Lecture II-10: Quantum gauge theories in two dimensions and intersection theory on moduli spaces

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10.1. The partition function in two dimensional Yang-Mills theory.

We consider gauge theories in two dimensions with a simple gauge group G. The spacetime is a compact Riemann surface Σ of genus g with no boundary. To apply our methods to intersection theory on moduli spaces, we shall need to consider the case $G = SU(n)/\mathbb{Z}_n$ and consider bundles on Σ for which the transition functions do not lift to $SU(n)$: in mathematical terms we are considering the moduli space of holomorphic vector bundles on Σ with rank n and degree d.

The theory we consider is *physical Yang-Mills theory.* The fields are a G connection A on a G bundle P on Σ, with curvature F_A; the Lagrangian is

$$\mathcal{L} = \frac{1}{4e^2}\int d^2x \left(| * F_A|^2 \right). \tag{1}$$

The partition function is thus

$$Z = \frac{1}{\mathrm{Vol}(G)}\int DA e^{-\mathcal{L}}. \tag{2}$$

We introduce a scalar field ϕ with values in $\mathrm{ad}(P)$ and rewrite the path integral for the partition function as

$$Z = \frac{1}{\mathrm{Vol}(G)}\int DAD\phi \exp\{i\int \mathrm{Tr}\phi F - \frac{e^2}{2}\int d\mu \mathrm{Tr}\phi^2\}. \tag{3}$$

This theory is invariant under area preserving diffeomorphisms since it has no explicit dependence on a Riemannian metric on Σ but only on the *measure* $d\mu = *(1)$. It is clear that the theory depends only on the coupling constant e and the area $\mathbf{a}$ of Σ only through the combination $e^2\mathbf{a}$. The path integral (3) is well behaved under taking the limit as $e \to 0$.

We make a further modification to the path integral for the partition function by introducing a fermionic variable ψ which is a one-form in the adjoint representation (in other words ψ is an element of of $\Omega^1(\Sigma)\otimes\Gamma(\mathrm{ad}P)$.

The field ψ should be thought of as lying in the tangent space to the space $\mathcal{A}$ of connections on Σ. The path integral becomes

$$Z = \frac{1}{\mathrm{Vol}G} \int DAD\phi D\psi \exp\{ i \int_\Sigma \mathrm{Tr}(\phi F) + \frac{1}{2} \int \mathrm{Tr}(\psi \wedge \psi) - \frac{e^2}{2} \int d\mu \mathrm{Tr}(\phi^2)\}. \tag{4}$$

Since the Lagrangian in (4) contains no terms which involve both ψ and the other fields ϕ and A, we can integrate out ψ and recover the earlier expression (3).

We can now define a supersymmetry operation δ on the space of fields:

$$\delta A = i\psi, \tag{5}$$

$$\delta\psi = -d_A\phi \tag{6}$$

$$\delta\phi = 0 \tag{7}$$

It follows that $\delta^2 A = -id_A\phi$, in other words $\delta^2 = 0$ up to the action of a gauge transformation. (Here, d_A refers to the de Rham differential d twisted by a connection A on Σ.) We may check the invariance of the action under δ: we have

$$\delta \int \mathrm{Tr}(\phi F) = -i \int \mathrm{Tr}(\phi D_A \psi), \tag{8}$$

while

$$\delta \int \mathrm{Tr}(\psi \wedge \psi) = - \int \mathrm{Tr}(D_A\phi \wedge \psi). \tag{9}$$

After integrating by parts, we find that the action appearing in (4) is invariant.

10.2. A finite dimensional analogue: the Cartan model.

Our path integral is a path integral over the infinite dimensional space $\mathcal{A}$ of connections which is acted on by the gauge group $\mathcal{G}$ with Lie algebra Lie($\mathcal{G}$). The *Cartan model* is used to treat a finite dimensional analogue of this situation: for a more detailed description see Chapter 7 of [**BGV**]. The analogue of $\mathcal{A}$ is a finite dimensional manifold M equipped with the action of a compact Lie group H (which is the analogue of $\mathcal{G}$). Functions of A and ψ correspond to differential forms on M, while the analogue of functions of A, ψ and ϕ are elements of $\Omega^*(M) \otimes \mathrm{Fun}(\mathbf{h})$. In fact we restrict to the H-invariant subspace $\left(\Omega^*(M) \otimes \mathrm{Fun}(\mathbf{h})\right)^H$ where H acts in the obvious way on $\Omega^*(M)$ and acts on $\mathbf{h}$ (and hence on $\mathrm{Fun}(\mathbf{h})$) via the adjoint action. Here $\mathrm{Fun}(\mathbf{h})$ denotes an appropriate class of functions on $\mathbf{h}$: in the literature one most usually restricts to polynomial functions $\mathrm{Pol}(\mathbf{h}) = S(\mathbf{h}^*)$ (the symmetric algebra on $\mathbf{h}^*$).

The set

$$\Omega^*_H(M) = \left(\Omega^*(M) \otimes S(\mathbf{h}^*)\right)^H$$

has a natural grading: it is the differential form grading plus two times the polynomial grading (in other words a linear function on $\mathbf{h}$ is assigned grading 2). With this grading one sees that both terms in the operator[1]

$$D = d - i\iota_{V(\phi)} \tag{10}$$

increase the grading by 1. (Here d is the de Rham differential and $\iota_{V(\phi)}$ is the interior product with the vector field $V(\phi)$ induced on M by the action of $\phi \in \mathbf{h}$.) We may write

$$\iota_{V(\phi)} = \sum_a \phi^a \iota_{V^a},$$

introducing a basis $\{\phi^a\}$ for $\mathbf{h}$. It is easy to check (since we have restricted to H-invariant elements of $\Omega^*(M) \otimes S(\mathbf{h}^*)$) that $D^2 = 0$, so one may take the cohomology with respect to D: this cohomology is identified with the H-equivariant cohomology $H^*_H(M)$ of M.[2] If H acts freely on M the topological quotient M/H is a manifold (and M is a principal H-bundle over M/H), and the equivariant cohomology is identified with the ordinary cohomology $H^*(M/H)$ of the quotient.

We shall start with classes in the D-cohomology of $\Omega^*_H(M)$. One type of classes come from $(S(\mathbf{h})^*)^H$ (in other words, the polynomials on the Lie algebra $\mathbf{h}$ which are invariant under the adjoint action): this is identified with the H-equivariant cohomology of a point, or in other words with the cohomology $H^*(BH)$ of the classifying space BH. If M is a principal H-bundle over M/H (or equivalently if the action of H on M is free) then each invariant polynomial on $\mathbf{h}$ corresponds to a characteristic class of principal bundles with structure group H. Under the isomorphism $H^*_H(M) \cong H^*(M/H)$, the invariant polynomial S on $\mathbf{h}$ is identified with the corresponding characteristic class of the principal H-bundle M. (This is given in Chern-Weil theory as $S(F_A)$ where A is a connection on the bundle $M \to M/H$ and F_A is its curvature, which is a 2-form on M with values in $\mathbf{h}$).

10.3 Infinite dimensional Cartan: the descent equations

[1]Mathematicians normally use a convention in which the i in (10) is omitted.

[2]Throughout this lecture all cohomology groups will be assumed to have complex coefficients.

We shall now outline the analogue of the Cartan model in our infinite dimensional situation: this material is covered in Section 3.3 of **[W2]**. The space M is the infinite dimensional vector space $\mathcal{A}$ of connections on a G bundle P over a Riemann surface Σ. We start with an Ad-invariant polynomial S on the Lie algebra $\mathfrak{g}$; from this we shall construct an operator $\mathcal{O}_S^{(0)}$ in two dimensional Yang-Mills theory, which corresponds to a cohomology class in the moduli space $\mathcal{M}$. Recall that the field theory contained a field ϕ with values in $\mathfrak{g}$. We define

$$\mathcal{O}_S^{(0)} = S(\phi). \tag{11}$$

The object $\mathcal{O}_S^{(0)}$ is thus a function on Σ: we shall see that up to the supersymmetry differential δ, the operator $\mathcal{O}_S^{(0)}(p) = S(\phi(p))$ is independent of the choice of a point $p \in \Sigma$. We decompose the field ϕ into components $\{\phi^b\}$ (where b indexes a basis for $\mathfrak{g}$). Then

$$d\mathcal{O}_S^{(0)} = \sum_b \frac{\partial S}{\partial \phi^b} d_A \phi^a \tag{12}$$

$$= i\delta(\sum_b \frac{\partial S}{\partial \phi^a} \psi^b), \tag{13}$$

in terms of the supersymmetry operator δ. Thus if we define

$$\mathcal{O}_S^{(1)} = -\sum_b \frac{\partial S}{\partial \phi^b} \psi^b,$$

we have proved

$$d\mathcal{O}_S^{(0)} = -i\delta\mathcal{O}_S^{(1)}. \tag{14}$$

Here $\mathcal{O}_S^{(1)}$ should be viewed as a 1-form on Σ.

We can iterate this procedure: we find that

$$d\mathcal{O}_S^{(1)} = \sum_{a,b} \frac{\partial^2 S}{\partial \phi^a \partial \phi^b} D_A \psi^a \wedge \psi^b + \sum_a \frac{\partial S}{\partial \phi^a} D_A \psi^a. \tag{15}$$

Observing that $D_A \psi = \delta F_A$, we can convert (15) into $d\mathcal{O}_S^{(1)} = -i\delta\mathcal{O}_S^{(2)}$, where we have defined

$$\mathcal{O}_S^{(2)} = \frac{1}{2} \sum_{a,b} \frac{\partial^2 S}{\partial \phi^a \partial \phi^b} \psi^a \wedge \psi^b + i \sum_a \frac{\partial S}{\partial \phi^a} F_a. \tag{16}$$

These equations may be summarized as follows:

$$(d + i\delta)(\mathcal{O}_S^{(0)} + \mathcal{O}_S^{(1)} + \mathcal{O}_S^{(2)}) = 0. \tag{17}$$

(The point is that in an appropriate double complex the total differential is $d + i\delta$ and $\mathcal{O}_S^{(0)} + \mathcal{O}_S^{(1)} + \mathcal{O}_S^{(2)}$ is closed.)

We shall now use this to construct cohomology classes on the moduli space

$$\mathcal{M} = \mathcal{A}^{\text{flat}}/\mathcal{G}. \tag{18}$$

If we choose a q-cycle C in Σ, we find (using Stokes' theorem) that

$$Q_S^{(q)}(C) \stackrel{\text{def}}{=} \int_C \mathcal{O}_S^{(q)}$$

satisfies

$$\delta Q_S^{(q)}(C) = 0, \tag{19}$$

and if $C = \delta B$ is a boundary, then

$$Q_S^{(q)}(C) = \delta T \text{ for some } \; T. \tag{20}$$

Provided that the group $\mathcal{G}$ of gauge transformations acts freely on $\mathcal{A}$, the $Q_S^{(q)}(C)$ correspond to the generators of the cohomology of $\mathcal{M}$: they are cohomology classes on $\mathcal{A}/\mathcal{G}$ which we will restrict to the moduli space $\mathcal{A}_{\text{flat}}/\mathcal{G}$. These generators are given in Section 2 of **[AB]**: they are produced by taking the slant product of the characteristic classes of the *universal bundle* over $\mathcal{M} \times \Sigma$ with classes in the homology of Σ. The quantum field theory will compute a generating functional

$$\int_{\mathcal{M}} \exp\Big\{\alpha Q_S^{(0)}(p) + \sum_{j=1}^{2g} \beta_j Q_{S'_j}^{(1)}(C_j) + \gamma Q_{S''}^{(2)}(\Sigma)\Big\} \tag{21}$$

which will encode all intersection numbers in the cohomology of $\mathcal{M}$. A mathematical proof of these formulas for intersection numbers is given in **[T]** for the case $G = SU(2)$ and in **[JK2]** in the case $G = SU(n)$ (in those cases where the moduli space $\mathcal{M}$ is smooth). Here, p is a point in Σ, the C_j are $2g$ cycles in Σ corresponding to the homology $H_1(\Sigma)$, and the S, S'_j and S'' are arbitrary invariant polynomials on $\mathfrak{g}$ (which are in general distinct). For simplicity we shall mostly treat the case $G = SU(2)$, for which the ring of invariant polynomials on $\mathfrak{g}$ is a polynomial ring on one generator S given by

$$S(\phi) = \text{Tr}(\phi^2).$$

10.4 Equivariant integration and localization

We now return to the finite-dimensional situation of Section 10.2. We would like to define a map $\mathcal{I} : Z_H^*(M) \to \mathbb{C}$ (where $Z_H^*(M)$ are the D-closed elements in $\Omega_H^*(M)$) by integrating over $M \times \mathbf{h}$. In order to define a convergent integral, we introduce a convergence factor $e^{-\varepsilon \mathrm{Tr}(\phi^2)}$. (In fact in the mathematical treatment of this convergence factor one may replace the Gaussian $e^{-\varepsilon \mathrm{Tr}(\phi^2)}$ by any collection of rapidly decreasing functions $\{f_\varepsilon\}$ on $\mathbf{h}$ which (as $\varepsilon \to 0$) represent the Dirac delta distribution: see **[JK1]** for a mathematical treatment of equivariant integration in the Cartan model.) For $\alpha \in Z_H^*(M)$ we define

$$\mathcal{I}(\alpha) = \int_{\phi \in \mathbf{h}} \int_M d\phi_1 \dots d\phi_s e^{-\varepsilon \mathrm{Tr}(\phi^2)} \alpha(\phi). \tag{22}$$

In fact in order to ensure convergence of the integral (22) we must place some hypotheses on α. A useful class of equivariantly closed forms are obtained in the special case when M is a symplectic manifold with symplectic form ω, and the action of H on M is Hamiltonian with moment map $\mu : M \to \mathbf{h}^*$: We see readily that

$$\tilde{\omega} = -i \mathrm{Tr}(\phi \mu) + \omega \tag{23}$$

satisfies

$$D\tilde{\omega} = 0$$

(where D is the Cartan model differential defined by (10)).

The computation of the integral (22) will be governed by a localization principle similar to that which leads to the Duistermaat-Heckman formula for an oscillatory integral over a symplectic manifold with a Hamiltonian torus action. We choose an H-invariant 1-form λ for which

$$D\lambda = d\lambda + -i(V(\phi), \lambda); \tag{24}$$

where $V(\phi)$ is the vector field given by the action of ϕ (which of course depends linearly on ϕ). The integral (22) is not changed by replacing $\exp(-\varepsilon \mathrm{Tr}(\phi^2))$ by $\exp(-\varepsilon \mathrm{Tr}(\phi^2) + itD\lambda)$ since $\exp(itD\lambda)$ expands as $1 + itD\lambda + \dots$ Because $-tD\lambda = it(V(\phi), \lambda) - t(d\lambda)$, as $t \to \infty$ the integral localizes around points where $(V(\phi), \lambda) = 0$. We may see this by alternatively computing the partition function by completing the square and performing the integral over ϕ: to leading order in t this gives a factor

$$\exp - \frac{t^2}{4\varepsilon} \sum_a |(V_a, \lambda)|^2,$$

so by considering the behaviour for $\epsilon \to 0$ we see that the integral localizes around points where $(V_a, \lambda) = 0$ for all a.

We now specialize to the case where M is a symplectic manifold equipped with the Hamiltonian action of a compact group H: the moment map will be denoted μ. We define $I = |\mu|^2 : M \to \mathbb{R}$, the norm squared of the moment map. We may choose an almost complex structure $J \in \mathrm{End}(TM)$ compatible with the symplectic structure, and let $\lambda = J(dI)$ be the invariant one-form defined above. The points where $(V_a, \lambda) = 0$ are readily identified with the critical points of I. We see this because

$$dI = 2\sum_a \mu_a d\mu_a,$$

so since $V_a = \omega^{-1} d\mu_a$ (regarding the symplectic form ω as an isomorphism $T_xM \to T_x^*M$ for each $x \in M$, with inverse $\omega^{-1} : T_x^*M \to T_xM$) we have

$$\omega^{-1} dI = 2\sum_a \mu_a V_a.$$

Here $dI \in \Omega^1(M)$, so $\omega^{-1}dI$ should be regarded as an element of $\Gamma(TM)$. Thus $(V_a, \lambda) = 0$ implies $(\omega^{-1}dI, \lambda) = 0$, in other words

$$(\omega^{-1}dI, J(dI)) = 0$$

(where both sides of this equation are in $\Gamma(TM)$). This implies $dI = 0$ since ω and J combine to give a nondegenerate symmetric bilinear form g on the cotangent space, defined by

$$g(X, Y) = \omega^{-1}(X, JY)$$

(for $X, Y \in \Gamma(T^*M)$). Thus our integral formally localizes on the critical points of I. One obvious source of such critical points is $\mu^{-1}(0)$, but there are other critical points where $I \neq 0$.

We now give a derivation of (one version of) the Duistermaat-Heckman formula. As noted above, the form (23) $\tilde{\omega} = -i\mathrm{Tr}(\phi\mu) + \omega$ is equivariantly closed. We may thus compute

$$\frac{1}{\mathrm{Vol} H} \int_{\phi \in \mathbf{h}} d\phi_1 \dots d\phi_n \int_M \exp\Big(-i\mathrm{Tr}(\phi\mu) + \omega\Big). \tag{25}$$

(Note that if we did not include the integration over ϕ in the formula (25), the oscillatory integral over M would correspond to a sum over the components of the fixed point set of T: see for instance Chapter 7 of **[BGV]**.) If however we first integrate over ϕ we obtain a delta function $\delta(\mu)$, so the formula becomes

$$\frac{1}{\mathrm{Vol} H}\int_M \exp\omega\delta(\mu) = \int_{\mu^{-1}(0)/H} e^{\omega}$$

in other words the symplectic (Liouville) volume of the symplectic quotient $\mu^{-1}(0)/H$.

We wish, however, also to include a factor $\exp(-\varepsilon \mathrm{Tr}(\phi^2))$ in the integrand. Thus we wish to compute

$$Z(\varepsilon) \stackrel{\mathrm{def}}{=} \frac{1}{\mathrm{Vol} H}\int_{\phi\in\mathbf{h}} d\phi_1\dots d\phi_n \int_M \alpha\exp\Big(-i\mathrm{Tr}(\phi\mu)+\omega-\varepsilon\mathrm{Tr}(\phi^2)\Big), \qquad (26)$$

where $\alpha = \alpha(\phi)$ is an equivariantly closed form on M which has *polynomial* dependence on ϕ. We assume H acts freely on $\mu^{-1}(0)$; near $\mu^{-1}(0)$, the equivariantly closed class $\mathrm{Tr}(\phi^2)$ (which comes from the H-equivariant cohomology of a point in the Cartan model) is the pullback of a class Θ on the symplectic quotient $\mu^{-1}(0)/H$, which is the class which would be evaluated in Chern-Weil theory as $\mathrm{Tr}(F_B^2)$ if B is a connection on the principal H-bundle $\mu^{-1}(0)\to\mu^{-1}(0)/H$ and F_B is its curvature. Similarly, near $\mu^{-1}(0)$, α restricts to the pullback of a class $\hat{\alpha}$ on $\mu^{-1}(0)/H$.

We find that

$$Z(\varepsilon) = \int_{\mu^{-1}(0)/H}\hat{\alpha}\exp(\omega-\varepsilon\Theta)+O(e^{-b/\varepsilon}),$$

where b is the smallest nonzero critical value of I. To see heuristically why the nonzero critical value b of I should give a contribution to the integral (26) which is of order $e^{-b/\varepsilon}$, we compute (26) by first integrating out ϕ (assuming for simplicity that $\alpha = 1$). We compute the integral over ϕ by completing the square in ϕ, which gives

$$Z(\varepsilon) = \int_M \exp\left(\omega-\frac{1}{\varepsilon}\mathrm{Tr}(\mu^2)\right).$$

10.5 Equivariant integration: the infinite dimensional case

We now pass to the infinite dimensional case and study the path integrals corresponding to equivariant integration. Recall that we had defined the generating functional (21). We restrict to $G = SU(2)$ or $SO(3)$; we will pick the invariant polynomials S, S'_j and S'' to all be equal to the generating polynomial (denoted S), namely $S(\phi) = \mathrm{Tr}(\phi^2)$. (We have picked an Ad-invariant inner product on $\mathfrak{g}$, which is denoted Tr; a corresponding measure is defined on G.) We assume $\gamma = 1$ and $\alpha = -e^2/2$ in (21), and observe that for $S = \mathrm{Tr}(\phi^2)$ we have

$$Q_S^{(1)}(C_j) = \int_{C_j} \mathrm{Tr}(\phi\psi).$$

Thus the path integral that gives rise to (21) becomes (where p is a point in Σ)

$$Z(e, \{\beta_j\}) = \frac{1}{\mathrm{Vol}G} \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\phi e^{\{-\frac{e^2}{2}\mathrm{Tr}(\phi^2)(p) + \sum_{j=1}^{2g} \beta_j \int_{C_j} \mathrm{Tr}(\phi\psi) + \int_\Sigma Q_S^{(2)}\}}. \quad (27)$$

(Note that when $S(\phi) = \mathrm{Tr}(\phi^2)$, the object $\int_\Sigma Q_S^{(2)}$ corresponds to the symplectic form on $\mathcal{A}$, and becomes identified with the standard symplectic form on the moduli space $\mathcal{M}$.)

We can now see how the integral (27) can be rewritten in such a way that the generating functional (21) involving the $Q_S^{(q)}$ for $q = 0, 1, 2$ is equated with a corresponding generating functional involving only $Q_S^{(q)}$ for $q = 0, 2$. This is accomplished by performing the integral over ψ, which is Gaussian: completing the square to perform this integral we see that we pick up a term

$$\frac{1}{\mathrm{Vol}G} \int \mathcal{D}A\mathcal{D}\phi \exp\Big\{ i \int_\Sigma \mathrm{Tr}(\phi F_A) - \varepsilon \mathrm{Tr}(\phi^2)(p), \Big\} \quad (28)$$

where $\varepsilon = e^2/2 - \sum_{i<j} \beta_i\beta_j \#(C_i \cap C_j)$. In other words we can eliminate the odd-dimensional generators of the cohomology ring of the moduli space corresponding to $Q_S^{(1)}$ (the non-algebraic cycles in the moduli space). A mathematical version of this argument is given in the work of Thaddeus **[T]**.

10.6 The partition function of Yang-Mills theory

The following material is covered in Section 4 of **[W2]**. We have now reduced the formula we need to compute to the partition function (2) of Yang-Mills theory without the fermionic variable ψ included. To compute the partition function we quantize the theory. This is accomplished by replacing the Riemann surface by a cylinder $C \times [0, T]$ of length T (where $C \cong S^1$ should be regarded as space and is assumed to have length L, while $[0, T]$ is a time interval of length T). The space of fields is equal to the space of connections $\mathcal{A}_C$ on the circle C, so the Hilbert space is

$$\mathcal{H} = C^\infty(\mathcal{A}_C)^{\mathcal{G}},$$

the gauge invariant functions on $\mathcal{A}_C$. Since up to gauge equivalence all connections are classified by their holonomy around the curve C, we find that the Hilbert space is identified with

$$\mathcal{H} = C^\infty(G)^G,$$

the ring of functions on G which are invariant under the adjoint action (or equivalently the functions on the maximal torus T which are invariant under the action of the Weyl group). The Hilbert space is thus identified with the irreducible (complex) representations R of G (or equivalently with their characters which are denoted χ_R or Tr_R): we take the characters χ_R to form a basis of $\mathcal{H}$, or

$$\mathcal{H} = \oplus_R \mathbb{C}_{(R)}.$$

For every irreducible representation R the basis element χ_R is identified with a function Ψ_R on $\mathcal{A}_C$, namely

$$\Psi_R(A) = \chi_R(\mathrm{Hol}_C A).$$

The Hamiltonian of the theory is

$$H = -\frac{e^2}{2}\int_C \mathrm{Tr}|F_{01}|^2$$

(where F_{01} is the curvature of the connection), which becomes (identifying the Hamiltonian as an operator on $\mathcal{H}$ in terms of the position variables $A^b(\theta)d\theta$ on $\mathcal{A}_C$ and the corresponding momentum variables $(\partial A^b(\theta)/\partial t)d\theta$ which are identified with $-i\partial/\partial A^b(\theta)$ under canonical quantization)

$$H = \frac{e^2}{2}\int_C \mathrm{Tr}(\frac{\delta}{\delta A})^2.$$

The action of the operator $\delta/\delta A^a(\theta)$ on Ψ_R gives

$$\frac{\delta}{\delta A^a(\theta)}\mathrm{Tr}_R \mathrm{Hol}_C(A) = \mathrm{Tr}_R T^a \mathrm{Hol}_C(A)$$

in terms of a basis T^a for the Lie algebra $\mathfrak{g}$ of G. We thus see that

$$\sum_a \frac{\delta}{\delta A^a(\theta)}\frac{\delta}{\delta A^a(\theta)}\mathrm{Tr}_R \mathrm{Hol}_C(A) = \mathrm{Tr}_R(\sum_a T^a T^a \mathrm{Hol}_C(A)). \qquad (29)$$

The element $\sum_a T^a T^a$ is the *quadratic Casimir* of G, whose evaluation in the representation R is denoted

$$\mathrm{Tr}_R \sum_a T^a T^a \stackrel{\mathrm{def}}{=} c_2(R).$$

Thus we see that

$$H\Psi_R = \frac{e^2 c_2(R) L}{2} \Psi_R. \tag{30}$$

We thus see that

$$< \Psi_R | e^{-HT} | \Psi_R >= \exp(-e^2 c_2(R) LT/2) = \exp(-e^2 \mathbf{a} c_2(R)/2), \tag{31}$$

in terms of the area $\mathbf{a} = LT$ of the cylinder. The answer depends only on the combination $e^2\mathbf{a}$.

In fact the effect of renormalization in this quantum field theory is to introduce a factor of the form

$$\exp(\alpha \int_\Sigma d\mu \mathcal{R} + (e^2/2)\beta \int_\Sigma d\mu) \tag{32}$$

into the integrand (where $\mathcal{R}$ is the Riemann curvature and α and β are appropriately chosen real-valued constants). The effect of the term involving β is to change the quadratic Casimir c_R to a renormalized value $\tilde{c}_R$ which differs from c_R by addition of a constant. The factor α multiplies the Euler characteristic of Σ: it will eventually be adjusted to ensure that the overall normalization of the partition function and its dependence on the genus g agrees with that found by other methods (for instance by computing the partition function when $\varepsilon = 0$ using Reidemeister-Ray-Singer torsion, as in Section 4 of **[QYM]**).

We must also consider the field theory on the Riemann surface $\mathcal{P}$ which is a sphere with three disks removed (the trinion or pair of pants: see Figure 1). If we insert the operator $\hat{Q}_S^{(0)}(p)$ corresponding to the observable $Q_S^{(0)}(p)$ (where p is some point in $\mathcal{P}$) we see using (30) and its generalizations that the state Ψ_R is an eigenstate of $\hat{Q}_S^{(0)}(p)$ with some eigenvalue $C(S,R)$:

$$\hat{Q}_S^{(0)}(p)\Psi_R = C(S,R)\Psi_R. \tag{33}$$

Here, if S is an invariant polynomial of degree l on $\mathfrak{g}$, $C(S,R)$ is the corresponding l-th order Casimir of the representation R. By considering the

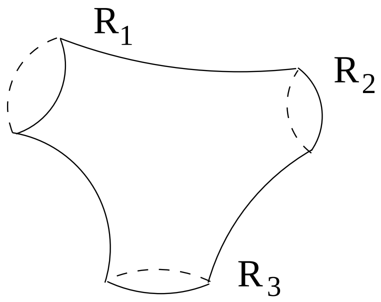

Figure 1

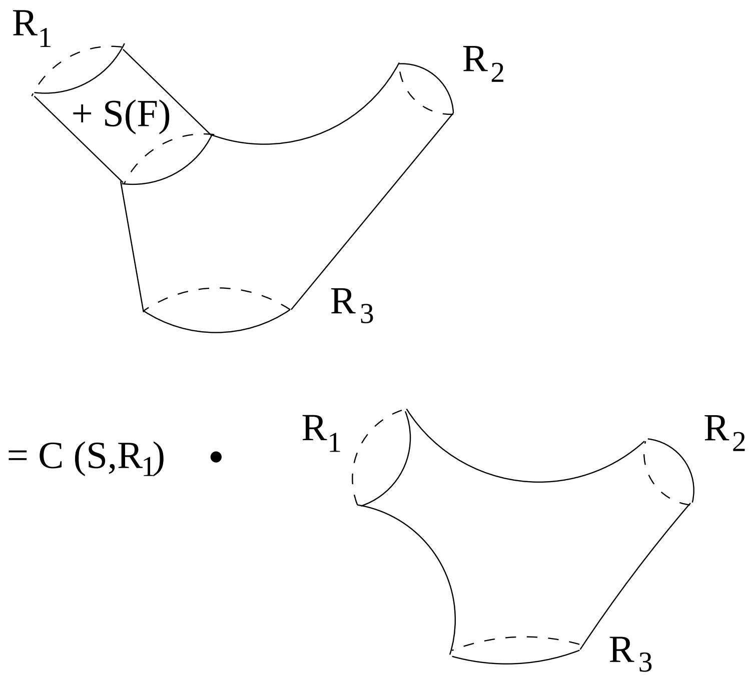

Figure 3

one point function determined by the Riemann surface $\mathcal{P}$ with the operator $\hat{Q}_S^{(0)}(p)$ inserted near the j-th boundary component (see Figure 3), we find that this one point function is equal to $C(S, R_j) W_{R_1 R_2 R_3}$ (for $j = 1, 2, 3$) if $W_{R_1 R_2 R_3}$ is the partition function for $\mathcal{P}$ with states Ψ_{R_1}, Ψ_{R_2}, Ψ_{R_3} along the three boundary components. This does not depend on the boundary component j (since this field theory is invariant under area preserving diffeomorphisms): it follows that the Casimirs $C(S, R_j)$ are all equal, and since this is true for all invariant polynomials S, we must have $R_1 = R_2 = R_3$ if $W(R_1, R_2, R_3) \neq 0$.

We are thus reduced to computing the partition function of $\mathcal{P}$ with boundary conditions determined by the external state Ψ_R along all three boundary components. Denote the value of this partition function in the limit of zero area by w_R: more generally for a pair of pants $\mathcal{P}$ of area $\mathbf{a}$ one obtains

$$Z_R = w_R \exp(-\tilde{c}_2(R) \frac{\mathbf{a} e^2}{2}). \tag{34}$$

We now consider the problem of computing the partition function for a Riemann surface of genus g with no boundary. Such a surface may be formed by gluing together $2g-2$ copies $\mathcal{P}_j$ of $\mathcal{P}$ along $3g-3$ boundary circles C_γ. We may factor the path integral for the partition function according to the values of the fields restricted to the boundary circles C_γ. If A' denotes the value of a connection on the boundary circles $\coprod_\gamma C_\gamma$ of the $\mathcal{P}_j$, and $\mathcal{A}_{A'} = \{A \in \mathcal{A} : A|_{\coprod_\gamma C_\gamma} = A'\}$ is the set of all connections which restrict to a given boundary value A', we have

$$\int \mathcal{D}A e^{-\mathcal{L}} = \int \mathcal{D}A' \int_{\mathcal{A}_{A'}} e^{-\mathcal{L}} \tag{35}$$

(cf. **[QYM]**, Section 4.5). Once we have fixed the boundary values A', the space $\mathcal{A}_{A'}$ is the product of $2g-2$ copies of the space of connections on $\mathcal{P}$ (with prescribed boundary values). In quantization the partition function of $\mathcal{P}$ is

$$Z(\mathcal{P}, A|_{\partial \mathcal{P}}) = \sum_R w_R \prod_{\gamma=1}^{3} \Psi_R(A|_{C_\gamma}). \tag{36}$$

To recover the partition function for the closed Riemann surface Σ of genus g, we multiply $2g-2$ copies of $Z(\mathcal{P})$ and integrate over the boundary values A'. Using the orthogonality relations for the group characters which correspond

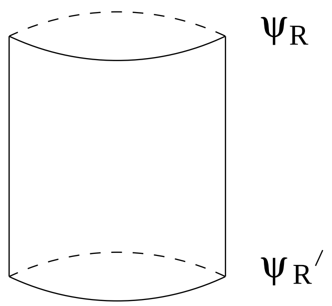

Figure 2

to the states Ψ_R, we find

$$Z(\Sigma) = \sum_R w_R^{2g-2} \exp(-\frac{e^2 \mathbf{a} \tilde{c}_2(R)}{2}). \tag{37}$$

We now explain how to compute the w_R. We observe that the partition function for a disk D of area $\mathbf{a}$ with an external state Ψ_R on the boundary is

$$Z(D) = v_R \exp(-\frac{e^2 \mathbf{a} \tilde{c}_2(R)}{2}). \tag{38}$$

The partition function corresponding to a cylinder with boundary conditions determined by wavefunctions Ψ_R (resp. $\Psi_{R'}$) on the two boundary components is equal to 0 if $R \neq R'$, since Ψ_R is an eigenfunction of the Hamiltonian H. (See Figure 2.) The partition function of a cylinder $S^1 \times I$ of area $\mathbf{a}$ with external states Ψ_R on both boundary components is

$$Z(S^1 \times I) = 1 \cdot \exp(-\frac{e^2 \mathbf{a} \tilde{c}_2(R)}{2}) \tag{39}$$

(since the Ψ_R are eigenstates of the Hamiltonian H). We may decompose the cylinder as in Figure 4: this yields the partition function of the cylinder with external states Ψ_R on both boundary components as the product of the partition function of $\mathcal{P}$ (with external states Ψ_R on all boundary components) and that of D, so that

$$w_R \cdot v_R = 1. \tag{40}$$

To determine v_R, it suffices to consider a disk D of very small area. If we fix the holonomy of a connection around the boundary of D to take the value

$U \in G$, the partition function for the disk (restricting to connections on D with boundary holonomy U) is (via quantization)

$$Z(U) = \sum_R v_R \chi_R(U) \tag{41}$$

(where as above χ_R denotes the character of the representation R). If we instead compute the partition function via the path integral, we write the action as

$$I(U) = \int_A \mathrm{Tr}(|F|^2), \tag{42}$$

and

$$Z(U) = \int DA \exp(-\frac{1}{e^2}\int \mathrm{Tr}(|F|^2)), \tag{43}$$

(where we have restricted to connections for which the boundary holonomy is U). This gives

$$Z(U) = \exp \alpha \delta(U-1), \tag{44}$$

(since the Euler-Lagrange equation implies that the dominant contribution comes from flat connections, which necessarily have trivial holonomy around the boundary of D). Here, α is the (as yet undetermined) constant which appeared in (32) multiplying the Euler characteristic of the Riemann surface. Hence we see that

$$\exp(\alpha)\delta(U-1) = \sum_R v_R \chi_R(U). \tag{45}$$

If we multiply both sides of this equation by $\chi_{R'}(U)$ and integrate over U, using the orthogonality relations for group characters we find

$$v_R = \exp(\alpha) \dim R, \tag{46}$$

and hence

$$w_R = \frac{\exp(-\alpha)}{\dim R}. \tag{47}$$

Thus the partition function is

$$Z(\Sigma^g, e^2\mathbf{a}) = \sum_R w_R^{2g-2} \exp(-\frac{\tilde{c}_2(R)e^2\mathbf{a}}{2}) \tag{48}$$

$$= \sum_R \frac{e^{-\alpha(2g-2)}\exp(-\tilde{c}_2(R)\varepsilon/2)}{(\dim R)^{2g-2}} \tag{49}$$

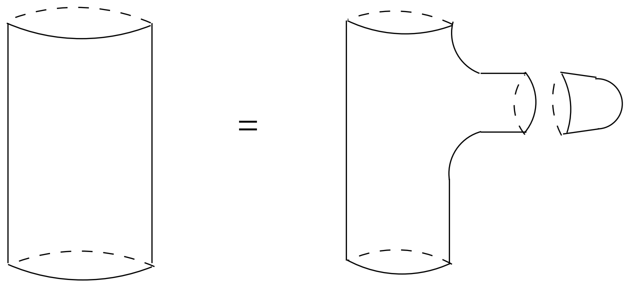

Figure 4

where we have introduced $\varepsilon = e^2\mathbf{a}$.

The computation we have just performed gives the sum of partition functions corresponding to bundles of all possible topological types (recall that if G is not simply connected there will in general be several topological types of bundles over Σ). To study the contribution of one particular topological type, we take G to be simply connected and pick an element $\zeta \in Z(G)$. Choosing a point $p \in \Sigma$ we restrict to connections on $\Sigma - \{p\}$ such that the holonomy around p is equal to ζ. (Such a connection will descend to a flat connection on a quotient bundle with structure group $G/Z(G)$, whose topology is determined by ζ.) In this setting we repeat the analysis above: we find that the path integral over connections with holonomy ζ around p is given (via quantization) by

$$\sum_R u_R \chi_R(U) \tag{50}$$

and (via the path integral) by

$$\exp(\alpha)\delta(U - \zeta). \tag{51}$$

If we multiply by $\chi'_R(U)$ and integrate over U we find

$$u_R = \exp(\alpha)\chi_R(\zeta) = \exp(\alpha)(\dim R)\hat{\chi}_R(\zeta) \tag{52}$$

where $\hat{\chi}_R(\zeta) \in U(1)$ is the normalized character evaluated at ζ. Thus the partition function becomes (since the Riemann surface is now decomposed into $2g - 1$ copies of $\mathcal{P}$ and one copy of the disk D)

$$Z(\varepsilon, g, \zeta) = \sum_R w_R^{2g-1} e^{-\tilde{c}_2(R)e^2\mathbf{a}/2} u_R \tag{53}$$

$$= \sum_R \frac{e^{-\alpha(2g-2)} e^{-\varepsilon \tilde{c}_2(R)/2}}{(\dim R)^{2g-2}} \hat{\chi_R}(\zeta). \tag{54}$$

This is the partition function for a particular class of G bundles corresponding to flat connections on a Riemann surface of genus g with one boundary component and with holonomy ζ around this boundary component. The partition function for the corresponding class of $G/Z(G)$ bundles on a Riemann surface of genus g with no boundary is obtained by dividing (54) by a factor $(\#Z(G))^{2g-2}$, since this component of the moduli space of flat $G/Z(G)$ connections is an unbranched cover (of order $(\#Z(G))^{2g-2}$) of the moduli space of flat G connections on a Riemann surface of genus g with one boundary component around which the holonomy is constrained to take the central value ζ.

We conclude this section by studying the example $G = SU(2)$, $\zeta = -1$. For each integer $n \geq 1$ there is a unique representation R_n of dimension n with $\hat{\chi_n}(\zeta) = (-1)^{n+1}$. The quadratic Casimir of the representation R_n is

$$c_2(R_n) = (n^2 - 1)/2;$$

this is renormalized to

$$\tilde{c}_2(R_n) = n^2/2.$$

The partition function corresponding to an $SO(3)$ bundle of $w_2 \neq 0$ is given (up to an overall multiplicative normalization constant independent of the genus g) by

$$Z(\varepsilon) = e^{-\alpha(2g-2)} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^{2g-2}} e^{-\varepsilon n^2/4}. \tag{55}$$

Let us now verify that this is the sum of a term which is a polynomial in ε (the polynomial dependence being expected from (21)) and a second term which is exponentially decaying of order $e^{-b/\varepsilon}$ for some positive constant b. We do this by differentiating the series to find that

$$\left(e^{\alpha(2g-2)} 4^{g-1} \frac{\partial}{\partial \varepsilon} \right)^{g-1} Z(\varepsilon) = F(\varepsilon), \tag{56}$$

where we have defined

$$F(\varepsilon) = \sum_{n \geq 1} (-1)^{n+1} e^{-\varepsilon n^2/4}.$$

We have that

$$2F(\varepsilon) - 1 = -\sum_{n\in\mathbb{Z}} (-1)^n e^{-\varepsilon n^2/4},$$

and using the Poisson summation formula this is given by

$$2F(\varepsilon) = 1 - \sqrt{4\pi/\varepsilon} \sum_{m\in\mathbb{Z}} e^{-(2\pi)^2(m+1/2)^2/\varepsilon}. \qquad (57)$$

(The terms in the sum indexed by m correspond to the contribution of the critical points of the Yang-Mills functional arising from unstable bundles of the form $\mathcal{L}_m \oplus \mathcal{L}_{-m+1}$, where $\mathcal{L}_m$ is a line bundle over Σ of degree m.) We may now apply (56) to integrate (57), showing that $Z(\varepsilon)$ is the sum of a polynomial in ε (of degree $g-1$) plus a term exponentially decaying like $G(\varepsilon)e^{-b/\varepsilon}$ for a positive constant b (where $G(\varepsilon)$ is a polynomial in $\sqrt{\varepsilon}$ and $1/\sqrt{\varepsilon}$).

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