fermion – called the Goldstone fermion. What you have just checked is a special case of the analog of Goldstone's theorem for supersymmetry.

(2) Now we want to do gauge theory, but again in $\mathbf{R}^{(3|2)}$ to begin with.

First we will do pure gauge theory. G is a compact gauge group and A is a connection on a G bundle P over $\mathbf{R}^{(3|2)}$. The curvature F is defined in the usual way. One can freely write down all kinds of supersymmetric actions by integrating over $\mathbf{R}^{(3|2)}$ an appropriate polynomial in F and its derivatives. However, such constructions (like the generic classical field theory) are "wrong," that is the bosonic part does not have positive energy, for instance. We will verify the analogous statement later in a case that may be a bit easier.

To get a good theory, we consider connections such that $F(D_A, D_B) = 0$. (Bernstein expressed this somewhat differently, I think.) An important fact is that this condition on the connection can be solved locally, without imposing any differential equations (but reducing the number of independent functions in the description of the connection).

Let $\mathcal{D}_A$ be the gauge-covariant extension of D_A and $\mathcal{D}_{AB}$ the gauge-covariant exension of $\partial/\partial y^{AB}$.

The basic curvature invariant is then

$$\lambda_A = \epsilon^{BC}\left[\mathcal{D}_A, \mathcal{D}_{BC}\right]. \tag{5}$$

Show that the whole curvature can be expressed in terms of λ and its covariant derivatives.

The basic gauge theory Lagrangian is then

$$L_2 = \int d^3y\, d^2\theta\ \epsilon^{AB}\mathrm{Tr}\,\lambda_A\lambda_B. \tag{6}$$

Here Tr should be understood simply as a gauge-invariant quadratic form on the Lie algebra.

(a) Show that this Lagrangian has for its bosonic reduction ordinary Yang-Mills theory. Identify what the fermions are. (Hint: you might use the relation between $d\theta^A$ and $\mathcal{D}_A$.)

You should see that this model is equivalent to one of the simplest models in three dimensions of Yang-Mills plus fermions; we might have studied it as a simple example of such without noticing it was supersymmetric.

What is the conserved supercurrent?

(b) Suppose that G acts on the manifold X considered in (1) above and reinterpret Φ as a section of $X \times_G P$. L_0 and L_1 are defined by the same formulas as in problem (1) above.

Show that, in terms of ordinary fields, L_0 and L_1 are now the obvious gauge-covariant extensions of what you had before.

(3) Now we move on to four bose dimensions. We cannot work in $\mathbf{R}^{(4|2)}$ because although $SO(3,1)$ has two-dimensional spinor representations, they are not real; we would not be able to get real Lagrangians in Minkowski space.

So we introduce spinors of both kinds and work on $\mathbf{R}^{(4|4)}$. The following will be written in Euclidean signature without imposing any reality condition. We write θ^A, $A = 1, 2$ and

$\bar{\theta}^{\dot{A}}$, $\dot{A} = 1, 2$ for the two kinds of spinors[1] and $x^{A\dot{A}}$ for the bose coordinates. Under the action of $SO(4) \cong SU(2) \times SU(2)$, the undotted indices A, B, C transform in the two-dimensional representation of the first $SU(2)$ and the dotted indices $\dot{A}, \dot{B}, \dot{C}$ transform in the two-dimensional representation of the third.

The supersymmetry generators are

$$\begin{aligned} Q_A &= \frac{\partial}{\partial \theta^A} + i\bar{\theta}^{\dot{A}} \frac{\partial}{\partial x^{A\dot{A}}} \\ \bar{Q}_{\dot{A}} &= \frac{\partial}{\partial \bar{\theta}^{\dot{A}}} + i\theta^{A} \frac{\partial}{\partial x^{A\dot{A}}} \end{aligned} \tag{7}$$

and commute with

$$\begin{aligned} D_A &= \frac{\partial}{\partial \theta^A} - i\bar{\theta}^{\dot{A}} \frac{\partial}{\partial x^{A\dot{A}}} \\ \bar{D}_{\dot{A}} &= \frac{\partial}{\partial \bar{\theta}^{\dot{A}}} - i\theta^{A} \frac{\partial}{\partial x^{A\dot{A}}} \end{aligned} \tag{8}$$

which are used in writing Lagrangians.

Now, the most obvious kind of field, imitating what we did in (1) above, would appear to be a field Φ which is simply a function on $\mathbf{R}^{(4|4)}$. This is called an ordinary superfield.

However, *it is impossible to write a sensible Lagrangian for such a field*, where again sensible means that the bosonic part of the energy is positive.

(a) Write some supersymmetric Lagrangians for an ordinary superfield and show that (if you specialize to Lorentz signature) the bosonic part of the energy is not positive.

Instead, we use a *chiral superfield* Φ, which is a function on $\mathbf{R}^{(4|4)}$ such that

$$\overline{D}_{\dot{A}}\Phi = 0. \tag{9}$$

Likewise $\overline{\Phi}$ is called an antichiral superfield if

$$D_A\overline{\Phi} = 0. \tag{10}$$

If X is a complex manifold, a chiral map from $\mathbf{R}^{(4|4)}$ to X is a map such that local holomorphic functions on X pull back to chiral superfields, and antiholomorphic functions pull back to antichiral superfields.

[1] $\bar{\theta}^A$ and $\theta^{\dot{A}}$ will never appear (except as a result of a possible misprint - in that case remove bars from undotted θ's and add them to dotted ones). In Lorentz signature if one imposes a reality condition, then $\bar{\theta}^{\dot{A}}$ is the complex conjugate of θ^A. In Euclidean signature there is no reality condition and no relation between these two. Similar statements hold for other objects introduced below such as the W's.

If K is a real analytic function on X, and $\Phi : \mathbf{R}^{(4|4)} \to X$ is a chiral map, then one defines a Lagrangian

$$\mathcal{L}_0 = \int d^4x\, D^4\theta \Phi^*(K). \tag{11}$$

(I will use $\mathcal{L}$ for four-dimensional Lagrangians, and L for somewhat similar three-dimensional ones.)

(b) Show that this is invariant under $K \to K + f + \overline{f}$ where f is a holomorphic function on X.

(c) Show that $\mathcal{L}_0$ is equivalent to an ordinary sigma model Lagrangian on $\mathbf{R}^4$, with target X – the metric on X being a Kahler metric with K as the Kahler potential – plus fermion terms. Describe the fermion terms.

Notice that $\mathcal{L}_0$, after doing all (or as you should be able to see, just half) of the θ integrals, makes sense globally for maps to any Kahler manifold X – even though the description here required the choice, and in particular the existence, of a Kahler potential, which of course exists locally but may not exist globally. Problem: find a good and economical global construction of $\mathcal{L}_0$. (To some extent this can be done using gauge theory as developed below.)

(d) Show that this theory has a $U(1)$ symmetry (classically - it may be anomalous) coming from a symmetry of $\mathbf{R}^{(4|4)}$ that acts only on the odd coordinates and acts by an outer automorphism of the super-Poincaré algebra.

A symmetry that acts by outer automorphisms of super-Poincaré and in superspace acts only on the odd coordinates is called an R symmetry, a term one will often encounter.

(e) Show that if one imposes invariance under one translation in $\mathbf{R}^4$ (by a vector that is not a null vector – if you are doing Lorentz signature it should be a spatial translation, here and in other cases below), $\mathcal{L}_0$ reduces to L_0 as considered in (1) above – for the special case that the target is actually Kahler.

It is therefore true, given also results from problem sets 1-3 in term one, that if one imposes invariance under three translations in $\mathbf{R}^4$, to reduce to $\mathbf{R}^{(1|4)}$, one gets the Lagrangian whose quantization gives Hodge theory of differential forms on X.

Deduce that there is an action of $SU(2) \times U(1)$ on differential forms on a Kahler manifold, commuting with the Hamiltonian, where the $U(1)$ was seen in (d) and the $SU(2)$ comes from spatial rotations of $\mathbf{R}^4$.

(f) The following is something that we will need to know about if we really want to do Donaldson theory.

Suppose that X is hyper-Kahler. Then the Lagrangian $\mathcal{L}_0$ has twice as much supersymmetry as for a Kahler X.

It can be shown comparatively easily as follows:

After writing down "in components" the sigma model action, show that the $U(1)$ found in (d) actually extends to $SU(2)$ in case X is hyper-Kahler. (The $SU(2)$ only acts on fermions. This should be obvious - there is no universal action of $SU(2)$ on a hyper-Kahler manifold.)

This $SU(2)$ does *not* act by outer automorphisms on the super-Poincaré algebra of $\mathbf{R}^{(4|4)}$. Can you describe what algebra is generated by the $SU(2)$ together with the automorphisms of $\mathbf{R}^{(4|4)}$? Hint: it can be naturally realized geometrically as a supergroup of automorphisms of an $\mathbf{R}^{(4|8)}$. It is called the $N = 2$ supersymmetry algebra in four dimensions, another term that is likely to be encountered frequently.

Unfortunately, there is no reasonable way known to make a superspace construction starting from $\mathbf{R}^{(4|8)}$. That is why I asked you to construct this $N = 2$ theory by looking at it as an $N = 1$ theory that just happens to have extra supersymmetry. The above computation is thus analogous to the way that mathematicians usually find the extra symmetries of Hodge theory on a Kahler manifold "by hand," not by starting with a geometrical construction of maps from $\mathbf{R}^{(1|4)}$.

(g) Deduce from the above that there is an action of $SU(2) \times SU(2)$ on the differential forms on a hyper-Kahler manifold, commuting with the Hamiltonian. Actually, this extends to an action of $Sp(2) = Spin(5)$. That is because there is a supersymmetric model in six dimensions in which the data consist of maps of $\mathbf{R}^6$ to a hyper-Kahler manifold plus fermions. This theory is one of the limiting cases permitted by Nahm's theorem (that is, six is the maximum dimension for sigma models according to Nahm's theorem). It does not have a known superspace construction.

Write down the Lagrangian for this model by using the fact that it must reduce in four dimensions to a Lagrangian found above. (For extra credit do the following not very hard exercise: Verify supersymmetry of this Lagrangian.)

(4) Going back to the case that X is only Kahler, let W be a holomorphic function on X – called the superpotential.

Let

$$\mathcal{L}_1 = \int d^4x\, d^2\theta \Phi^*(W) + \int d^4x\, d^2\overline{\theta} \Phi^*(\overline{W}). \tag{12}$$

(a) Show that this is supersymmetric.

(b) Reduce to an ordinary Lagrangian and compute the potential energy function on X and fermion mass terms.

(c) Upon imposing invariance under translation of one coordinate, show that $\mathcal{L}_1$ becomes equivalent to L_1 as constructed in 1(b) above with $h = \mathrm{Re}(W)$.

(d) Under what conditions on X and W does the R symmetry – the outer automorphism of super-Poincaré that comes from a symmetry of $\mathbf{R}^{(4|4)}$, as found above, lift to a symmetry of this theory? (That is, a classical symmetry - quantum mechanically it may be anomalous.)

(5) Now we want to do gauge theory. Once again, we introduce a compact connected Lie group G and G bundle P over $\mathbf{R}^{(4|4)}$. A is a connection on P and F is the corresponding curvature.

One can write all kinds of supersymmetric Lagrangians for such a general connection, but not of much interest – the usual classical pathology is non-positive energy for the bosons.

For the good construction, one restricts A by requiring the vanishing of some components of the curvature, in fact $F(D_A, D_B) = F(D_A, \overline{D}_{\dot{A}}) = F(\overline{D}_{\dot{A}}, \overline{D}_{\dot{B}})$. (Or more succinctly, F vanishes when evaluated on any two fermionic D's)

(a) Show that these conditions can be solved locally, without differential equations.

(b) With these conditions, the basic curvature invariants are

$$W_A = \epsilon^{\dot{B}\dot{C}}[\overline{\mathcal{D}}_{\dot{B}}, \mathcal{D}_{A\dot{C}}], \tag{13}$$

and

$$\overline{W}_{\dot{A}} = \epsilon^{BC}[\mathcal{D}_B, \mathcal{D}_{C\dot{A}}]. \tag{14}$$

where as before $\mathcal{D}$ is the gauge-covariant extension of D.

Show that W_A is a chiral superfield (with values in one of the two spin bundles[2] and $\overline{W}_{\dot{A}}$ is antichiral, and that all components of the curvature can be expressed in terms of the W's and their covariant derivatives by using the Bianchi identity.

(c) The basic Lagrangian is

$$\mathcal{L}_2 = \tau \int d^4x\, d^2\theta \epsilon^{AB} \mathrm{Tr} W_A W_B + \overline{\tau} \int d^4x\, d^2\overline{\theta} \epsilon^{\dot{A}\dot{B}} \mathrm{Tr} \overline{W}_{\dot{A}} \overline{W}_{\dot{B}}. \tag{15}$$

[2] In this flat model the spin bundle has a flat connection. A chiral superfield with values in a flat bundle is defined in the obvious way.

τ is a complex conjugate and (to get a real Lagrangian in Lorentz signature) $\overline{\tau}$ is its complex conjugate.

Reduce this to an ordinary Lagrangian. It is equivalent to a simple special case of four dimensional gauge theory coupled to fermions. Which special case?

In order to do this, you may have made contact with the formalism in Wess and Bagger, chapter 6, where a field that they call V is introduced. It will probably be necessary to make contact with that in order to solve an important part of a later exercise.

(d) Show that this theory at the classical level possesses the R symmetry that was discussed in the previous exercise.

Now the one quantum question in these exercises. Using your knowledge of determinant line bundles, do you believe that this R symmetry is valid quantum mechanically?

If you claim that there is an anomaly breaking the $U(1)$ R symmetry to a finite subgroup of $U(1)$, can you identify the subgroup in question?

(e) If you impose invariance under one of the spatial translations, this theory does *not* reduce to L_2 as formulated in the three dimensional case in problem 2 above. However, it does reduce to one of the problems of gauge theory plus matter formulated in problem 2 (that is, a Lagrangian of the form $L_0 + L_2$). Can you identify which special case?

(6) Now we want to consider a Kahler manifold X with a G action. To begin with, we suppose that $X = \mathbf{C}^n$ for some n with a linear action of G on X, leaving fixed the origin. We pick a G-invariant Kahler potential K on X.

We consider a theory in which the data are a connection A on a G bundle P (obeying the curvature constraints discussed in the last problem) and a chiral section Φ of $X \times_G P$.

(a) Explain what it means for Φ to be chiral.

(b) Convince yourself that the Lagrangian

$$\mathcal{L}_0 = \int d^4x d^4\theta \, \Phi^*(K) \tag{16}$$

makes sense in this situation and is, in fact, equivalent to the obvious gauge covariant extension of $\mathcal{L}_0$ as defined above.

(c) Consider the Lagrangian $\mathcal{L}_0 + \mathcal{L}_2$ (with $\mathcal{L}_2$ as defined above). Determine the moduli space of classical states of minimum energy. You should meet some notions of "symplectic quotient" (related to geometric invariant theory) and you should find that the moment map you get vanishes at the origin in $\mathbf{C}^n$.

In fact, it is possible (in case G has $U(1)$ factors – if G is semi-simple the following generalization is vacuous) to add an additional supersymmetric interaction that adds a constant to the moment map.

In the notation of Wess and Bagger, for $G = U(1)$, the additional interaction is

$$\mathcal{L}_3 = r \int d^4x \, d^4\theta \; V. \tag{17}$$

Here r is a real constant. ($\mathcal{L}_3$ is commonly called the Fayet-Iliopoulos term or D term and is introduced by Wess and Bagger starting on p. 52.) Show that this is supersymmetric (a trivial consequence of what in my edition of Wess and Bagger is eqn. 6.4). I do not know in four dimensions a more intrinsic way to write (17) – my apologies for using a formula that depends on Wess-Zumino gauge (which leads to the definition of V). After reduction to two dimensions there is a nice formula that we will see later.

Show that replacing $\mathcal{L}_0 + \mathcal{L}_2$ by $\mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_3$ has the effect of shifting the moment map by a constant r. The space of classical zeroes of the energy (or supersymmetric vacua) is the symplectic quotient with the new moment map. Of course, if G has several $U(1)$ factors, we add a term like (17) for each such factor, so we can get a general moment map.

(e) The global generalization of the above is as follows. Let X be any Kahler manifold with G action (and Kahler metric invariant under G) and such that there is a moment map for the G action; pick a moment map. For each such choice, an appropriate global version of $\mathcal{L}_0 + \mathcal{L}_3$ exists. (There is no canonical $\mathcal{L}_0$ just as in ordinary geometry there is no canonical choice of a moment map. There is the canonical family of $\mathcal{L}_0 + \mathcal{L}_3$'s with the variable constant r in the definition of $\mathcal{L}_3$.) The verification of this is in a paper by Bagger and Witten (which will be available from Paula Bozzay or Val Nowak). I do not know a natural global construction in superspace. (The above is not even a universal local construction since in identifying X at least locally with $\mathbf{C}^n$ with linear action of G we assumed that G has a fixed point. The supersymmetric theory exists for any G action on X, even if there is no fixed point, as long as there is a moment map.)

(f) Now, let W be a G-invariant holomorphic function on X and consider the superpotential term $\mathcal{L}_1$. Convince yourself that $\mathcal{L}_1$ makes sense in this gauge theory setting and describe the moduli space of classical vacua in the theory with Lagrangian $\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3$ (the inclusion of the $\mathcal{L}_3$ term with a constant r in $\mathcal{L}_3$ just means, of course, that you are to work with a general moment map).

(g) (We will need this when we get to Donaldson theory.) Consider the case that G is n-dimensional and $X = \mathbf{C}^n$ with G acting on X via the adjoint representation. Consider the Lagrangian $\mathcal{L}_0+\mathcal{L}_2$. By writing it out in terms of ordinary fields, show that the $U(1)$ R symmetry extends to to an $SU(2)$ symmetry which does *not* act by outer automorphisms of super-Poincaré, so that super-Poincaré is extended to a super-algebra with twice as many odd generators. (In fact, in this case, unlike the sigma model case, there is a rather nice realization via gauge theory on $\mathbf{R}^{(4|8)}$.)

More generally, if $X=\mathbf{C}^n\times Y$, where (i) G acts on $\mathbf{C}^n$ via the adjoint representation; (ii) Y is hyper-Kahler; (iii) G acts on Y preserving the metric and with a hyper-Kahler moment map, then (for every choice of all this data including the moment map) there is a gauge-invariant sigma model with the data being the connection A and a section of $X\times_G P$. One again needs this generalization for Donaldson theory; however, I won't explain it here. (The moduli space of classical supersymmetric vacua is the hyper-Kahler quotient of Y by G, which therefore must naturally be a hyper-Kahler manifold; that is how the hyper-Kahler quotient was discovered.)

(7) The supersymmetric theories that I will lecture about in two dimensions are actually all obtained by dimensional reduction from $\mathbf{R}^{(4|4)}$, that is, by imposing invariance under two translations to get a theory on $\mathbf{R}^{(2|4)}$. However, some properties that appear upon reduction to $\mathbf{R}^{(2|4)}$ deserve special discussion.

First of all, the rotation of the two coordinates that are removed in going from four to two dimensions acts as an outer automorphism of the two-dimensional super-Poincaré. Since there already was such an outer automorphism in four dimensions, there are two of them in two dimensions, making up a group $U(1)\times U(1)$ of R symmetries. (It is not the case that all the theories we can write admit the full $U(1)\times U(1)$ even classically, since in four dimensions we had the possibility of breaking the R symmetry classically with the choice of W.)

The bosonic coordinates in two dimensions are, say, x^+ and x^-, of weights 1 and -1 under Lorentz boosts. The fermionic coordinates consist of two of weight $1/2$ and two of weight $-1/2$.

(a) Show – just using the geometrical properties of $\mathbf{R}^{(4|4)}$ – that of the R symmetry group $U(1)\times U(1)$ in two dimensions, one $U(1)$ acts only on the weight one half odd coordinates and the other $U(1)$ acts only on the weight minus one half odd coordinates.

I label the weight one half odd coordinates as θ^+, $\overline{\theta}^+$ where one of the two $U(1)$'s in the R symmetry group acts by $\theta^+\to e^{i\alpha}\theta^+$, $\overline{\theta}^+\to e^{-i\alpha}\overline{\theta}^+$. Likewise I label the weight minus one half odd coordinates as θ^-, $\overline{\theta}^-$ with analogous action of the second $U(1)$. (As you know well, in Lorentz signature the barred objects are complex conjugates of the unbarred ones, but in Euclidean signature as always there is no reality condition.)

For the supersymmetries I write

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial\theta^+} + i\overline{\theta}^+ \frac{\partial}{\partial x^+} \\ \overline{Q}_+ &= \frac{\partial}{\partial\overline{\theta}^+} + i\theta^+ \frac{\partial}{\partial x^+}. \end{aligned} \tag{18}$$

Thus $Q_+^2 = \overline{Q}_+^2 = 0$ and $\{Q_+, \overline{Q}_+\}$ is a multiple of $\partial/\partial x^+$. Q_- and $\overline{Q}_-$ are defined in the same way with $+ \to -$.

The supersymmetries commute with vector fields D_+, $\overline{D}_+$, D_-, $\overline{D}_-$ that are defined by the same formulas with $i \to -i$. (Or if you've suppressed the i's, then reverse the sign of $\partial/\partial x^\pm$.)

Mirror symmetry is the automorphism of $\mathbf{R}^{(2|4)}$ that exchanges θ^- and $\overline{\theta}^-$ while doing nothing to other odd or even coordinates. Thus it reverses one of the R symmetries and commutes with the other. More precisely, two field theories defined in $\mathbf{R}^{(2|4)}$ are said to be mirrors if the stated automorphism of $\mathbf{R}^{(2|4)}$ lifts to an isomorphism between the given theories.

(b) A chiral superfield is a function Φ on superspace such that $\overline{D}_+\Phi = \overline{D}_-\Phi = 0$. An antichiral superfield $\overline{\Phi}$ obeys $D_\pm\overline{\Phi} = 0$.

A twisted chiral superfield is a function Σ on superspace such that $\overline{D}_+\Sigma = D_-\Sigma = 0$. A twisted antichiral superfield $\overline{\Sigma}$ obeys $D_+\overline{\Sigma} = \overline{D}_-\overline{\Sigma} = 0$.

The sigma model of a chiral superfield is our friend

$$\mathcal{L}'_0 = \int d^2x\, d^4\theta\, K(\Phi, \overline{\Phi}) \tag{19}$$

and the sigma model of a twisted chiral superfield is defined exactly analogously. One can also straightforwardly define sigma models containing both chiral and twisted chiral superfields, but we will not study them.

Likewise, if W is a holomorphic function of a chiral superfield, we have

$$\mathcal{L}'_1 = \int d^2x d^2\theta \Phi^*(W) + \int d^2x d^2\overline{\theta} \Phi^*(\overline{W}), \tag{20}$$

as in four dimensions.

(c) Some novelties occur if one considers gauge theory. We want to consider gauge theory with chiral superfields. For this, we consider a connection A on G bundle P such that the curvature F obeys $F(u,v)=0$ where u,v are either both linear combinations of D_+ and D_- or both linear combinations of $\overline{D}_+$ and $\overline{D}_-$. (If we wanted to use twisted chiral superfields, then we would pair D_+ and $\overline{D}_-$ in the constraint on the curvature.)

For a connection of this kind, the basic curvature invariants are $\Sigma = F(\overline{D}_+, D_-)$ (or equivalently $\Sigma = \{\overline{\mathcal{D}}_+, \mathcal{D}_-\}$ where the $\mathcal{D}$'s are gauge-covariant extensions of the D's) and $\overline{\Sigma} = F(D_+, \overline{D}_-)$.

Show that all components of the curvature can be expressed in terms of Σ, $\overline{\Sigma}$, and their covariant derivatives.

Show that Σ is a twisted chiral superfield and that $\overline{\Sigma}$ is a twisted antichiral superfield.

(d) The basic Lagrangian for the gauge multiplet (that is, the "supermultiplet" of fields that includes the connection) is

$$\mathcal{L}_2' = \frac{1}{4e^2}\int d^2x d\theta^+ d\overline{\theta}^- \mathrm{Tr}\Sigma^2 + \frac{1}{4e^2}\int d^2x d\theta^- d\overline{\theta}^+ \mathrm{Tr}\overline{\Sigma}^2. \tag{21}$$

Here Tr is to be understood as any invariant quadratic form on the Lie algebra.

Either by reducing this to an ordinary Lagrangian or in any other way of your choosing, show that this is equivalent to the reduction to two dimensions of our four dimensional Lagrangian $\mathcal{L}_2$.

In particular, it describes gauge fields, fermions, *and scalars*, in contrast to the three and four-dimensional gauge actions L_2 and $\mathcal{L}_2$ constructed earlier, which describe gauge fields and fermions only.

(e) This will be very important in applications. If $G = U(1)$, one has the additional possible term

$$\mathcal{L}_3' = r\int d^2x d\theta^+ d\overline{\theta}^- \Sigma + r\int d^2x d\theta^- d\overline{\theta}^+ \overline{\Sigma} \tag{22}$$

with r a constant. More generally, for any G one has such a term for any $U(1)$ factor in G. In other words, Σ can be replaced by any gauge-invariant linear function of the Σ"s. (One can more generally use any gauge invariant holomorphic function of the Σ"s, obviously.)

$\mathcal{L}_3'$ arises by dimensional reduction from the four-dimensional interaction $\mathcal{L}_3$, but as you can see, in two dimensions a much nicer and more intrinsic description is possible.

This also enables one to give a nicer global description of the sigma model of maps to a Kahler manifold in two dimensions, at least if the Kahler class is integral.

The supersymmetric Lagrangians we will study in two dimensions will be special cases of $\mathcal{L}_0' + \mathcal{L}_1' + \mathcal{L}_2' + \mathcal{L}_3'$.