Lecture II-12: $N = 2$ SUSY theories in Dimension Two: Part I

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I Introduction

In this lecture and the next two we consider $N = 2$ supersymmetric field theories in dimension two. Today, after some general introductory remarks about such theories, we consider two-dimensional $U(1)$-gauge theories with complex-valued chiral superfields charged under the gauge group. These theories depend on two functions – a superpotential W and a twisted superpotential $\widetilde{W}$. The superpotential W is a complex polynomial in the superfields. This polynomial must satisfy a certain 'weight' conditions, but is otherwise free. For gauge group $U(1)$, the twisted superpotential $\widetilde{W}$ of the sort of model we will look at is determined by a single complex parameter $t = -ir + \frac{\theta}{2\pi}$, or more precisely by $\exp(-2\pi it)$. Today we will keep the superpotential is fixed and we allow the twisted superpotential to vary. Thus, for each (generic) complex polynomial W satisfying the weight conditions, we get a family of theories parameterized by the point $\exp(2\pi it)$ in the cylinder $\mathbf{C}/\mathbf{Z}$. We are interested in what happens at the ends of the cylinder $\mathrm{Im}(t) \mapsto +\infty$ and $\mathrm{Im}(t) \mapsto -\infty$. In the cases we consider, the limit as $\mathrm{Im}(t) \mapsto -\infty$ is described in terms of a σ-model on the projective variety X defined by the polynomial W giving the superpotential. In this region, $-\mathrm{Im}(t)$ is the Kahler class of X, and the fact that there is only one parameter in the superpotential is related to the fact that $H^{1,1}(X)$ (or at least the part of it that is pulled back

*Notes by John Morgan and David R. Morrison from a certain ambient weighted projective space) is one-dimensional. The limit as $\mathrm{Im}(t) \mapsto +\infty$ is described in terms of an orbifold version of a Landau-Ginzburg model.

When $c_1(X) = 0$, we will find that the beta function vanishes for all t, and we get a family of conformally invariant theories. For $c_1(X) \neq 0$, we get instead a non-trivial renormalization group flow in t. The flow increases $\mathrm{Im}\, t$ when $c_1(X) > 0$ and decreases $\mathrm{Im}\, t$ when $c_1(X) < 0$. (Furthermore, in these latter two cases there are "extra" vacua at the end of the cylinder to which one flows in the infrared.) This allows us to see various Landau-Ginzburg models as infra-red limits of σ-models and other σ-models as infra-red limits of Landau-Ginzburg models. This is interesting because the descriptions of these types of models are very different at the classical level. Most of today's lecture will deal with special cases (the cases that $c_1(X) = 0$, and some of the cases with $c_1(X) > 0$ with flow to a massive infrared theory) that have been extensively studied, but the general picture that will be presented at the end of the lecture is actually new, as far as I know.

II Generalities on $N = 2$ SUSY Theories in Two Dimensions

II.1 The β-function of $N = 2$ SUSY σ-models

The first remark concerns $N = 2$ supersymmetric σ-models with target a compact Kähler manifold X, which we assume has a Kähler-Einstein metric. In Gawedski's lectures we have seen that the one-loop β-function is determined by the sign of the Ricci curvature of the metric and in fact when the Ricci tensor is positive or negative-definite, the sign of the one-loop β-function is opposite that of the Ricci curvature. Thus, if $c_1(X) > 0$ then the one-loop β-function is negative. Thus, at small coupling the β-function is negative so that the σ-model under consideration is a 'good' fundamental theory which is asymptotically free. Such a theory is well-defined at the quantum level. The question arises of what it flows to in the infrared. Today we shall see examples of such X for which the infra-red theory is massive and trivial, and also examples with flow to a non-trivial infra-red fixed point. In the latter case, the limiting superconformal theories that we will get can be described in terms of what are known as supersymmetric Landau-Ginzburg theories. The Landau-Ginzburg models are sometimes explicitly soluble by algebraic methods.

On the other hand, if $c_1(X) < 0$, then the one-loop β-function is positive and hence β is positive at weak coupling. Thus, we have no reason to expect σ-models for such X to be 'good' fundamental theories. Rather in this case the σ-model is an effective theory, free in the infra-red, and one should ask whether one can describe a concrete, well-defined ultraviolet theory that flows to these sigma models in the infrared. We will solve this problem for this class of sigma models by taking $\mathrm{Im}t \to \infty$, where we will find a Landau-Ginzburg theory (which is superconformal), which after a relevant perturbation (to $\mathrm{Im}\,t$ large and positive but not ∞) flows to the sigma model with target space X (plus some extra massive vacua).

The case when $c_1(X) = 0$, i.e., the case when X is Calabi-Yau, is different. Here, the one loop β-function vanishes and one-loop contributions do not change the Kähler metric. It turns out that there are higher loop corrections to the β-function so that it is not identically zero and hence the σ-model on X (with its Ricci-flat metric) is not conformal. Nevertheless, standard invariant theory arguments show that there is no higher loop correction to the Kähler class of the metric. Using Yau's theorem about existence of Kähler metrics on X with prescribed Ricci curvature ($\partial\overline{\partial}$-exact) we can modify (at least order-by-order in perturbation theory) the original Ricci-flat Kähler metric without changing the complex structure or the Kähler class until the σ-model for the new Kähler structure on X has vanishing β-function. This process will converge if the Kähler class is sufficiently ample and far from the boundaries of the Kähler cone; for today's models these conditions simply mean that $-\mathrm{Im}\,t$ should be large enough. What we will be exploring today is really the question of what happens when $-\mathrm{Im}\,t$ is not large.

Today we shall see examples of X with $c_1(X) = 0$ for which there is a family of theories, parametrized by a punctured cylinder that approaches the σ-model on X as $\mathrm{Im}t \mapsto -\infty$ and approaches a Landau-Ginzburg model as $\mathrm{Im}t \mapsto +\infty$. To get such a family, the cylinder must be punctured at one point, where the theory has a sort of pole. It is believed that the punctured cylinder parametrizes a smooth family of theories, which flow in the infrared to a punctured cylinder of superconformal field theories. As one check of this belief, we argue that the central charge of the Landau-Ginzburg theory at $+\infty$ and the central charge of the σ-model at $-\infty$ are equal. We will present more evidence for the scenario described today in a later lecture.

II.2 Reasons for considering $N = 2$ SUSY theories

One advantage of studying $N = 2$ supersymmetric theories is that they are much more rigid than theories with less (or no) supersymmetry. For example if we are considering two-dimensional σ-models into a compact Riemannian manifold X, then without supersymmetry we can add terms to the Lagrangian of the form

$$\int d^2y\, \varphi^*(h)$$

for a potential function $h: X \to \mathbf{R}$. (Here, $\varphi: \Sigma \to X$ is the basic field in the Lagrangian.) In $N = 1$ supersymmetric σ-models, such a term is not possible, but there can still be a term

$$\int d^2y\, d^2\theta\, \Phi^*(h)$$

where again h is a real-valued function on X (called the superpotential) and Φ is the basic superfield in the Lagrangian. In both cases this is a relevant perturbation. When we require $N = 2$ supersymmetry, the Riemannian manifold must be Kähler and the possible potential term can still exist but must be of the form $h = \mathrm{Re}(f)$ for f a holomorphic function. Since our Riemannian manifold is assumed to be compact, this means that f is constant and hence, after performing the θ integrals, this term vanishes. Simple power counting shows that at the classical level this is the only possible relevant perturbation over a $2|4$-superspace.

II.3 Revelant and Marginal Perturbations of the conformal two-dimensional σ-model

Let us study the marginal perturbations of these conformal σ-models. Part of the data of the σ-model is the complex structure on X. Varying this is a marginal perturbation of the theory. Let us examine this perturbation in local coordinates. The metric tensor on X is a $(1,1)$-tensor of the form $h_{i\overline{j}} dz^i \otimes d\overline{z}^j$. A change in the complex structure corresponds to a perturbation of the metric by a term of the form

$$\delta h_{\overline{i}\overline{j}} + \delta h_{ij}$$

of type $(2,0)+(0,2)$. Of course, this form must be real in order that the new form be a metric tensor, and $\delta h_{\overline{i}\overline{j}}$ must be $\overline{\partial}$-closed in order that the new almost complex structure is integrable. Let us compute the change in the Lagrangian that goes with this perturbation of the Kähler metric. We write things in local coordinates using bosonic coordinates $y^{\alpha}, \alpha = 1,2$ on the Riemann surface and fermionic coordinates $\theta^{\pm}, \overline{\theta}^{\pm}$. We write superfields as $X = X(y,\theta,\overline{\theta})$, chiral superfields as $\Phi(y,\theta)$ and anti-chiral superfields as $\overline{\Phi}(y,\overline{\theta})$. The change in the Lagrangian is given by

$$\Delta\mathcal{L} = \left(\int d^2y\, d^2\overline{\theta}\delta h_{\overline{i}\overline{j}}\overline{D}_+\overline{X}^{\overline{i}}\overline{D}_-\overline{X}^{\overline{j}}\right) + \text{c. c.}.$$

It is a nice exercise to verify that the argument of the $\overline{\theta}$ integral in $\Delta\mathcal{L}$ is anti-chiral. (This uses the fact that $\delta h_{\overline{i}\overline{j}}$ is $\overline{\partial}$-closed.) This is one of the two possible types of marginal perturbations.

The other type of marginal perturbation is to fix the complex structure X and vary the complexified Kähler class of the Kähler metric. One piece of this deformation is an ordinary variation $\delta h^{\text{herm}}_{\overline{i}j} d\overline{z}^{\overline{i}} \wedge dz^j$ by a hermitian symmetric closed two-form. This ordinary metric perturbation is the real part of a complex perturbation whose imaginary part is called the "B-field" perturbation (which, as we shall see, is analogous to the θ-angle which appears in gauge theories). Thus, we consider a perturbation $\delta h = \delta h_{\overline{i}j} d\overline{z}^{\overline{i}} \wedge dz^j$ where δh is a closed complex two-form. This leads to a perturbation of the Lagrangian of the form

$$\Delta\mathcal{L} = \left(\int d^2y d\overline{\theta}^+ d\theta^- (\delta h_{\overline{i}j}\overline{D}_+\overline{X}^{\overline{i}} D_- X^u\right) + \text{c. c.}.$$

Other than the marginal deformations of the theory described above, there can only be deformations of the form

$$\int d^2y d^4\theta(\text{something}).$$

For these the θ-integration has weight 2, and hence all such terms are marginal (of dimension two) classically. (Notice this count requires four θ's and hence uses the fact that we have $N = 2$ SUSY). But at the quantum level, they have anomalous dimensions (proportional to nonzero eigenvalues of the Laplacian and so strictly positive) and are irrelevant. These types of deformations can affect the Kähler metric without changing the complex structure or Kähler class.

These computations show that for the σ-model with $c_1(X) = 0$, the theory has no relevant perturbations and the marginal perturbations are obtained by deforming the complex structure and complexified Kähler metric.

Let us begin with such a σ-model with a Kahler metric that is smooth and has a large radius of curvature close to, but not equal to, the metric that actually gives zero beta function. The renormalization group flow will bring us to a metric that gives conformal invariance, as all other deformations are irrelevant. Were there relevant perturbations these would dominate in the infrared limit and the limit would be in general impossible to control.

II.4 Central Charge

Let us review some material on central charges in conformal and super-conformal field theories in dimension two. There are two basic types of $N = 0$, i.e., of ordinary, conformal field theories – those with central charge $c < 1$, and those with central charge $c \geq 1$. The former are called minimal models, exist only for a discrete set of values of c, and are explicitly algebraically described in terms of representation theory of the Virasoro algebra. The theories with $c = 1$ can also be explicitly described, in terms of a free boson. When $c \leq 1$, the Virasoro algebras of left- and right- movers act almost irreducibly in the quantum Hilbert space, which is why these theories are known rather explicitly. For $c > 1$, the Virasoro action is very far from being irreducible (the number of highest weight vectors grows exponentially as one goes to higher energies), and conformal field theory has a completely different flavor.

When we go to the $N = 1$ super-conformal algebra in dimension two, the central charge again lies either in a discrete set or in a continuous part. The lowest value of the central charge in the continuous part is realized by the free scalar super-field. Being the sum of a free scalar boson and a free scalar fermion, considered as a representation of the ordinary conformal algebra this field has central charge $c = 3/2$. For this reason one usually defines $\widetilde{c} = 2c/3$ so that the continuous values for $\widetilde{c}$ begin at $\widetilde{c} = 1$.

The story is similar for the $N = 2$ super-conformal algebra in dimension two. Here the lowest value in the continuous part is realized by the free chiral super-field. Since this field consists of a complex scalar boson and a complex fermion, its $N = 0$ central charge is 3. Hence, we set

$\widehat{c} = c/3$ so that once again the lowest value in the continuous part is $\widehat{c} = 1$. The discrete set of $N = 2$ theories with $\widehat{c} < 1$ are the "$N = 2$ minimal models" which we will consider in detail later on. Their central charges take the form $\widehat{c} = 1 - \frac{2}{n}$ for $n = 3, 4, 5, \ldots$, and they can be described algebraically and studied in a variety of attractive ways.

A conformal $N = 2$ supersymmetric σ-model with target space a k-dimensional complex manifold X of $c_1(X) = 0$ has central charge $\widehat{c} = k$. This is proved by using the fact that $\widehat{c}$ is constant in a family of conformal field theories, and expanding the metric until X can be approximated, in a local region, to any desired accuracy, to $\mathbf{C}^k$ with a flat metric. Thus, this sigma model has the same $\widehat{c}$ as a sigma model with target a flat $\mathbf{C}^k$. That is a free theory with k chiral superfields, and so has $\widehat{c} = k$, as claimed.

III The $U(1)$ Theories

Let us turn now to the study of $U(1)$-gauge theory with complex-valued chiral superfields $A_1, \ldots, A_n$. These superfields are charged under the $U(1)$ with charges $q_1, \ldots, q_n \in \mathbf{Z}$. Recall from the superhomework that connections we use on $2|4$-space are constrained to be flat in all pairs on odd directions except the pair $\{\overline{\theta}^+, \theta^-\}$ (and its complex conjugate). The basic invariant of such a connection is

$$\Sigma = \{\overline{\mathcal{D}}_+, \mathcal{D}_-\}$$

which is a section of the complexification of the adjoint bundle over $2|4$-space. The minimal pure gauge theory Lagrangian that we can write down is

$$\int d^2y d^4\theta \frac{1}{2e^2} \overline{\Sigma}\Sigma.$$

Adding in a kinetic term for the n chiral superfields, the minimal Lagrangian becomes

$$\mathcal{L}_0 = \int d^2y d^4\theta \left(\frac{1}{2e^2} \overline{\Sigma}\Sigma + \sum_{i=1}^{n} \overline{A}_i A_i \right). \tag{1}$$

Eventually, we will add a superpotential and a twisted superpotential term to this Lagrangian.

III.1 R-symmetries

The above $N=2$ supersymmetric gauge theory in two dimensions (or $2|4$ supertheory) comes by dimensionally reducing from $N=1$ gauge theory in four-dimensions (or $4|4$ supertheory). As such, there is an R-symmetry for the $N=2$ two-dimensional theory induced by spatial rotation in the remaining two dimensions in four-space. But, in addition, the $N=1$ supersymmetric version of $U(1)$-gauge theory in four-dimensions itself has an R-symmetry and that symmetry continues to exist even if we add a superpotential. This R-symmetry from dimension four dimensionally reduces to give a second R-symmetry for our 2-dimensional theory.

Independent of their sources, it is easy to see directly in the $2|4$-theory what the $U(1)\times U(1)$-group of R-symmetries is. We have J_R, the circle action

$$\theta^+ \mapsto e^{i\alpha}\theta^+$$

$$\overline{\theta}^+ \mapsto e^{-i\alpha}\overline{\theta}^+$$

with θ^- left fixed, and we have J_L, the circle action on θ^- with θ^+ left fixed. The R-symmetry that comes from the R-symmetry of four-dimensional gauge theory is the product of these two (i.e., it acts by the standard representation on both $\theta^\pm$). Spatial rotation in the omitted two dimensions induces the circle action which is the product of the first times the inverse of the second of these. Potentially there is an anomaly in one of these R-symmetries. To have a super-conformal theory this anomaly must vanish (or be cancelled) since these symmetries are part of the super conformal algebra.

Let us make more explicit the earlier remark that the anomaly of the σ-model is proportional to the first Chern class $c_1(X)$. The point is that when we write the Lagrangian out in coordinates we have an expression of the form

$$\mathcal{L} = \int d^2y \left(g_{i\overline{j}}\partial X^i \partial \overline{X}^{\overline{j}} + \overline{\psi}_-^{(0,1)} \partial\!\!\!/_+ \overline{\psi}_-^{(1,0)} + \psi_+^{(0,1)} \partial\!\!\!/_- \psi_+^{(1,0)} \right)$$

(plus a four-fermion term that will not affect the present discussion). One of the basic R-symmetries J_L rotates the $\psi_-^{(0,1)}$ and $\psi_-^{(1,0)}$ and the other J_R rotates in the same manner $\psi_+^{(0,1)}$ and $\psi_+^{(1,0)}$. Thus, under a chiral rotation generated by $a_R J_R + a_L J_L$, the measure of integration in the path integral $\mathcal{D}\psi_+\mathcal{D}\psi_-$ changes by an amount proportional to

$$a_R \mathrm{index}(\partial\!\!\!/_+) + a_L \mathrm{index}\partial\!\!\!/_-.$$

The operators $\partial\!\!\!/_\pm$ are Dirac operators of positive or negative chirality acting on the pullback to the Riemann surface of the holomorphic tangent bundle $T^{(1,0)}(X)$. By the index theorem both of these operators have index given by $c_1(X)$ evaluated on the fundamental class of the Riemann surface. Thus, we see that the R-symmetry induced from the R-symmetry in four-dimensions is not anomalous, since $\mathrm{index}\partial\!\!\!/_+ - \mathrm{index}\partial\!\!\!/_- = 0$, but that the R-symmetry coming from the spatial rotation has an anomaly: $\mathrm{index}\partial\!\!\!/_+ + \mathrm{index}\partial\!\!\!/_- = 2\mathrm{index}\partial\!\!\!/_+$.

We have just given a computation for the anomaly for the σ-model. Let us make an analogous computation in $U(1)$-gauge theories with chiral superfields $A_i, 1 \leq i \leq n$. The superfield A_i is a section of the q_i-power of the line bundle associated to the connection on $2|4$-space. It can be expanded in terms of ordinary fields, both bosons and fermions. The R-symmetries act as chiral symmetries on the fermions. So the same type of anomaly-index computation as above shows that the anomaly of J_R is given by

$$\sum_i q_i \int d^2y \frac{F_A}{2\pi}$$

where F_A is the curvature of the ordinary connection on the Riemann surface. (The anomaly for J_L is the negative of this number.) Thus, in order to get a superconformal field theory we need $\sum_i q_i = 0$.

IV One Example

To make things as simple and concrete as possible let us focus on one special case of our $U(1)$-gauge theory: $n+1$ chiral superfields

$$A_1, \ldots, A_n \quad \text{of weight} \quad q = 1$$

$$A_{n+1} = P \quad \text{of weight} \quad q = -n.$$

This choice of weights assures us that our R-symmetries are anomaly-free. Now it is time to supplement the minimal Lagrangian $\mathcal{L}_0$ of Equation 1 by a superpotential and a twisted superpotential. That is to say we consider Lagrangians of the form

$$\mathcal{L} = \mathcal{L}_0 + \left(\int d^2y d\theta^+ d\theta^- W(A,P) + \text{c. c.} \right) + \left(\int d^2y d\overline{\theta}^+ d\theta^- \widetilde{W}(\Sigma) + \text{c. c.} \right). \tag{2}$$

Here, W is called the superpotential; it must be a holomorphic function on $\mathbf{C}^{n+1}$. The function $\widetilde{W}(\Sigma)$ is called the twisted chiral superpotential term.

Let us examine the consequence of assuming that each of these two terms in the Lagrangian is invariant under the R-symmetries. We consider the twisted chiral superpotential term $\widetilde{W}$ first. Here, it is important to use the fact that our theory is obtained by dimensionally reducing a gauge theory on $4|4$-space. Starting with a connection $A = \sum_{\mu=0}^{3} A_\mu dx^\mu$ on $4|4$-space we form

$$\sum_{\alpha=0}^{1} dy^\alpha + \sigma(dx^2 + idx^3) + \text{c. c.}.$$

This shows us how Σ transforms under the R-symmetries. For the extra term in Equation 2 coming from the twisted superpotential to be invariant, it must have the same transformation law under the R-symmetries as Σ does. That is to say $\widetilde{W}(\Sigma)$ must be a linear function of Σ, i.e., $\widetilde{W}(\Sigma) = t\Sigma$ for some $t \in \mathbf{C}$. Writing

$$t = \frac{\theta}{2\pi} - ir,$$

we have

$$\left(\int d^2y d\overline{\theta}^+ d\theta^- t\Sigma \right) + c.\ c. = \theta \int \frac{F}{2\pi} + r \int D.$$

Thus, we see that adding such a linear twisted superpotential produces a Fayet-Iliopulous term $r \int D$ plus a θ-angle term which is θ times an integral topological quantity. In this case the Fayet-Iliopulous term corresponds to adding a constant to the moment map for the $U(1)$-action on $\mathbf{C}^{n+1}$.

Now let us turn to the superpotential $W(A,P)$. For the term

$$\int d^2y d\theta^+ d\theta^- W(A,P)$$

to be invariant, we need $W(A,P)$ to transform with weight two under the R-symmetry induced from the R-symmetry in four dimensions and be invariant under the R-symmetry induced by spatial rotation in four-space. In order to arrange this, we choose the action of the R-symmetry groups on the line bundles of which the A_i and P are sections as follows: the line bundles L_i

with sections A_i are invariant under the R-symmetries and the line bundle L_{n+1} with section P is invariant under the R-symmetry induced by spatial rotation and has weight two under the R-symmetry coming from the R-symmetry of four-space. With this choice of R-symmetry actions any superpotential of the form

$$W = P \cdot F(A_i)$$

for F an arbitrary polynomial of $A_1, \ldots, A_n$, is invariant under the R-symmetries. There is of course, one extra condition, namely that W must be gauge invariant. This implies that F must be homogeneous of degree n. Thus, any W of this form will produce a gauge invariant term in the Lagrangian which is invariant under the R-symmetries. This is the most general interaction with these symmetries.

IV.1 Classical analysis

Now that we have specified our theory in detail, we want to determine what this theory is, particularly in the limits $r \to \pm\infty$. We begin with a classical analysis. Write each of the superfields appearing in our Lagrangian in components

$$A = a + \theta(\cdots) + \cdots$$

$$P = p + \theta(\cdots) + \cdots$$

$$\Sigma = \sigma + \theta(\cdots) + \cdots$$

The superfield Σ includes auxiliary fields D and F which can be eliminated by means of their equations of motion.

The bosonic potential is a function $V(a, p, \sigma)$ of the bosonic components of these fields. For the Lagrangian we have described, the potential takes the form

$$V(a, p, \sigma) = \frac{1}{2e^2} D^2 + |dW|^2 + |\sigma|^2 \left(\sum |a_i|^2 + n^2 |p|^2 \right). \tag{3}$$

The last term in eq. (3) arises as follows: in four dimensions, a chiral field $\Phi = \phi + \theta(\cdots) + \cdots$ leads to a term in the potential of the form

$$\sum_{\mu=0}^{3} \left| \frac{D\phi}{Dx^\mu} \right|^2$$

which upon reduction to two dimensions becomes

$$\sum_{\alpha=0}^{1} \left| \frac{D\phi}{Dx^\alpha} \right|^2 + n^2 |\sigma|^2 |\phi|^2$$

if ϕ has charge n.

In general, the "D-term" (the auxiliary field in Σ), is equal by its equations of motion to $\sum q_i |\phi_i|^2 - r$. This is actually a familiar function mathematically; it is the "moment function" generating the $U(1)$ gauge action on the flat Kahler manifold (a copy of $\mathbf{C}^s$, with s the number of chiral superfields) in which the chiral superfields take their values. In our example, this becomes

$$D = \sum |a_i|^2 - n|p|^2 - r. \tag{4}$$

Since $W = P \cdot F(A)$, we also have

$$|dW|^2 = |F|^2 + |p|^2 |dF|^2. \tag{5}$$

We will focus on the case in which F is *transverse*, i.e., $F = dF = 0$ only at the origin.

We want to find all classical zero-energy solutions, in other words, to solve $V = 0$. Thanks to the form of the potential, this requires that (i) $D = 0$, (ii) $|dW|^2 = 0$ and (iii) either $\sigma = 0$ or $\sum |a_i|^2 + n^2 |p|^2 = 0$. We will also need to take the gauge equivalence under $U(1)$ into account.

First, note that thanks to eq. (5), setting $|dW|^2 = 0$ when $p \neq 0$ implies $F = dF = 0$ so by our transversality condition, all $a_i = 0$. Thus, there are two cases: $p = 0$ or $a_i = 0$ for all i. On the other hand, if $r \neq 0$, then by eq. (4) since $D = 0$ we cannot have both $p = 0$ and $a_i = 0$ for all i. (In fact the sign of r will determine which one of these holds.) Thus, when $r \neq 0$ we have $\sum |a_i|^2 + n^2 |p|^2 \neq 0$ and so σ must be 0.

Consider now the case $r > 0$. Our equations for classical vacua become $p = 0$, $\sum |a_i|^2 = r$, and $F = 0$, and we must divide by the action of the gauge group $U(1)$. This gives the hypersurface X defined by the equation $F = 0$ in $\mathbf{CP}^{n-1}$, with Kähler class r. Thus, classically our theory can be described as a nonlinear σ-model whose target space is this hypersurface X, using a metric whose Kähler class has volume proportional to r^{n-2}. As we noted above, such a theory has central charge $\widehat{c} = n - 2$.

This classical description will be a good approximation to the quantum theory when $r \gg 0$, since the nonlinear σ-model is weakly coupled there. In fact, since the nonlinear σ-model is stable (in the sense of having no relevant perturbations), and marginal perturbations merely vary the complexified Kähler and complex structures, we should expect our quantum theory to coincide with one member of this family of σ-models.

On the other hand, when $r < 0$, the space of classical vacua satisfies $a_i = 0$ and $n|p|^2 = -r$. We can use a gauge transformation to fix $p = \sqrt{-r/n}$, leaving a residual gauge invariance of $\mathbf{Z}_n$, i.e., the original $U(1)$ gauge group is broken to $\mathbf{Z}_n$. This will therefore be what is known as an "orbifold" theory, in which the $\mathbf{Z}_n$ shows up in a global analysis of the theory (in a manner which we shall describe later). The local description of the theory (for which we can ignore the orbifolding issue) is this: for $r \ll 0$, the field P has a large mass and can be integrated out, leaving an effective theory of n massless chiral superfields $A_1, \ldots, A_n$ with an effective interaction

$$W_{\text{eff}} = \text{constant} \cdot F(A_1, \ldots, A_n),$$

where F is a homogenous polynomial of degree n.

Such a theory of n massless fields with a polynomial interaction, is called a *Landau–Ginzburg theory*. It should apparently flow in the IR to a conformal field theory with $\widehat{c} = n - 2$. (We expect this since that is what happened for $r \gg 0$, and neither the conformality nor the central charge of the IR theory should change when we vary r.)

Let's check this prediction, following work of Greene, Vafa, Warner, and Martinec among others. Consider a special F of "Fermat type:"

$$F(A_1, \ldots, A_n) = \sum A_i^n.$$

The Landau–Ginzburg theory with this interaction will (at least locally) factor as a product of n identical theories, each with a simple A^n interaction. Our prediction implies that each of these theories must have $\widehat{c} = \frac{n-2}{n}$. But that is exactly the central charge of a minimal model! So if we can identify the minimal model with this Landau–Ginzburg theory, we will have established the result. Note that we would have been in difficulties if we had obtained a value of $\widehat{c}$ less than one and not of the form $1 - 2/k$ for some integer k! Greene, Vafa, Warner and Martinec presented evidence that the Landau–Ginzburg theory with superpotential A^n does flow in the infrared to an $N = 2$ minimal model. More arguments have been found subsequently. We will see some of the evidence for this claim in Lecture II-14.

IV.2 Orbifolding

As we pointed out above, the original gauge symmetry group of $U(1)$ of our theory was broken to $\mathbf{Z}_n$ in the effective theory when $r \ll 0$. Thus, if we consider the path integral for the low energy effective theory on a Riemann surface Σ, the $U(1)$ gauge group is reduced to $\mathbf{Z}_n$, and we need only sum over flat $\mathbf{Z}_n$ bundles. On any surface Σ, a flat $\mathbf{Z}_n$ bundle can be specified by its holonomies $\gamma_1, \ldots, \gamma_k \in \mathbf{Z}_n$ around various loops in Σ. The process of summing over all flat $\mathbf{Z}_n$ bundles in a theory with a $\mathbf{Z}_n$ symmetry is known as *orbifolding* the theory.

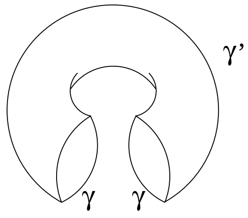
In a Hamiltonian approach, formulated on a circle S^1, we need to specify the holonomy $\gamma \in \mathbf{Z}_n$ around the circle as part of the data determining a state. Let $\mathcal{H}_\gamma$ be the Hilbert space of states whose holonomy is γ. (Then Riemann surfaces with boundary on which flat $\mathbf{Z}_n$ bundles have been specified will determine operators mapping among these various Hilbert spaces, depending on the holonomies on the boundary circles.) $\mathbf{Z}_n$ acts on each $\mathcal{H}_\gamma$, since as the group $\mathbf{Z}_n$ is abelian, twisting as one goes around the circle by an element of $\mathbf{Z}_n$ is an operation that commutes with $\mathbf{Z}_n$.

We claim that the Hilbert space for the orbifolded theory is

$$\mathcal{H} = \bigoplus_\gamma (\mathcal{H}_\gamma)^{\mathbf{Z}_n}.$$

To see this, consider a cylinder with holonomy γ on the ends:

There is still as we have noted an action of $\mathbf{Z}_n$ on each sector $\mathcal{H}_\gamma$; let $\gamma' \in \mathbf{Z}_n$ determine an operator $\gamma' : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$. The trace of this operator can be evaluated as a partition function $Z_{\gamma'\gamma}$ on a torus



constructed by using γ' to twist the identification of bundles on the boundary circles of the cylinder, i.e.,

$$Z_{\gamma'\gamma} = \mathrm{Tr}_{\mathcal{H}_\gamma}\, \gamma' q^H .$$

If we sum the operators γ', the result is $n\pi$ where π is the projection onto the $\mathbf{Z}_n$-invariant subspace:

$$\frac{1}{n}\sum_{\gamma'} Z_{\gamma'\gamma} = \mathrm{Tr}_{\mathcal{H}_\gamma}\, \pi q^H = \mathrm{Tr}_{(\mathcal{H}_\gamma)^{\mathbf{Z}_n}}\, q^H .$$

Now the partition function for the $\mathbf{Z}_n$ gauge theory[1] is obtained by summing further on γ:

$$\frac{1}{n}\sum_{\gamma,\gamma'} Z_{\gamma'\gamma} = \mathrm{Tr}_{\oplus(\mathcal{H}_\gamma)^{\mathbf{Z}_n}}\, q^H ,$$

from which we conclude that the Hilbert space for our problem is indeed $\mathcal{H} = \bigoplus_\gamma (\mathcal{H}_\gamma)^{\mathbf{Z}_n}$.

IV.3 Interpolation from positive to negative r

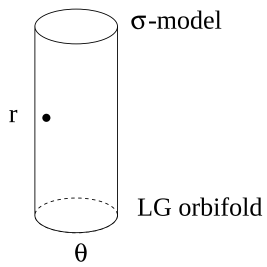
Our classical analysis can be summarized as follows: we needed to set $D = dW = 0$ and divide by $U(1)$. The step of setting $D = 0$ and dividing by $U(1)$ is the familiar mathematical operation of *symplectic reduction*, in which $D = 0$ defines a level set for the moment map of the $U(1)$ action (with the choice of r specifying the level). There is another mathematical interpretation

[1] In performing the path integral in a gauge theory, one is supposed to divide by the volume of the gauge group. When the gauge group is a Lie group of positive dimension, the group of gauge transformations is infinite-dimensional and making sense of "dividing by the volume of the gauge group" requires the Faddeev–Popov construction and introduction of ghosts. In the present case, we are considering a low energy effective theory with gauge group $\mathbf{Z}_n$. The only gauge transformations on Σ are constant transformations by elements of $\mathbf{Z}_n$; the volume of the gauge group is therefore n, the number of elements of $\mathbf{Z}_n$. The only gauge connections are the flat connections. So in the formula below, the sum over γ and γ' is the path integral over the $\mathbf{Z}_n$ connections, and the factor of $1/n$ results from dividing by the volume of the gauge group.

of this process, as a quotient in the sense of Geometric Invariant Theory (GIT): we complexify the group $U(1)$ to $\mathbf{C}^*$, and consider the action of $\mathbf{C}^*$ on $\mathbf{C}^{n+1}$ with the same weights as before (the A_i's have weight 1 and P has weight $-n$). There are two possible quotients (topologically): for $r > 0$ the quotient can be interpreted as the total space of the bundle $\mathcal{O}_{\mathbf{P}^{n-1}}(-n)$ (in which p serves as a fiber coordinate), while for $r < 0$ the quotient is $\mathbf{C}^n/\mathbf{Z}_n$.

In either case we must still impose $dW = 0$: in the $r > 0$ case, $W = PF(A_1, \ldots, A_n)$ is a nondegenerate function (in a generalized sense introduced by Bott) and the the space of critical points in $\mathcal{O}_{\mathbf{P}^{n-1}}(-n)$ is the variety in $\mathbf{P}^{n-1}$ defined by the equation $F = 0$; in the $r < 0$ case we have a homogenous polynomial $F(A_1, \ldots, A_n)$ with a highly degenerate critical point at the origin of $\mathbf{C}^n/\mathbf{Z}_n$.

The natural parameter in our Lagrangian is $t = -ir + \frac{\theta}{2\pi}$. It is the possibility of going to $\theta \neq 0$ that will enable us to interpolate from positive r to negative r without meeting the singularity that one would find in classical geometry.



Classically, there is a singularity at $r = 0$ with arbitrary θ, and interpolation is not possible. Quantum mechanically, we claim that the singularity will be isolated, located at $r = r_0$ (for some r_0) and $\theta = 0$. (Classically, the singularity is at $r = 0$, but quantum mechanically, it is shifted away from $r = 0$ by a one-loop correction.) It is believed that there are no other singularities; we will not prove this claim rigorously, but only explain some of the reasons physicists believe it to be true. The more rigorous part of our analysis will be the demonstration that there *is* a singularity at this location; the less rigorous part will be the argument that this is the only singularity.

To make this argument, we again work on a circular space S^1 of finite radius. The compactness means that the story is rather similar to ordinary quantum mechanics in zero space dimensions, so let us first recall what happens in ordinary quantum mechanics of a point particle. We suppose that the particle is moving on a manifold X with a potential function V. If X is compact, or if V grows at the "ends" of X, then one can vary the parameters upon which X and V depend without meeting a singularity in the ground state of the quantum mechanics. However, one will get a singularity if one varies in the parameters to a point at which V no longer grows at infinity. For instance, if $X = \mathbf{R}$, and $V = \frac{1}{2}kx^2$ (with x a linear function on $\mathbf{R}$), then the ground state wave is a smooth function of k as long as $k > 0$, but develops a singularity at $k = 0$. The $k = 0$ problem has no normalizable ground state wave function.

Something like that will happen in our problem on a circle. Recall that the form of our potential is

$$V = |P|^2|F|^2 + |dF|^2 + \frac{e^2}{2}\left(\sum |A_i|^2 - n|P|^2 - r\right)^2 + |\sigma|^2\left(\sum |A_i|^2 + n^2|P|^2\right).$$

For generic r, this theory has the property that the potential grows at infinity. This ensures that the theory has a discrete spectrum, and a ground state wave function that varies smoothly with the parameters. However, at $r = 0$, we see that one can go to infinity in σ, at no cost in energy, as long as A_i and P vanish. This suggests that at $r = 0$ there might be a continuous spectrum and a singularity of the ground state wave function.

To explore this question in more detail, we need to understand how the theory behaves in the dangerous region, that is, very large σ with other fields small. When A and P are very close to zero, and σ is large, A and P have large masses (on the order of $|\sigma|^2$) and they can be integrated out, leaving us with a pure gauge theory with an effective bosonic potential which at the classical level is merely a constant

$$V_{\text{eff}} = \frac{e^2}{2}r^2. \tag{6}$$

If this is the correct answer, quantum states decay exponentially in the large σ region if their energy is less than $e^2r^2/2$. If so, this region could be dangerous for a supersymmetric ground state of zero energy – and could lead to a singularity in the wave function of such a state – only if $r = 0$.

To understand the situation better, we must make the analogous argument quantum mechanically. For this, we consider the superfield to which σ belongs, i.e., we write

$$\Sigma = \sigma + \theta\lambda + \theta\overline{\theta}(F + iD).$$

The action for the supersymmetric gauge theory is

$$\frac{1}{2e^2}\int d^4\theta\,\overline{\Sigma}\Sigma + \left(\int d\overline{\theta}^+\,d\theta^-\,t\Sigma + \text{c.c.}\right). \tag{7}$$

If we perform the θ-integrals in the effective Lagrangian (7), we get

$$\frac{1}{e^2}\int d^2x\left(\frac{F^2}{2} + |d\sigma|^2 + \overline{\lambda}\partial\!\!\!/\lambda\right) + \int d^2x\left(\frac{e^2r^2}{2}\right) + i\theta\int_\Sigma \frac{F}{2\pi},$$

which includes a curvature term (multiplied by a "θ-angle") as well as the bosonic potential previously discussed.

Luckily, we know the θ-dependence of energy in $U(1)$ gauge theories in two dimensions as computed in lecture II-4:

$$\frac{e^2}{2}\min_{n\in\mathbf{Z}}\left(n - \frac{\theta}{2\pi}\right)^2.$$

So as $\sigma \to \infty$, we have $V(\sigma) \sim \frac{e^2}{2}|\widetilde{t}|^2$ where $\widetilde{t} \equiv -ir + \frac{\theta}{2\pi} \pmod{\mathbf{Z}}$. In other words, while the classical energy at large σ is $e^2r^2/2$, quantum mechanically r^2 is replaced by $|\widetilde{t}|^2$. Thus, while classically the singularity is at $r = 0$ and θ is invisible, quantum mechanically the singularity is at $\widetilde{t} = 0$. For any other value of $\widetilde{t}$, there is a positive energy for $\sigma \to 0$, and this region is not dangerous for supersymmetric ground states.

Their is actually one more quantum effect of relevance: a one loop correction gives a finite renormalization of r, and shifts the position of the singularity from $\widetilde{t} = 0$ to $\widetilde{t} = -ir_0$ for a certain constant r_0. This correction is important for comparing to certain predictions of classical geometry, but not important for what we will say today.

V Another example: flops

As a second example, we consider a model with gauge group $G = U(1)$, chiral fields A_1 and A_2 of charge 1, chiral fields B_1 and B_2 of charge -1, with *no* superpotential, and no "P." The

Lagrangian takes the form

$$\int d^2x\, d^4\theta \left(\frac{1}{2e^2}\overline{\Sigma}\Sigma + \sum(|A_i|^2 + |B_i|^2) \right) + \left(t \int d\overline{\theta}^+\, d\theta^-\, \Sigma + \text{c.c.} \right).$$

The D term is

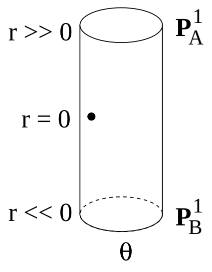
$$D = \sum(|A_i|^2 - |B_i|^2) - r,$$

and the bosonic potential also contains a term

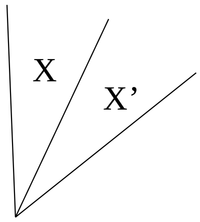
$$|\sigma|^2 \sum(|A_i|^2 + |B_i|^2).$$

As before, we must set $D = 0$ and divide by $U(1)$. When $r > 0$ this yields the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbf{P}^1_A$, whereas when $r < 0$ this yields the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbf{P}^1_B$. Here $\mathbf{P}^1_A$ is a copy of $\mathbf{P}^1$ obtained by requiring A_1, A_2 to be not both zero, and dividing the pair (A_1, A_2) by $\mathbf{C}^*$. Likewise $\mathbf{P}^1_B$ is a copy of $\mathbf{P}^1$ obtained by projectivizing (B_1, B_2).

The transformation from the bundle over $\mathbf{P}^1_A$ to the bundle over $\mathbf{P}^1_B$ is a simple model of a birational transformation known as a "flop." As in the previous example, the singularity only occurs at $r = \theta = 0$ and one can interpolate between these two models. This contrasts with classical geometry, where one passes through a singularity in going from one model of the quotient $\mathbf{C}^4/\mathbf{C}^*$ to the other.



More generally, consider two Calabi–Yau manifolds X and X' which are birationally equivalent, in such a way that their ample cones meet.



For simplicity assume $h^{2,0} = 0$, and note that birational transformations won't affect the value of $h^{p,0}$. Classically, to pass from X to X', one goes through a singularity on the wall of the Kahler cone. The singularity is rather similar to the singularity just found in our "flop" example at $r = 0$. In quantum field theory, one can go around the singularity by taking $\theta \neq 0$; thus one can smoothly continue from X to X' by varying the parameters of the conformal field theory.

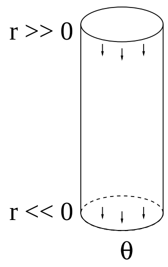
For example, we might construct such a pair of Calabi–Yau manifolds with $h^{1,1} = 2$ by starting with a gauge group $G = U(1) \times U(1)$, and a superpotential of the form $W = PF(A_i)$. If one sets things up correctly, one of the $U(1)$'s puts us in the world of projective varieties X and X', and by varying the moment map for the other $U(1)$, we can make a "flop" between X and X' that is quite like our above discussion with the four chiral superfields fields A_i and B_j.

VI Cases in which $c_1 \neq 0$

We now wish to consider linear sigma models in which the R-symmetry is anomolous, so that we expect a nontrivial renormalization group flow.

VI.1 Negative β-function

We begin with the case of negative β-function, which for nonlinear sigma models corresponds to target spaces X for which $c_1(TX) > 0$. We can build examples of these exactly as before, with $U(1)$ acting on $n+1$ fields A_i, P, but this time we give P charge $-k$ and take a superpotential $W = P \cdot F$ with F homogeneous of degree k. The beta function of the sigma model is negative, zero, or positive for $k < n$, $k = n$, or $k > n$. For $k \neq n$, even though there is still classically a singularity at $r = 0$, the quantum theory has no singularity on the t-cylinder. (This is shown by analyzing the behavior at large σ, as we do presently.) For $k \neq n$, there is a nontrivial renormalization group flow on the cylinder. The flow is holomorphic and singularity-free, so it is a constant flow "downward" or "upward" on the cylinder, depending on the sign of $n-k$.



When we repeat the previous analysis, we find that the R-symmetry which comes from the four-dimensional R-symmetry is anomalous, although the one which comes from rotations in the missing directions is anomaly-free.

The classical analysis can also be repeated: for $r \gg 0$ we find a space of classical vacua desribed by $\{F=0\}$ in $\mathbf{P}^{n-1}$, and we expect the nonlinear sigma model on this hypersurface to be a good approximation to our theory. This hypersurface has positive first Chern class if $k<n$, so we expect a well-defined quantum field theory from looking at the nonlinear sigma model. On the other hand, since $k<n$, the sigma model is infrared-unstable. The infrared flow will take us away from the sigma model in the infrared and toward the theory we will find at the $r \to -\infty$ limit. Note that the sigma model has an $n-2$ dimensional target space, so for $k<n$ the effective central charge in the ultraviolet limit is $\widehat{c}=n-2$.

On the other hand, if $r \ll 0$ then on the space of classical vacua $P \neq 0$ but $A_i=\sigma=0$. Giving a nonzero expectation value to P breaks $U(1)$ to $\mathbf{Z}_k$, and (in the $r \to -\infty$ limit) we get a Landau-Ginzburg orbifold with an effective interaction

$$W_{\text{eff}}(A_i)=F(A_i).$$

The central charge of this model is

$$\widehat{c}=n(1-\frac{2}{k}).$$

This is less than $n-2$ if $k<n$. We will argue shortly that, for $k<n$, the Landau-Ginzburg theory is the infrared limit of the renormalization group flow from the sigma model; and the fact

that its central charge is smaller than that of the sigma model illustrates a general theorem by Zamolodchikov, which asserts that the central charge always diminishes along a renormalization group flow: This infrared theory is behaving as if it had fewer degrees of freedom

$$c_{\mathrm{IR}} \leq c_{\mathrm{UV}}.$$

The intuition behind this theorem is that the central charge measures the total number of degrees of freedom of the theory. As one flows toward the infrared, massive degrees of freedom are "integrated out," and no longer contribute to the central charge, which therefore can only diminish.

As we will see, the Landau-Ginzburg model is only one possibility for what the infrared flow can lead to for $k < n$. There are in fact $n - k$ other vacuum states at $r = -\infty$, with a mass gap, which can't be seen classically but will require a 1-loop calculation. Being massive, they have $\widehat{c} = 0$, which is certainly less than $n - 2$.

VI.2 The $\mathbf{CP}^{n-1}$ model

An extreme case of the situation we are considering is the case $k = 0$, i.e., the case of no P field and no superpotential. This is a $U(1)$ gauge theory with n free charged superfields $A_1, \ldots, A_n$ (all of charge 1). The D-term takes the form

$$D = \sum |A_i|^2 - r,$$

and we find the supersymmetric $\mathbf{CP}^{n-1}$ model which we studied earlier. It will be very instructive to re-examine this case in detail before going back to the general case.

At first glance, there would appear to be no supersymmetric vacua at all in the $r \to -\infty$ limit of this $\mathbf{CP}^{n-1}$ model, since the moment map is strictly positive. However, it cannot be so that there are no supersymmetric vacua at all in this limit. An obstruction is provided by the supersymmetric index $\mathrm{Tr}\,(-1)^F$. This, when computed for large positive r, is seen to coincide with the Euler characteristic of $\mathbf{CP}^{n-1}$, which is n. The supersymmetric index must be invariant under deformations of r, so there must be n supersymmetric vacua even if r is large and negative. We must find them somewhere!

To see what is happening, we must consider the quantum mechanics more carefully. The classical treatment of the $r \to -\infty$ limit is valid in any compact region of field space, so if there are quantum mechanical vacua for arbitrarily negative r that cannot be seen classically, they must disappear from the classical field of view by going off to infinity in field space for $r \to -\infty$. So again we must consider the quantum-mechanical behavior at large σ. For our CP^{n-1} model, the bosonic potential takes the form

$$V = \frac{e^2}{2} D^2 + |\sigma|^2 \sum |A_i|^2.$$

The theory is weakly coupled for $r \ll 0$.

What happens if $|\sigma| \to \infty$? As earlier, the fields A_i will then be massive and can be integrated out to yield an effective theory which is an abelian gauge theory with Lagrangian

$$\mathcal{L}_{\mathrm{eff}} = \int d^2x\, d^4\theta\, \frac{1}{2e^2}\, \overline{\Sigma}\Sigma + \left(\int d^2x\, d\overline{\theta}^+\, d\theta^-\, \widetilde{W}_{\mathrm{eff}}(\Sigma) + c.c. \right).$$

The effective twisted superpotential depends on the renormalization mass scale μ, and we claim that it takes the unusual (R-symmetry violating) form

$$\widetilde{W}_{\mathrm{eff}}(\Sigma) = t\Sigma + \frac{in}{2\pi} \left(\Sigma - \Sigma \ln(\Sigma/\mu) \right) \tag{8}$$

which we shall verify by making a 1-loop calculation below. Notice that a change in the choice of logarithm in eq. (8) can be compensated for by changing the value of t by an integer; such a change has no effect on the physics. Of course, we must keep Σ away from zero in order to define this term, but that is consistent with our assumption that σ is large.

With this new term in the twisted superpotential, the equation for a critical point of $\widetilde{W}_{\mathrm{eff}}$ is

$$0 = t - \frac{in}{2\pi} \ln(\Sigma/\mu),$$

or in components,

$$0 = -ir + \frac{\theta}{2\pi} - \frac{in}{2\pi} \ln(\sigma/\mu) \quad (\mathrm{mod}\ 1).$$

This has n solutions

$$\sigma = \mu \exp\left(-\frac{2\pi r}{n} - \frac{i\theta}{n} + \frac{2\pi i k}{n} \right), \tag{9}$$

$k = 0, \ldots, n-1$, corresponding to n different vacua in the $r \to -\infty$ limit. n is of course, the expected number, the Euler characteristic of $\mathbf{CP}^{n-1}$. Notice that $|\sigma|$ is indeed large when

$r \to -\infty$, so the assumptions made in deriving these vacua hold. The fact that the n vacua go off to infinity for $r \to -\infty$, along with the fact that their existence depends on the one-loop quantum correction, explains why one cannot see them classically.

Another very important consequence of the logarithm is that it means that for $\sigma \to \infty$, the energy in the quantum theory grows like $|\ln \sigma|^2$. Hence, in particular, low energy states are limited to a bounded region of field space, and there is no singularity in the vacuum behavior for any value of t on the cylinder. As we will see, the models with $n \neq k$ all have such a logarithm, and hence they are all nonsingular throughout the t cylinder.

VI.3 The 1-loop calculation

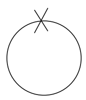
To understand the origin of the term $\frac{in}{2\pi}(\Sigma - \Sigma \ln(\Sigma/\mu))$ in the effective twisted superpotential, note that its presence is equivalent to the effective Lagrangian having a term

$$\int d^2x \, r_{\text{eff}} \, D,$$

where

$$r_{\text{eff}} = r + \frac{n}{2\pi} \ln(|\sigma|/\mu).$$

In this theory, we have $D = \sum \overline{A}_i A_i - r$. Classically, $A_i = 0 = \overline{A}_i A_i$ for large σ, but quantum mechanically $\overline{A}_i A_i$ has an expectation value at large σ; this expectation value can be interpreted in an effective classical description as a shift in $-r$. So we need to calculate the expectation value of $\overline{A}A$ by means of a 1-loop diagram



which contributes

$$\int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + m_A^2} = \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + |\sigma|^2}$$

where we got the mass m_A from the term $|\sigma|^2 |A|^2$ in the potential V. Cutting off the divergent integral, we find

$$\int^{\Lambda} \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + |\sigma|^2} = -\frac{1}{2\pi} \ln \frac{|\sigma|}{\mu}.$$

This is the shift in $-r$, and accounts for the claimed formula for r_{eff}. There are, by the way, no higher loop corrections to this formula. This can be proved using holomorphy, or by noting that, in this superrenormalizable theory, higher loop diagrams would involve negative powers of σ and would vanish for large σ.

Let us now repeat this for a general case with chiral superfields B_i of charges q_i. (The B_i are to include all of the chiral superfields, including the P field of the sigma model discussion, if it is present.) The D-term is $\sum q_i |B_i|^2$ and for its expectation value at large σ we get the 1-loop correction

$$\sum q_i \int^{\Lambda} \frac{d^2k}{(2\pi)^2} \frac{1}{k^2+q_i^2|\sigma|^2} = -\frac{1}{2\pi}\sum q_i \ln|q_i| - \frac{1}{2\pi}(\sum q_i)\ln\frac{|\sigma|}{\mu},$$

leading to an effective twisted superpotential

$$\widetilde{W}_{eff}(\Sigma) = \left(t - \frac{i}{2\pi}\sum q_i \ln|q_i|\right)\Sigma + \frac{i\sum q_i}{2\pi}(\Sigma - \Sigma\ln(\Sigma/\mu)).$$

In the Calabi-Yau case, $\sum_i q_i = 0$ and the coefficient of the logarithm vanishes. Hence $\widetilde{W}_{eff}$ is linear in Σ, as a result of which its contribution to the potential energy is independent of σ. As a result of this, it is possible for σ to go to infinity at no cost in energy if t has the correct value. At this value of t, one gets a singularity. The correct value is the value at which the coefficient of Σ in the twisted chiral superpotential vanishes; the condition is not $\mathrm{Im}\, t = 0$, as one would expect classically, nor even $t = 0$, but rather $t - (i/2\pi)\sum_i q_i \ln|q_i| = 0$. This is the shift in the position of the singularity that was mentioned in our discussion of the Calabi-Yau case. In order to compare the present discussion to, for instance, the counting of rational curves on the Calabi-Yau manifold, it is important to know about this shift. When $\sum_i q_i \neq 0$, the logarithm prevents σ from going to infinity, and hence prevents a singularity on the t cylinder, as we have already asserted.

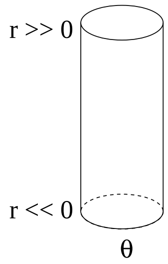
On the other hand, if $\sum q_i \neq 0$, then there are $|\sum q_i|$ extra vacua (as in the $\mathbf{CP}^n$ model) with σ-values

$$\sigma = \mu \exp\left(-\frac{2\pi r}{\sum q_i} - \frac{i\theta}{\sum q_i} - \frac{\sum q_i \ln q_i}{\sum q_i} + \frac{2\pi i k}{\sum q_i}\right), \tag{10}$$

However, the prediction of these vacua is only valid if they occur at large σ. Hence, these vacua are only trustworthy for $r \to -\infty$ if $\sum_i q_i > 0$, or for $r \to +\infty$ if $\sum_i q_i < 0$.

VI.4 Hypersurfaces

Let us now return to the situation of a hypersurface defined by a homogeneous polynomial F of degree k in n variables. Thus $\sum_i q_i = n - k$. Hence, vacua at large σ are present for $r \to -\infty$ if $n > k$, or at $r \to +\infty$ if $n < k$. The vacua at large σ are the ones that one cannot see classically. In addition to those, one has vacua that one can see classically – a sigma model for $r >> 0$ and a Landau-Ginzburg orbifold for $r << 0$.



Let us now consider the two cases in detail. For $k < n$, the sigma model has for its target space a Fano variety, a hypersurface in $\mathbf{CP}^{n-1}$ of degree $k < n$. This is a good description in the ultraviolet, and gives a well-posed quantum field theory whose infrared behavior we would like to determine. The infrared flow brings us to $r \ll 0$, where we find two types of vacua: (1) vacua which are conformal field theories of central charge $\widehat{c} = n(1 - \frac{2}{k})$, and (2) $n - k$ new vacua at large σ. The corresponding effective twisted superpotential for large σ is

$$\widetilde{W}_{\mathrm{eff}} = t\Sigma + \frac{i}{2\pi}(n-k)(\Sigma - \Sigma \ln \Sigma).$$

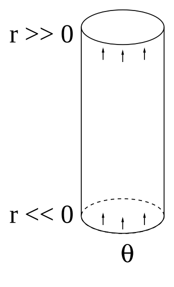
Both of these types of vacua have central charge less than the value $\widehat{c} = n - 2$ of the UV theory. Thus, in this example, a single quantum field theory in the ultraviolet can flow to quite different possibilities in the infrared. Renormalization group flow for $N = 2$ sigma models in two dimensions has many invariants (such as $\mathrm{Tr}\,(-1)^F$, the elliptic genus, and some of their cousins). To reproduce the invariants of the UV sigma model with Fano target, one must sum over the infrared vacua of types (1) and (2).

VI.5 The case $c_1 < 0$

Consider now the opposite case $k > n$. A hypersurface of degree $k > n$ in $\mathbf{CP}^{n-1}$ is an algebraic variety of general type. In this case $\beta > 0$, and there is not a well-defined quantum field theory of maps from spacetime to this hypersurface. However, the sigma model with this hypersurface as target makes sense as an effective infrared theory. One can ask what well-defined ultraviolet theory can flow to this effective theory in the infrared.

For this question, we will now give an answer. For $k > n$, the ultraviolet stable end of the cylinder is the Landau-Ginzburg end. The Landau-Ginzburg model has central charge $\widehat{c} = n(1-2/k)$, which for $k > n$ is greater than the value $\widehat{c} = n-2$ of the sigma model. Thus the Zamolodchikov theorem permits a renormalization group flow from the LG model to the sigma model. That is precisely what happens, since for $n < k$ the renormalization group flow on the cylinder is "upwards," away from the LG model and towards the sigma model.

If $k > n$, there are no large σ vacua at the LG end of the cylinder. Instead, $k-n$ such vacua accompany the sigma model. Thus, a well-defined ultraviolet fixed point, given by the Landau-Ginzburg model, can flow in the infrared either to the sigma model of the hypersurface, or to a nonclassical massive vacuum at large σ. Just as for $k < n$, to equate renormalization group invariants such as $\mathrm{Tr}\,(-1)^F$ between the ultraviolet and infrared theories, one must sum over both types of vacua in the infrared limit.



Note that, regardless of the sign of $n-k$, the nonclassical vacua always appear in the infrared, not the ultraviolet. There is a good intuitive reason for this. In the ultraviolet, where we are defining our theory, we can select what theory we wish to study (from all of the possible UV fixed points), so the UV theory has a vacuum of just one type. Going to the infrared means solving the equations, and at this stage things are out of our hands: a definite quantum field theory may have more than one solution for its infrared behavior. That is what we have found today. In fact, we have seen that a definite UV field theory (based on a Fano variety if $k < n$ or a Landau-Ginzburg model if $k > n$) can flow to quite different possibilities in the infrared.

Both of the $k \neq n$ cases that we have studied today are of considerable methodological interest. For $k < n$, we have seen how the quantum theory with a Fano target gives, in general, an answer which is a sort of mixture of the behavior of a Calabi-Yau manifold and the behavior of complex projective space. For $k > n$, we have found, in a concrete situation, a well-defined UV fixed point with flow to the effective theory based on an algebraic variety of general type.