Addendum To Superhomework

The discussion in and after the last lecture suggested that the following exercises on representations of super-Poincaré in four dimensions would be helpful.

Recall that unitary irreducible representations of super-Poincaré are constructed from certain equivariant vector bundles supported on a cone or hyperboloid in momentum space. The equivariant vector bundles in question are classified by representations of the subgroup H_p of super-Poincaré that leaves fixed a chosen vector p on the cone or hyperboloid. Moreover, one can replace H_p by its maximal reductive subgroup, whose bosonic part is $Spin(2)$ if we are on the cone and $Spin(3)$ if we are on the hyperboloid.

The $Spin(2)$ Lie algebra has one generator which we call J; its eigenvalues are in $\mathbf{Z}/2$. The $Spin(3)$ Lie algebra has one representation R_n of every positive dimension n.

The $N = 1$ super-Poincaré Lie algebra in four dimensions is (with notation as in the superhomework)

$$\{Q_A, Q_{\dot{A}}\} = P_{A\dot{A}} \tag{1}$$

with other anticommutators vanishing and Poincaré acting on the odd generators Q in the usual fashion.

Upon fixing a vector on the cone or hyperboloid, the $P_{A\dot{A}}$ become numbers, say $p_{A\dot{A}}$. Unitary representations of (1) are analyzed in a fashion we should know from last fall: Q's that are in the kernel of the quadratic form in (1) are represented by zero, and the remaining Q's generate a Clifford algebra.

(1) Show that if p is lightlike, the quadratic form in (1) has a two-dimensional null space, and that quantization of the Clifford algebra gives a two-dimensional representation with J eigenvalues $j, j + 1/2$, for arbitrary $j \in \mathbf{Z}/2$. We will call the $0, 1/2$ and $-1/2, 0$ representations the chiral representations C_+ and C_- and the $1/2, 1$ and $-1, -1/2$ representations the vector representations V_+ and V_-.

Show that quantization of a chiral superfield gives a sum of C_- and C_+ for the one particle states, and quantization of a vector superfield gives similarly a sum of V_+ and V_-.

(2) Show that if p is timelike, the quadratic form is nondegenerate. Its irreducible representation is four-dimensional and is (as a $Spin(3)$ respresection) $C' = R_1 \oplus R_1 \oplus R_2$. An irreducible representation of H_p is then $C' \otimes R$ where R is any irreducible representationa of $Spin(3)$.

We call C' the massive chiral representation and we call $V' = C' \otimes R_2$ the massive vector representation.

(3) Show that when a null vector p is perturbed to be timelike, $C_+ \oplus C_-$ can be deformed to C'. Now do a bit of field theory: show that the representation theory fact just stated corresponds to the fact that it is possible to add a bare mass for a massless chiral superfield.

(4) The analogous statement for the vector representation is a bit more complex (as we should expect from our study of the Higgs mechanism in the bosonic case). Show that the representation $V_+ \oplus V_- \oplus C_+ \oplus C_-$ can be perturbed to V'.

Now show how this can be realized in field theory.[1] Consider a theory with a vector superfield and a chiral superfield Φ of charge 1. Take the Lagrangian to be the kinetic energy for both superfields plus the "Fayet-Iliopoulos D-term"

$$r \int d^4x D \tag{2}$$

(related to the constant term in the moment map) discussed in the superhomework.

Show that (a) for $r = 0$, we get the representation $V_+ \oplus V_- \oplus C_+ \oplus C_-$ at tree level, (b) for $r \neq 0$ one gets at tree level depending on the sign of r either supersymmetry breaking, or a supersymmetric vacuum with mass gap and representation V'. The occurrence of the latter is called the super-Higgs effect.

(5) Now we consider the $N = 2$ supersymmetry algebra. In the absence of a central term, the algebra reads

$$\{Q^I_A, Q_{\dot{A}J}\} = \delta^I_J P_{A\dot{A}}, \quad I, J = 1, 2 \tag{3}$$

with other anticommutators vanishing.

Classify the unitary representations as follows:

(a) In the massless case, the "helicities" or J eigenvalues in an irreducible representation are $j, j + 1/2, j + 1/2$, and $j + 1$ for any $j \in \mathbf{Z}/2$.

(b) In the superhomework you met two kinds of $N = 2$ fields. Show that the massless hypermultiplet gives on quantization a representation H which is the sum of two copies of the $j = -1/2$ representation (helicities from $-1/2$ to $+1/2$; there are two copies because of the quaternionic structure of the hypermultiplet) while the vector multiplet gives a

[1] I write the following in a slightly naive way; the model presented here is actually anomalous. One can add additional fields to cancel the anomaly without affecting the occurrence of the super-Higgs mechanism. (The supersymmetry breaking that occurs in one case in the text may be destroyed.)

representation V that is the sum of a representation with $j = -1$ (helicities from -1 to 0) and $j = 0$ (helicities from 0 to 1). We call the $j = -1/2$ representation H_0.

(c) In the massive case, the basic representation H', obtained by quantizing the non-degenerate Clifford algebra in (3), has helicities ranging from -1 to 1 with multiplicities $1, 4, 6, 4, 1$. [2] Any irreducible unitary representation of the little group H_p is $H' \otimes R$ with R an irreducible representation of $Spin(3)$.

The above facts might seem to imply, for instance, that a bare mass for the hypermultiplet is impossible (why?) but it is in fact possible to add a supersymmetric bare mass term. In the presence of the bare mass, the $N = 2$ super-Poincaré is deformed to the most general possibility allowed by the Haag-Sohnius-Lopuszanski theorem (the subject of Bernstein's first lecture). In the process, the R symmetry group of outer automorphisms, isomorphic to $U(2)$ (and discussed in the superhomework), is reduced to $SU(2)$.

Concretely, the deformation of the algebra is as follows. (3) is unchanged, but instead of the other anticommutators vanishing one has

$$\{Q_A^I, Q_B^J\} = \epsilon^{IJ}\epsilon_{AB}Y, \tag{4}$$

where the ϵ's are the antisymmetric tensors invariant under the action of Poincaré and of the $SU(2)$ group of R symmetries. Y is a "central charge," that is an operator that commutes with the whole super-Poincaré, including its adjoint $\overline{Y}$ which appears in the remaining anticommutator:

$$\{Q_{\dot{A}I}, Q_{\dot{B}J}\} = \epsilon_{AB}\epsilon_{IJ}\overline{Y}. \tag{5}$$

(6) Consider a representation of mass M and $Y = y$ (y is a complex number; as Y is central, in an irreducible representation it can be set to a constant). We have already analyzed the representations for $y = 0$. Show that more generally the representations with $M > |y|$ are similar (for instance, deformable to) the massive representations with $y = 0$ and the representations with $M = |y|$ are likewise similar to the representations with $M = y = 0$.

(7) There are two important cases of this in practice. One is the hypermultiplet, for which it is in fact possible to add a bare mass. Recall from the superhomework that a single hypermultiplet has for its bosonic fields a map to the flat hyper-Kahler manifold $\mathbf{H} = \mathbf{R}^4$. $\mathbf{H}$ has a group $SU(2)$ of symmetries that preserve the hyper-Kahler structure

[2] In the massive case, by helicity we mean the weight under the generator of a maximal torus of $Spin(3)$.

(and so commute with super-Poincaré). Show that it is possible to add a bare mass, in such a way that Y is equal to an arbitrary element of the $SU(2)$ Lie algebra. (This is easy if you have the right frame of mind, but perhaps not otherwise. In any event it is less immediately necessary than the results stated in the next exercise, which are essential for doing Donaldson theory.)

(8) The other important case is the vector multiplet. First we have to discuss the Higgs mechanism for $N = 2$. We recall that the bosonic fields of a vector multiplet for gauge group G are a connection A and a complex scalar ϕ in the adjoint representation. There may also be charged scalars that transform in hypermultiplets; this means that one has for bosonic fields not only the vector multiplet but also a map from Minkowski space to a hyper-Kahler manifold X (on which G acts).

There are two basic cases of the Higgs mechanism for $N = 2$. We describe these in a language that is tied to weak coupling (so we can speak informally of the expectation value of a charged field):

(i) The scalar fields that have vacuum expectation values are all in the vector multiplet; this means that the vacuum is determined by a point in X that is G-invariant.

(ii) The vacuum corresponds to a non-trivial orbit in the action of G on X, for instance a free orbit.

In this context:

(a) Consider case (i), which is really essential for Donaldson theory. You may as well assume there are no hypermultiplets. Then because of the term $\mathrm{Tr}[\phi,\bar{\phi}]^2$ term in the potential (recall the superhomework) the unbroken subgroup of G has the same rank as G and is in fact, for generic ϕ, a maximal torus. By going from $\phi = 0$ to $\phi \neq 0$, most of the representations get mass, but the representations do not change in size; the massive representations we get are obviously representations with $M = |Y|$, as those are the massive representations to which the massless vector representation can be deformed. The relevant Y is computed in the Olive-Witten paper that was distributed previously.

(b) In this same situation, magnetic monopoles, which arise from solutions of the BPS equation $F = *D\phi$, are likewise in "small" representations of supersymmetry. This is shown as follows: (1) The classical solution is invariant under half of the Q's. (2) The other half of the Q's give fermion zero modes, four of them to be precise. There are no other fermion zero modes. (3) Quantization of four fermion zero modes gives a four-dimensional Clifford algebra with a four-dimensional irreducible representation, which is what we called above representation H_0. The monopole *of fixed electric charge*[3] transforms in the representation H_0. The antimonopole (obtained by quantizing the solution of $F = - * D\phi$, and with opposite electric charge) gives another copy of H_0. Together they make the hypermultiplet representation H; this will be important in Donaldson theory.

(c) Now show that a Higgs effect of type (ii) gives a representation of $M > |Y|$. To show that this is so generically it is enough to consider a special case, and I would recommend working through in detail the following example. Take $G = U(1)$, with a linear action on the quaternions $\mathbf{H}$ (preserving the hyper-Kahler structure) and let $X = \mathbf{H}^2$. A Higgs mechanism is possible, since the hyper-Kahler quotient of X by $U(1)$ is non-trivial and in fact is four-dimensional. (What is it?) For a generic classical vacuum, show that the spectrum at tree level is as follows: a massless hypermultiplet and a copy of the basic massive $Y = 0$ representation H'. We recall that the latter has helicities running from -1 to 1 with multiplicities $1, 4, 6, 4, 1$. (As a $Spin(3)$ representation it is the sum of R_3, four copies of R_2, and five copies of R_1.)

[3] We have seen in one of the lectures that monopoles have an "arbitrary" electric charge. We recall that the electric charge of the monopole is obtained by quantizing a bosonic zero mode.