Lecture II-13: $N = 2$ SUSY theories in Dimension Two, Part II: Chiral Rings and Twisted Theories

Edward Witten*

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This lecture is again concerned with $N = 2$ supersymmetric theories in dimension 2. Using one of the supersymmetries Q, we shall define the chiral ring of local operators and the Q-cohomology of states for theories defined in flat Minkowski or Euclidean space. We then discuss how to twist theories so that the supersymmetry Q has global meaning at least over a large class of $2|4$ supermanifolds. Once Q has been globalized in a theory over Σ we have the notions of the ring of chiral local operators for that theory. We can define what are called descendants of Q-invariant operators, and we show that the integrals of these descendants over cycles in the manifolds lead to correlation functions which have topological meaning. To compute these functions one does integrals over the Q-invariant fields.

We then consider in detail two examples of twisted theories with a global supersymmetry Q. The first is pure $N = 2$ gauge theory in dimension 2 with a simple Lie group G. Here, we see that the ring of chiral functions is identified with the invariant polynomials on the Lie algebra. Classically, the space of Q-invariant states is the usual space of flat connections (plus an extra parallel section of the adjoint bundle in the case of reducible connections). The correlation functions of the Q-invariant operators are identified with the integrals over the moduli space of flat connections of the usual classes derived from the invariant polynomials in the Lie algebra. (There are extra complications from reducible connections.)

The second example is $N = 2$ supersymmetric two-dimensional σ-models into a Kähler

*Lecture notes by John Morgan

manifold. It turns out that there is one way of twisting that leads to a chiral ring which is the topological cohomology of the Kähler manifold. The space of Q-invariant states is the space of holomorphic maps from the riemann surface into X. In this twisted theory the path integrals which commute the topological correlation functions are the integrals in quantum cohomology. When the Kähler manifold is Calabi-Yau, there is a second, inequivalent way of twisting which leads to a different theory. In this theory the chiral ring is $H^*(X, \wedge^* T^{1,0}(X))$ and the Q-invariant fields are constant maps from the riemann surface to X. Thus, the path integrals which compute the topological correlation functions are integrals of these objects over X – these integrals are related to variation of Hodge structures on X.

I R-symmetry revisited

Most of the $N = 2$ supersymmetric examples we shall consider will have (at least classically) both left- and right-moving R-symmetries – the R-symmetries denoted by J_L and J_R in the last lecture. It is possible to carry out today's discussion with only one of these R-symmetries, but we must have at least one. But we will suppose that the group of R-symmetries is $U(1) \times U(1)$ so that we have both J_R and J_L. The supermanifold on which the theory is defined in $\mathbf{R}^{2|4}$. We shall also eventually allow a certain class of split $2|4$ super-riemann surfaces with a Euclidean metric (and $2|4$-supermanifolds with a flat Minkowski metric). The space of odd directions is a copy of the two-dimensional representation for J_L and the two dimensional representation for J_R. We expect a symmetry between J_L and J_R obtained by reversing the orientation of the worldsheet.

One operation we can attempt to do to a symmetry of a Lagrangian is to gauge it. If the symmetry group to be gauged is H, then the new Lagrangian will involve gauge fields with gauge group H times the gauge group (if any) of the original theory. Intuitively, the process of gauging involves replacing fields with values in a constant space by fields with values in an H-bundle and adding a new field – the H-connection. In order to carry out this program the original Lagrangian must be a local expression in the fields and the symmetry must be also be local in the sense that it maps local fields to local fields at the same point. With these reasonable assumptions on the Lagrangian and its symmetry, it is always possible to form a new, gauged, Lagrangian. We replace ordinary derivatives in the Lagrangian by covariant derivatives. Quite often there is a natural way to carry out this process, but in any case different choices for gauging the symmetry lead to Lagrangians which differ by gauge invariant terms up to a total derivative. There is however the issue of whether the gauged theory makes sense as a quantum theory. If the original theory has this property then the necessary and sufficient condition that the gauged theory does as well is that the symmetry being gauged be anomaly-free.

In our case it is never possible to gauge the entire $U(1) \times U(1)$ group of R-symmetries to produce a theory that makes sense as a quantum theory. In fact the only combinations of J_L and J_R that can be gauged in this way are $J_L \pm J_R$. These are referred to as the vector and axial currents

$$J_V = J_L + J_R$$

$$J_A = J_L - J_R.$$

If one of these is gauged, then the other develops anomalies and stops being a symmetry of the quantum theory. To explain this, we need some general observations about anomalies. The statement that a theory has a symmetry, generated by a current J_L, means that correlation funcions obey the identity

$$d_x \langle J_L(x) \mathcal{O}(y_1) \ldots \rangle = 0 \quad (*)$$

as long as x is distinct from the y_i. Here d_x is the exterior derivative in the x variable. If one extends the correlation functions (as distributions) across the diagonal, this identity will pick up extra terms supported on the diagonal. A very important special case is the two point function of the current itself (or, in $2k$ spacetime dimensions, the $k+1$ point function). For this the general structure is

$$d_x \langle J_L(x) J_L(y) \rangle = \frac{k}{2\pi} d_y \delta(x-y) \quad (**)$$

with some constant k, which may or may not be zero. The condition that the global symmetry generated by J_L can be gauged is precisely that $k=0$. To see this, let us try to gauge the symmetry by adding to the theory a $U(1)$ gauge field A. The Lagrangian obtains an extra term $\int A \wedge J_L$. (There may be more terms, higher order in A, if J_L itself contains derivatives of charged fields.) Now, let us try to determine whether the theory with this coupling actually has the symmetry generated by J_L, that is whether the identity of equation $(*)$ holds for this theory. For this we must look, in the original theory, at

$$d_x\langle J_L(x)\mathcal{O}(y_1)\ldots\exp(\int A\wedge J_L)\rangle$$

The exponential is included to express correlation functions of the new theory in terms of the old theory. If we expand this in powers of A, the term linear in A will contribute, upon using $(**)$:

$$\langle\mathcal{O}(y_1)\ldots\cdot kF(x)\rangle$$

with F the curvature. This can be expressed as an operator relation

$$dJ_L=\frac{k}{2\pi}F,$$

asserting that after gauging J_L no longer generates a symmetry.

More generally, if there are several currents $J^1,\ldots,J^s$, generating a Lie group G of dimension s, with Lie algebra $\mathcal{G}$, then the two point functions obey

$$d_x\langle J^a(x)J^b(y)\rangle=\frac{k^{ab}}{2\pi}d_y\delta(x-y)$$

with k^{ab} some invariant quadratic form on $\mathcal{G}$. The quadratic form k has no positivity (since for instance left and right moving massless fermions make contributions of opposite signs) but does obey an integrality condition. If we gauge a subgroup H of G, with Lie algebra $\mathcal{H}$, then by a reasoning along the lines of the above, the quantum theory only makes sense if the quadratic form k vanishes when restricted to $\mathcal{H}$. If this is so, then the "global symmetries" are generated by currents in $\mathcal{G}$, not in $\mathcal{H}$, that are orthogonal to $\mathcal{H}$ with respect to k.

In our problem, with the two currents J_L, J_R, the structure of k is typically that $k^{LL}=-k^{RR}\neq 0$, $k^{LR}=0$. This quadratic form has two null vectors, $J_L\pm J_R$, so either of those two linear combinations of the R-symmetries can be gauged. If one is gauged, the second becomes "anomalous" – it ceases to be conserved – since the two null vectors are, of course, not orthogonal to each other. "Luckily," these options agree with the choices that we will need below for other reasons.

What we are really interested in is not just gauging but twisting theories by R-symmetries. This process requires first gauging the R-symmetry and then twisting it using a homomorphism from the orthogonal group determined by the metric to the symmetry group. It turns out that there will be two equivalent ways of going about this process. But before we embark on twisting, we must understand the local story. That is the subject of the next section.

II Q-cohomology of operators

We consider $\mathbf{R}^{2|4}$ with its standard Euclidean or Minkowski metric. Suppose that our $N = 2$ supersymmetry theory in dimension two has the following supersymmetries:

characters of $U(1)_L$	characters of $U(1)_R$
Q_-	Q_+
$\overline{Q}_-$	$\overline{Q}_+$.

These supersymmetries obey the relations

$$Q_+^2 = \overline{Q}_+^2 = 0$$

$$\{Q_+, \overline{Q}_+\} = P_+$$

with symmetric relations with $+$ replaced by $-$. Also,

$$\{Q_+, Q_-\} = \{\overline{Q}_+, Q_-\} = \{Q_+, \overline{Q}_-\} = \{\overline{Q}_+, \overline{Q}_-\} = 0.$$

We will consider at the same time four closely related possibilities. We let Q be one of the following:

$$Q_+ + Q_-, \quad Q_+ + \overline{Q}_-, \quad \overline{Q}_+ + Q_-, \quad \overline{Q}_+ + \overline{Q}_-.$$

(The first two possibilities are inequivalent and the last two are hermitian conjugate to the first two and would give equivalent theories.) To fix notation let us choose the generators so that $Q = Q_+ + Q_-$. Letting $\mu = \pm$ we see that P_μ can be written as a commutator

$$P_\mu = \{Q, T_\mu\} \tag{1}$$

where T_μ is a linear combination of $Q_\pm, \overline{Q}_\pm$. We are interested in the cohomology of Q acting as a differential (by superbracketing) on the space of local operators (and more generally on products of such operators). Thus, a local operator $\phi(x)$ will define a Q-cohomology class if

$\phi(x)$ is Q-closed, i.e., if $\{Q, \phi(x)\} = 0$. A simple computation using Equation 1 shows that for a Q-closed operator $\phi(x)$ we have

$$d_\mu \phi = \{Q, [T_\mu, \phi]\}.$$

(This derivative is interpreted to be the covariant derivative if the super-riemann surface is not flat.) This equation means that the operator $d\phi$ is Q-exact, and hence that the class of $d\phi$ in the space of Q-cohomology classes of operators is trivial. This means that as we vary the point x in the riemann surface the Q-cohomology class of the local operator $\phi(x)$ is unchanged. It then follows that in Q-cohomology such Q-closed local operators commute – simply move the points to be space-like separated. That is to say, the space Q-cohomology of operators is a $\mathbf{Z}/2\mathbf{Z}$-graded commutative ring. We can always represent these operators by 0-form operators. This is an easy consequence of the independence of the Q-cohomology class on the metric and diffeomorphism invariance of the Q-cohomology class. These facts will be established later.

Let us recall a construction that we have seen before. Given a Q-invariant 0-form local operator $\phi = \phi^{(0)}$ we define a 1-form local operator by Cross reference?

$$\phi^{(1)}_\mu = [T_\mu, \phi^{(0)}]$$

and

$$\phi^{(1)} = \phi^{(1)}_\mu dx^\mu.$$

The same computation as we did in the previous paragraph shows that

$$d\phi^{(0)} = \{Q, \phi^{(1)}\}.$$

This means that if Σ_1 is a one-cycle in the super-riemann surface then

$$\int_{\Sigma_1} \phi^{(1)}$$

is Q-invariant. Let $\phi^{(2)} = [T, \phi^{(1)}]$. Then

$$d\phi^{(1)} = \{Q, \phi^{(2)}\}.$$

So in this way, beginning with a Q-invariant 0-form local operator $\phi^{(0)}$, we can build a tower of k-form local operators $\phi^{(k)}$, called the descendants of $\phi^{(0)}$, and hence produce more general

Q-invariant operators of the form

$$\int_{\Sigma_k} \phi^{(k)}.$$

Our computations show that for any Q-invariant local operator $\phi^{(0)}$ the operator $\phi^{(0)}+\phi^{(1)}+\phi^{(2)}$ is closed for the total differential $d-Q$.

III Twisting the theory to give it global meaning

So far we have been working on $R^{2|4}$ with fixed coordinates. If we rotate space, then $Q = Q_+ + Q_-$ is not invariant. Our goal is to modify things until Q is invariant under rotations and hence can be defined on more general Euclidean $2|4$-manifolds. Let us denote by $SO(2)_E$ the Euclidean rotation group. The entire group of symmetries at our disposal for the Euclidean theory is $SO(2)_E \times U(1)_L \times U(1)_R$. This group has the obvious projection onto $SO(2)_E$ with a splitting back into the first factor. We wish to find another splitting of this projection map, i.e., a group $SO(2)'_E$ inside the product projecting isomorphically onto $SO(2)_E$ but which has the extra property that it commutes with Q. It is easy to see that there exists a unique such splitting and it is given on the Lie algebra level by

$$K' = K \pm \frac{1}{2}J_L \pm \frac{1}{2}J_R$$

where K is the generator for $SO(2)_E$ and the signs depend on which of the four possibilities we have chosen for Q. In the case that we are concentrating on when $Q = Q_+ + Q_-$ we have

$$K' = K + \frac{1}{2}J_L - \frac{1}{2}J_R.$$

The fact that $1/2$ appears in these formulas means that the twisting actually involves the spin form of $SO(2)_E$.

Notice that in order to make such twisting in dimension n there must be non-trivial homomorphisms from $Spin(n)$ to the R-symmetry group. In dimension two it is possible to do this when the R-symmetry group is $U(1)$, e.g., for $N = 1$ supersymmetry. But for $n = 4$ it is necessary to have $N = 2$ supersymmetry so that the R-symmetry group is $SU(2) \times U(1)$ which has a non-trivial homomorphism from $Spin(4)$.

With this preliminary discussion in place, we are now ready to globalize our theory of Q-cohomology classes of local operators. Suppose that we have a $2|4$ super-riemann surface locally modeled on $\mathbf{R}^{2|4}$. The hypothesis we have in mind for the super-riemann surface is that it be a split $2|4$-manifold complex with the odd directions being

$$\left(\Pi K^{1/2} \oplus \Pi \overline{K}^{1/2}\right) \oplus \left(\Pi \overline{K}^{1/2} \oplus \Pi K^{1/2}\right).$$

Here $\theta^+, \overline{\theta}^+$ are sections of the first two line bundles and $\theta^-, \overline{\theta}^-$ are sections of the last two. The process of twisting changes the odd part of the underlying supermanifold of the theory. We twist the spinor bundles containing $\theta^\pm$ by the square root of the canonical bundle to the power $\pm 1/2$. The result of twisting replaces the above line bundles by

$$(\Pi \mathcal{O} \oplus \Pi K) \oplus (\Pi \mathcal{O} \oplus \Pi \overline{K}).$$

and makes the supersymmetry

$$Q = \frac{\partial}{\partial \theta^+} + \frac{\partial}{\partial \theta^-}$$

which is a global symmetry. Locally, this symmetry agrees with the symmetry Q on $\mathbf{R}^{2|4}$ that we have been discussing.

Once we have globalized the supersymmetry Q, all the previous discussion globalizes. In particular, for a Q-closed local operator $\mathcal{O}(p)$ the Q-cohomology class of $\mathcal{O}(p)$ is independent of the point p. Thus, the Q-cohomology classes of local operators on this global super-riemann surface generate a $\mathbf{Z}/2\mathbf{Z}$-graded commutative ring called the *chiral ring* of the theory.

III.1 Certain Correlation Functions and Their Basic Properties

We will consider path integrals of the following types of path integrals:

$$Z_{(\mathcal{O}_\alpha)} = \int \mathcal{D}\Phi e^{-\mathcal{L}} \prod_{\alpha=1}^{n} \int_{\Sigma_\alpha} \mathcal{O}_\alpha^{(k_\alpha)}$$

where for each α the operator $\mathcal{O}_\alpha$ is a Q-closed local operator, $k_\alpha = 0, 1, 2$ and Σ_α is a k_α-cycle in the super-riemann surface Σ.

Here are the basic properties of $Z_{(\mathcal{O}_\alpha)}$:

1). It is independent of the metric on the super-riemann surface.

The reason is that as you vary the metric by δg you get an insertion of $\int_\Sigma \delta g \cdot T$, where T is the stress-energy tensor. But as we have seen before T is Q-exact in the sense that there is a field $S_{\mu\nu}$ with $T_{\mu\nu} = \{Q, S_{\mu\nu}\}$. We then see that this insertion produces a term of the form $\{Q, \cdot\}$. The path integral of such a term is zero since since the fields are Q-invariant. (Notice also that it is necessary to assume that Q preserves the measure in the path integral. This will be so if the original theory was supersymmetric and the current $\frac{1}{2}J_L - \frac{1}{2}J_R$ was gaugeable. An important corollary of the fact that the correlation function is metric-independent is that the zero-form operators $\mathcal{O}_\alpha^{(0)}$ must be scalar operators, of spin zero; objects of higher spin would not be invariant under local rotations, and their insertion in a correlation could not be metric-independent.) cross ref?

2). It is independent of most of the parameters in $\mathcal{L}$.

First, the correlation function is independent of any term $\int d^2y d^4\theta(\cdots)$. The reason is that to evaluate such a term we differentiate by all four of the Q's. For the same reason, only one of the terms

$$\int d^2y d\theta^+ d\theta^- W + \text{c. c.}$$

and

$$\int d^2y d\overline{\theta}^+ d\theta^- \tilde{W} + \text{c.c.}$$

can be non-trivial depending on which combination of $Q_+, Q_-, \overline{Q}_+, \overline{Q}_-$ we are using. In fact the correlation function is independent of one of these terms and varies holomorphically with the other.

3). We can calculate $Z_{(\mathcal{O}_\alpha)}$ by using a fixed point theorem applied to the action of Q. We consider Q as acting (infinitessimally) on the function space of all the fields in the theory. If this action has no fixed points, then the value of the path integral would be zero. In general, we can reduce the computation to an integral over the fixed point set of Q. These of course are the field configurations which are invariant under the supersymmetry Q.

III.2 Q-cohomology of States

Now we pass from the local operators, path integrals and correlation functions world to the Hamiltonian framework. Our riemann surface is now a cylinder $S^1 \times \mathbf{R}$. We trivialize all the relevant spin bundles over the S^1. We have the Hilbert space $\mathcal{H}$ of states of the quantum theory over the circle and the Hamiltonian operator H on $\mathcal{H}$ associated with time translation. Since Q is acting on the theory it produces an operator, which we also call Q, on $\mathcal{H}$ commuting with H. Also, from the commutation relations in the superalgebra we see that

$$H = \{Q, \tilde{Q}\}$$

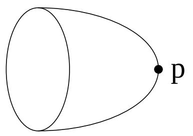
where $\tilde{Q}$ is the adjoint of Q. Of course, we still have the relation $Q^2 = 0$, so that we can take the cohomology of the action of Q on $\mathcal{H}$:

$$H^*(Q) = \ker Q / \mathrm{Im}\, Q.$$

If our theory is sufficiently well-behaved so that H has a discrete spectrum, then $H^*(Q)$ is identified with $\{\psi | H\psi = 0\}$. (This is not the case for the gauge theory example we discuss later today but it is true for the sigma models.) In general $H^*(Q)$ is bigraded by the action of the $U(1) \times U(1)$ R-symmetry group.

III.3 Comparison of Q-cohomology of local operators and Q-cohomology of states

We have seen two types of Q-cohomology – the first involving local operators and yielding the chiral ring and the second involving states on the circle. Let us construct a map between these Q-cohomologies. We consider a hemisphere



Let $\mathcal{O}$ be a local operator at p. Doing the path integral for the hemisphere over the space of fields with a given boundary value produces a number, and hence a function of the state on the boundary. This function is an element in the Hilbert space of the boundary. This then gives a map from local operators at p to $\mathcal{H}$. One sees easily that it commutes with the action of Q and hence induces a map from the Q-cohomology of local operators to the Q-cohomology of states on the circle:

$$\psi\colon (\text{chiral ring}) \to H^*(Q).$$

In examples we shall see that if the spectrum of Q on $\mathcal{H}$ (at least near zero) is discrete then ψ is an isomorphism.

If J_L and J_R are both R-symmetries classically, then the chiral ring and $\ker H$ are both bigraded by these symmetries. Because of the anomaly, the map ψ shifts the bigrading

$$\psi\colon H^{p,q}(Q) \to \ker H^{p\pm k, q\pm k},$$

where k is a measure of the anomaly. Notice that the bigrading need not be by integers, only by rational numbers whose denominators divide the order of the covering of $U(1) \times U(1)$ that actually acts.

IV A Gauge Theory Example

The first example we take is that of gauge theory of a simple Lie group G. We begin with $N = 1$ supersymmetric gauge theory in four dimensions and dimensionally reduce to an $N = 2$ supersymmetric theory in dimension two. As we saw in the last lecture, the basic field strength is the field

$$\Sigma = \{\overline{\mathcal{D}}_+, \mathcal{D}_-\},$$

which is a twisted chiral superfield with values in the adjoint bundle of the principal bundle. We use the expansion

$$\Sigma = \sigma + \cdots + \overline{\theta}^+ \theta^- (F + iD).$$

The bosonic fields are the connection A_μ and $\sigma, \overline{\sigma}$. Remember that the connection on four-space is

$$A^{(4)} = \sum_{\mu=0}^{1} A_\mu dy^\mu + \sigma(dy^2 - i dy^3) + \overline{\sigma}(dy^2 + i dy^3).$$

The two R-symmetries are:

$$J_A = J_L - J_R; \quad J_V = J_L + J_R.$$

The first is induced by rotation in the (y^2, y^3)-plane and the second comes from the chiral $U(1)$ R-symmetry in four dimensions. (This symmetry is anomalous in four dimensions but not in two dimensions – see the superhomework.) We now twist by J_V, so that $SO(2)'_E$ is generated by $K' = K \pm \frac{1}{2} J_V$. Then A_μ remains a connection one-form and $\sigma, \overline{\sigma}$ become 0-forms. In general, this twist eliminates spin bundles and turns sections of the spin bundles with differential forms. Then four spinor fields $\lambda_\pm, \overline{\lambda}_\pm$ get twisted into a 0-form $\eta = \psi_{(0)}$, a one form $\psi = \psi_{(1)}$ and a two-form $\chi = \psi_{(2)}$. (This new theory is of course a topological theory existing on a compact surface. It is not a physical theory in the sense that there is a Poincaré group action on the theory.)

In this example J_A is twice the rotation of (y^2, y^3)-space, normalized so that $[J_A, Q] = Q$. Let us give the J_A degrees of the various propagating fields in the Lagrangian, cf. Formulae 7.15 of the superhomework. (The reference to propagating fields, means that we have in mind using the equations of motion to set the auxiliary field $F + iD$ equal to some expression in the other fields (F is set equal to the curvature of A and D is set equal to zero since there are no matter fields.)

J_A-degree	-2	-1	0	1	2
field	$\overline{\sigma}$	$\psi_{(0)}, \psi_{(2)}$	A_μ	$\psi_{(1)}$	σ

Actually, the J_A-degree -1 and $+1$ terms could be reversed, but after appropriately normalizing things the degrees will be as indicated.

Recall from Lecture II-10[1] that the super symmetry Q lifts to a supersymmetry δ of the space of super connections. It is not the case that $\delta^2 = 0$, but only that δ^2 is a gauge transformation.

[1] In that lecture we had a real field ϕ instead of the complex field σ here, but that changes nothing.

In fact, $\delta^2 = [\sigma, \cdot]$. In terms of the coordinate fields it is given by:

$$\delta A = \psi_{(1)}$$

$$\delta\psi_{(1)} = -D_A \sigma$$

$$\delta\sigma = 0.$$

The symbol D_A of course refers to covariant derivative with respect to the connection A.

In Lecture II-10 using the analogous symmetry led us to the equivariant cohomology of the moduli space of connections, via the standard model of forms. We let $\mathcal{G}$ be the Lie algebra of the gauge group G and M the space on which this group acts. Then we have the complex

$$(\Omega^*(M) \otimes Func(\mathcal{G}))^G$$

with differential

$$D = d_M + i_{V(\phi)}.$$

Here, we use another closely related model which is not as well known but should be. We replace the Lie algebra $\mathcal{G}$ by its complexification $\mathcal{G}_{\mathbf{C}}$ and we form the complex

$$\left(\Omega^*(M) \otimes \Omega^{0,*}(\mathcal{G}_{\mathbf{C}})\right)^G.$$

Here complex linear function ϕ on $\mathcal{G}_{\mathbf{C}}$ have degree 2, the functions $\overline{\phi}$ have degree -2 and the $(0,1)$-form $d\overline{\phi}$ has degree -1. The differential in this complex is

$$\tilde{D} = d_M + i_{\tilde{V}(\phi)} + \overline{\partial}_{\mathcal{G}_{\mathbf{C}}}$$

where $\tilde{V}$ is a vector field on $M \times \mathcal{G}_{\mathbf{C}}$ giving the action. To see that this complex also computes the equivariant cohomology we view $\tilde{D}$ as a sum of $(d_M + i_{\tilde{V}(\phi)})$ and $\overline{\partial}_{\mathcal{G}_{\mathbf{C}}}$. We apply the usual spectral sequence for the double complex, doing $\overline{\partial}_{\mathcal{G}_{\mathbf{C}}}$ first. Since this cohomology vanishes except in degree zero, where it produces the polynomial functions on $\mathcal{G}_{\mathbf{C}}$, we see that one recovers the same answer as for the first complex. (Notice that this second approach leads to a Hodge theory version of equivariant cohomology, using the operator $\tilde{D}^*\tilde{D}$ as a "Laplacian"; it would be interesting to understand the resulting $\mathbf{L}^2$ version of equivariant cohomology.)

Let us complete the table describing the action of the supersymmetry δ on the space of propagating fields. The above model of equivariant cohomology translates into the following formulae for the supersymmetry:

$$\delta A = \psi_{(1)}$$

$$\delta\psi_{(1)} = -D_A\sigma$$

$$\delta\sigma = 0$$

$$\delta\overline{\sigma} = \psi_{(0)}$$

$$\delta\psi_{(0)} = [\sigma, \overline{\sigma}]$$

$$\delta\psi_{(2)} = F_A.$$

It follows from the last equation that the fixed points of δ will be field configurations where the connection is flat, the section σ is covariantly constant and commutes with $\overline{\sigma}$. Next we see that the anomaly of J_A is $\mathrm{dim}G\cdot(-\chi(\Sigma)) = (2g-2)\cdot\mathrm{dim}(G)$. The reason for this is that the term that contributes to the anomaly in the measure of integration for the path integral is $\mathcal{D}\psi_{(0)}\mathcal{D}\psi_{(1)}\mathcal{D}\psi_{(2)}$. The symmetry J_A of the Lagrangian acts by -1 on $\psi_{(0)}$ and $\psi_{(2)}$ and by $+1$ on $\psi_{(1)}$. Thus, the index computation for the anomaly becomes the dimension of the adjoint representation times $-\chi(\Sigma)$. By invariance the correlation function

$$\langle\prod_\alpha\int_{\Sigma_\alpha}\mathcal{O}_\alpha^{(k_\alpha)}\rangle = 0$$

unless $\sum_\alpha n_\alpha = (2g-2)\mathrm{dim}(G)$ where n_α is the J_A-degree of $\mathcal{O}_\alpha^{(k_\alpha)}$. Let us consider a local operator which is of the form $P(\sigma)$ for some gauge invariant polynomial on the Lie algebra of G. The ring structure on these operators is just the usual ring structure on invariant polynomials. It is easy to see that if P is of degree r, then the operator $\mathcal{O}_P^{(0)}$ has J_A-degree $2r$, and the descendants $\mathcal{O}_P^{(k)}$ have J_A-degree equal to $2r-k$. Notice that these degrees agree with the usual dimension of the analogous classes in Donaldson theory.)

Now let us calculate $\langle\prod_\alpha\int_{\Sigma_\alpha}\mathcal{O}_\alpha^{(k_\alpha)}\rangle$ via fixed point theory. Recall that the Lagrangian is

$$\mathcal{L} = \frac{1}{4e^2}\int d^2y d^4\theta \mathrm{Tr}\overline{\Sigma}\Sigma.$$

We will recover a theoretical formula for this correlation function. (We actually calculated the answer in Lecture II-10.) In coordinates we have

$$\mathcal{L} = \frac{1}{4e^2}\int d^2y\left(\mathrm{Tr}F^2 + \mathrm{Tr}D\sigma D\overline{\sigma} + \mathrm{Tr}([\sigma,\overline{\sigma}]^2) + \cdots\right). \tag{2}$$

By the fixed point theorem we need only calculate at points where the curvature is zero and $D_A\sigma = [\sigma,\overline{\sigma}] = 0$. There are two cases. If A is irreducible and $D_A\sigma = 0$ then $\sigma = 0$. These solutions give smooth points in the moduli space of classical solutions to the equations of motion and this part of the moduli space of classical δ-invariant solutions is simply the space of of irreducible representations of $\pi_1(\Sigma) \to G$. The other possibility is that A is reducible and $\sigma \neq 0$. These points produce singular points of the moduli space of classical solutions. Recall that σ takes values in the complexified adjoint bundle. Since $D_A\sigma = 0$, σ is covariantly constant, and it suffices to study its structure over one point of Σ. Since σ and $\overline{\sigma}$ commute, the σ can be diagonalized (conjugated into the Cartan subalgebra of the complexification of the Lie algebra of G) by an element of G.

Now let us examine how perturbation theory will compute these topological correlation functions. We will restrict to the smooth case where $\sigma = 0$. Of course, we are taking the coupling constant e in the Lagrangian, Equation 2 to be small. We are reduced to computing an integral over the moduli space, taken in the supersense, of classical solutions. Let us consider integrating out the odd variables at a point $[\rho]$ of the ordinary moduli space of flat connections. The fermion space is the space of zero modes of the $\psi_{(k)}$. These are the spaces of harmonic forms and hence are identified with the spaces $H^k(\Sigma, \mathrm{ad}(\rho))$. Since we are assuming that ρ is irreducible and that the moduli space is smooth at $[\rho]$ we have that $H^0(\Sigma,\mathrm{ad}(\rho)) = H^2(\Sigma,\mathrm{ad}(\rho)) = 0$ and $H^1(\Sigma,\mathrm{ad}(\rho))$ is the tangent space to the moduli space at $[\rho]$. So in fact the space of odd directions at $[\rho]$ is identified with $\Pi T\mathcal{M}_{[\rho]}$. Thus, at the open subset of irreducible connections, the supermanifold of classical solutions is $\Pi T\mathcal{M}$, the parity reversed tangent bundle over the moduli space of flat connection. As we have seen, the Berezinian of such a supermanifold is naturally trivialized so that one integrates functions over this supermanifold. We have also seen that a function on this supermanifold is the same thing as a differential form on $\mathcal{M}$ and integration of a function over $\Pi T\mathcal{M}$ is the same thing as the integral of the corresponding differential form over $\mathcal{M}$. (This requires an orientation of $\mathcal{M}$ which is determined by an orientation of Σ.)

Of course, we wish to compute correlation functions involving the operators $\int_{\Sigma_\alpha} \mathcal{O}_\alpha^{(k_\alpha)}$. These operators determine functions on $\Pi T\mathcal{M}$ and the correlation function of a product of them becomes the integral of the product of the functions. In this manner gauge invariant polynomials in σ are mapped to functions on $\Pi T\mathcal{M}$ and hence to differential forms on $\mathcal{M}$. A crucial step in the reduction from polynomials to differential forms is the observation that with fermions present, it is no longer the case that $\sigma = 0$. Rather the equations of motion give

$$\Box\sigma = -[\psi_{(1)}, \psi_{(1)}].$$

This is important because the right hand side is directly interpreted as a two-form on moduli space (as the zero modes of $\psi_{(1)}$ represent one-forms on the moduli space), so the formula enables us to express the function σ, which is formally of degree two in the quantum field theory formalism, in terms of an ordinary two-form on the moduli space. One evaluates the correlation functions by integrating the differential forms obtained in this way over the moduli space. (Of course, these computations have to be augmented by computations at the components given by reducible connections and a non-zero but parallel field σ.)

In this gauge theory example, because of the presence of reducible connections, the Hamiltonian has a continuous spectrum beginning at zero energy. As a result the map from Q-cohomology of operators to Q-cohomology of states is not very useful.

V A σ-model example

Let us consider the $N = 2$ supersymmetric σ-model

$$\Phi : \Sigma \to X$$

where X is a Kähler manifold and where Σ is a split $2|4$ manifold as in the previous gauge theory example. With this assumption, the fermions of the σ-model lie in

$$K_\Sigma^{\pm 1/2} \otimes T^{\pm} X$$

where K_Σ is the canonical bundle of Σ, $T^+X = T^{1,0}X$ and $T^-X = T^{0,1}X$ are the holomorphic and anti-holomorphic tangent bundle of X. We denote these fermions by $\chi^{\pm,\pm}$. The first sign refers to whether the fermion is left- or right-moving (i.e., in whether it is a section of K_Σ^+ or K_Σ^-) and the second sign refers to the type of the section (holomorphic or anti-holomorphic) on X. Once again we twist the theory for it to exist globally on Σ replacing K by

$$K' = K \pm \frac{1}{2} J_L \pm \frac{1}{2} J_R.$$

Depending on the choice of signs in this twisting we get two possible cases called the A-model (using $-\frac{1}{2}J_L + \frac{1}{2}J_R$) or the B-model (using $\frac{1}{2}J_L + \frac{1}{2}J_R$).

V.1 The A-model

The zero-forms are $\chi^{++} \in K^{1/2} \otimes T^{1,0}X$ and $\chi^{--} \in K^{-1/2} \otimes T^{0,1}$. The ring of 0-form operators is

$$F(\Phi, \chi^{++}, \chi^{--}) = F(\Phi, \partial\phi, \overline{\partial}\phi).$$

This is simply the ring of functions on ΠTX which is identified with the ring of differential forms on the Kähler manifold. Furthermore, with this identification of a differential form F with a corresponding operator $\mathcal{O}_F$, we have

$$Q\mathcal{O}_F = \mathcal{O}_{dF},$$

so that Q becomes the ordinary differential. The chiral ring is then $H^*(X)$, the usual topological cohomology of X, and the bigrading induced by J_L and J_R is the usual Hodge bigrading. In the space of maps the Q-fixed points are the holomorphic curves

$$\phi: \Sigma \to X.$$

Thus, to compute correlation functions in this model we must do integrals over the spaces of holomorphic curves in X. This will lead, as we shall see in the next lecture, to quantum cohomology.

V.2 The B-model

It is possible to construct this model only in the case X is Calabi-Yau, for this is the only case when the symmetry $(J_L + J_R)$ by which we wish to twist is not anomalous.[2] We fix a

[2] In other words, if $c_1(X)$ is nonzero, then $J_L + J_R$ is simply not a symmetry of the quantum theory, even before gauging. The discussion of the obstruction to gauging in the first part of this lecture assumed that J_L and

non-zero holomorphic n-form ω on X. In this case the zero-forms are $\chi^{+-} \in K^{1/2} \otimes T^{1,0}X$ and $\chi^{--} \in K^{-1/2} \otimes T^{0,1}X$. The ring of 0-form operators is

$$F(\Phi, \chi^{+-}, \chi^{--})$$

which is the ring

$$\Omega^{0,*}(X) \otimes \Omega^{0,*}(X) = \Omega^{0,*}\left(X, \wedge^* T^{1,0}(X)\right).$$

Under this identification we have

$$Q\mathcal{O}_F = \mathcal{O}_{\overline{\partial}F},$$

so that the chiral ring is

$$\oplus_{q,p} H^q(X; \wedge^p T^{1,0}X).$$

The Q-fixed points in the space of fields are the constant maps $\Sigma \to X$. Thus, the computations of correlation functions in this model will become integrals over X

$$\langle \prod_{i=1}^{s} \mathcal{O}_{F_i} \rangle = \int_X (\wedge_{i=1}^{s} F_i) \cdot \omega^{\otimes 2}.$$

These integrals lie squarely in classical mathematics; they have to do with variation of Hodge structures under deformation of complex structure.

J_R both generated symmetries prior to gauging, and thus does not apply to this case.