Lecture II-14, part I: The Landau–Ginzburg Description of $N = 2$ Minimal Models

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I Landau–Ginzburg models

The topic for the first part of today's lecture is a more detailed discussion of the known evidence for the predicted equivalence between IR limits of certain Landau–Ginzburg models, and the algebraically constructible conformal field theories known as "minimal models."

Consider first the simplest Landau–Ginzburg model: a two-dimensional N=2 theory of a single chiral superfield Φ with Lagrangian

$$\mathcal{L} = \int d^2x\, d^4\theta\, \overline{\Phi}\Phi + \left(\int d^2x\, d\theta^+\, d\theta^-\, \Phi^k + \text{c.c.}\right).$$

Our claim in lecture II-12 was that this model flows in the IR to a conformal field theory with $\widehat{c} = 1 - \frac{2}{k}$. If true, then the theory it flows to must be an algebraically solvable theory, due to the known classification of representations of the N=2 superconformal algebra with $\widehat{c} < 1$. More precisely, fixing $\widehat{c} = 1 - \frac{2}{k}$, there turn out to be a finite number of irreducible representations $\mathcal{H}_\alpha$ of the algebra of central charge $\widehat{c}$ (up to isomorphism); a conformal field theory must be built as a modular-invariant combination

$$\bigoplus(\mathcal{H}_\alpha \otimes \overline{\mathcal{H}}_\beta)^{e_{\alpha\beta}}$$

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of these representations. The simplest modular-invariant combination, and the one which will occur here, is the unweighted diagonal sum over all of the representations:

$$\bigoplus_{\alpha} \mathcal{H}_\alpha \otimes \overline{\mathcal{H}}_\alpha.$$

More general Landau–Ginzburg models involve several chiral superfields $\Phi_1, \ldots, \Phi_n$ and a quasi-homogeneous polynomial $W(\Phi_1, \ldots, \Phi_n)$ which serves as the superpotential of the theory. We should assume that the polynomial is transverse in the sense that $W = dW = 0$ holds only at the origin; the quasi-homogeneity condition can be expressed as the existence of positive rational numbers α_j such that

$$W = \sum \alpha_j \Phi_j \frac{\partial W}{\partial \Phi_j}.$$

The Lagrangian can be written

$$\mathcal{L} = \int d^2x\, d^4\theta\, \overline{\Phi}_j \Phi_j + \left(\int d^2x\, d\theta^+\, d\theta^-\, W(\Phi_j) + \text{c.c.} \right).$$

Intuitively, the quasi-homogenity is important because it allows the possibility of extending one of the R-symmetries, namely

$$\theta^+ \to e^{-i\varepsilon}\theta^+, \qquad \theta^- \to \theta^-,$$

to act on the chiral fields as $\Phi_j \to e^{i\varepsilon\alpha_j}\Phi_j$ so that the superpotential transforms as $W \to e^{i\varepsilon}W$ and hence the Lagrangian is invariant. If we *assume* that these microscopic R-symmetries *become* the R-symmetries of a limiting conformal field theory in the IR, then we can calculate the central charge $\widehat{c}$ of that limiting theory via operator product expansions.

This works as follows. Recall that the coefficient k which we encountered in lecture II-13 as measuring the anomaly which arises if one attempts to gauge the R-symmetry J_L appears in the operator product expansion as

$$J_L(x)\, J_L(x') = \frac{k}{(x^- - x'^-)^2} + \cdots.$$

The interpretation of k as the obstruction to gauging makes it clear that it is a topological invariant which can be reliably measured or computed at any scale. In particular, it can be calculated in the UV, where the superpotential doesn't matter. On the other hand, if J_L is the same R-symmetry which appears in the limiting $N = 2$ superconformal field theory, then the operator product expansion of the superconformal algebra asserts that

$$J_L(x)\,J_L(x') = \frac{\widehat{c}}{(x^- - x'^-)^2} + \cdots,$$

where $\widehat{c}$ is the central charge in the $N = 2$ algebra. Thus, $k = \widehat{c}$ and we can measure the IR central charge by computing in the UV.

To compute k in the ultraviolet, one treats the fields as free fields. The obstruction to gauging comes from the fact that the R-symmetries are chiral symmetries for the fermions in the free supermultiplets. One gets $k = \sum_j (1 - 2\alpha_j)$. Assuming the flow to an $N = 2$ theory in the infrared, with the same R-symmetry appearing in the $N = 2$ algebra, we have $k = \widehat{c}$ and hence $\widehat{c} = \sum_j (1 - 2\alpha_j)$. Notice that in the case of a single field with $W(\Phi) = \Phi^k$, we must have $\alpha = 1/k$ and so we get the desired result $\widehat{c} = 1 - 2/k$ for the minimal model.

We now want to consider a more sophisticated argument in favor of this equivalence, one which reveals more of the structure of the (limiting) conformal field theory. Recall the structure of the N=2 algebra:

$$\{Q_\pm, \overline{Q}_\pm\} = P_\pm$$

$$\{Q_\pm, Q_\mp\} = 0.$$

We have $\overline{Q}_+^2 = 0$ so we can consider the cohomology of $\overline{Q}_+$, i.e., consider local operators $\mathcal{O} = \mathcal{O}(x)$ such that $\{\overline{Q}_+, \mathcal{O}\} = 0$. Then

$$\partial_+ \mathcal{O} = \left[\left\{Q_+, \overline{Q}_+\right\}, \mathcal{O}\right] = \left[\overline{Q}_+, \{Q_+, \mathcal{O}\}\right],$$

i.e., in Euclidean language (identifying (x^+, x^-) with $(z, \overline{z})$) we would say that the class of $\mathcal{O}(x)$ in $\overline{Q}_+$-cohomology varies holomorphically.

Taking two such operators, using holomorphic language, we write

$$\mathcal{O}(z)\mathcal{O}'(z') = \sum f_k(z - z')\mathcal{O}_k(z') + \{\overline{Q}_+, \cdots\},$$

with the f_k's being holomorphic functions. From this formula, we can see that the $\overline{Q}_+$-cohomology has the general struture of a conformal field theory in which the operators are "purely left-moving," that is, they vary holomorphically.

An important remark is that the $\overline{Q}_+$-cohomology is invariant under Weyl rescaling, and hence it depends only on the complex structure of the Riemann surface (not the metric). To see this invariance, we must show that $T_{z\overline{z}}$ is $\overline{Q}_+$-exact. A natural way to prove this is to modify the usual definition of scale invariance, employing the quasi-homogeneity of our polynomial, so that the superpotential term is preserved under the modified scaling. Then the kinetic term in the action fails to be scale-invariant, but the change in the kinetic term is of the form $\int d^4\theta(\cdots)$ and hence is $\overline{Q}_+$-exact.

We have formulated this discussion in terms of $\overline{Q}_+$-cohomology in order to include very general $N{=}2$ theories. However, whenever there is a superspace realization of the theory (as is true of the Landau–Ginzburg theories we are studying), it is more convenient to study the $\overline{D}_+$-cohomology rather than the $\overline{Q}_+$-cohomology, where

$$\overline{D}_+ = e^{-\theta^+\overline{\theta}^+\partial_+}\,\overline{Q}_+\,e^{\theta^+\overline{\theta}^+\partial_+}.$$

In our Landau–Ginzburg theory, the equations of motion are

$$2\overline{D}_+\overline{D}_-\Phi^j = \frac{\partial W}{\partial \Phi^j}.$$

Consider the operator

$$\mathcal{T} = \sum_j \left(\left(\frac{1-\alpha_j}{2} \right) D_-\Phi^j\ \overline{D}_-\,\overline{\Phi}_j - i\alpha\overline{\Phi}_j\partial_-\Phi_j \right);$$

then $\overline{D}_+\mathcal{T} = 0$ so that this defines a $\overline{D}_+$-cohomology class. [1]

In the free theory, with $W = 0$, $\mathcal{T}$ is an operator of dimension $(1,0)$. If we have a superconformal theory, we will be able to write

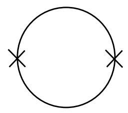
$$\mathcal{T} = J + \theta^- G + \overline{\theta}^-\overline{G} + \theta^-\overline{\theta^-}T.$$

If we do arrive at an $N{=}2$ conformal field theory in the limit, then we should expect an OPE of the form

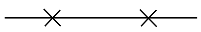
$$\mathcal{T}(x)\,\mathcal{T}(y) = \frac{\widehat{c}}{(x-y)^2} + \frac{\mathcal{T}}{(x-y)} + \text{holomorphic in } \mathcal{O}\text{'s}.$$

[1] We should worry about anomalies here, but in fact there is no quantum anomaly in the statement that $\overline{D}_+\mathcal{T} = 0$, though there are such anomalies in somewhat similar statements in two-dimensional models with $(0,2)$ supersymmetry. See E. Silverstein and E. Witten, "Global $U(1)$ R Symmetry And Conformal Invariance Of $(0,2)$ Models," Phys. Lett. **B328** (1994) 307.

We can check this structure in the free field theory: the first term comes from a diagram of the form



with two propagators exchanged between the two operators and the second term comes from a diagram of the form



with exchange of only one propagator.

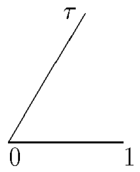
Note that, in computing these terms in the operator product expansion, we can set $W = 0$, since the superpotential term is too soft (superrenormalizable) to contribute to these singularities. Once this is done, the different superfields are decoupled, so the computation of $\widehat{c}$ is a sum over different superfields. So there is an immediate conclusion that $\widehat{c}$ is *additive* in the α_j's, that is, $\widehat{c} = \sum f(\alpha_j)$ for some function f.

In summary, even in our microscopic Lagrangian, we can see an operator $\mathcal{T}$, which at the level of $\overline{D}_+$ cohomology obeys at $N = 2$ superconformal algebra. So presumably this theory flows in the IR to a conformal field theory whose $\overline{D}_+$-cohomology contains the superconformal algebra.

If we have only one chiral superfield, so $\widehat{c} < 1$, the $N = 2$ algebra acts almost irreducibly in the quantum Hilbert space. Hence one should expect that $\mathcal{T}$ will generate the $\overline{Q}_+$-cohomology. For $\widehat{c} > 1$, the $N = 2$ algebra acts in a way that is far from irreducible, and one should expect to require additional generators.

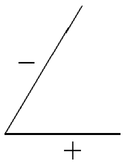
II The elliptic genus

Our refined evidence for the equivalence of these theories will come from a computation of the *elliptic genus*. We work on a Riemann surface of genus 1 with period lattice spanned by 1 and τ.



We let $q = e^{2\pi i \tau}$, and compute $\mathrm{Tr}\, q^{(H-P)/2} \overline{q}^{(H+P)/2} (-1)^{F_R}$.

The insertion of $(-1)^{F_R}$ means that the right-moving fermions ψ_+, $\overline{\psi}_+$ should be given the *odd* spin structure (the one periodic in both directions) while the left-moving fermions ψ_-, $\overline{\psi}_-$ should be be given the *even* spin structure (anti-periodic in one direction):



For N=1 supersymmetry, that was the right thing to compute—it turns out to be an index

$$\mathrm{Tr}\, q^{(H-P)/2} \overline{q}^{(H+P)/2} (-1)^{F_R} = \mathrm{index}(Q_+ + \overline{Q}_+),$$

where $(Q_+ + \overline{Q}_+)^2 = 2P_+ = H + P$.

The story is more interesting in the case of N=2 supersymmetry, since there is the possibility of inserting $e^{i\gamma J_L}$, which commutes with both Q_+ and $\overline{Q}_+$.[2] We introduce the function

$$F(q,\gamma) = \mathrm{Tr}\, q^{(H-P)/2} \overline{q}^{(H+P)/2} e^{i\gamma J_L} (-1)^{F_R}.$$

The insertion of $e^{i\gamma J_L}$ means: when calculating the trace, glue the data on the torus using $e^{i\gamma J_L}$.

The first observation is that the quantity $F(q,\gamma)$ is independent of any continuously variable parameters of the theory that preserve the $N = 2$ structure. This is so as it is a linear combination

[2] We use the same symbol J_L for the left R-current and the conserved charge derived from it.

of terms each of which is an index of an operator. For the same reason (or because the trace of the stress tensor of the theory is $\overline{Q}_+$-exact) $F(q,\gamma)$ is independent of the area of the torus, and only depends on the complex structure.

We want to calculate $F(q,\gamma)$ in two contexts: (i) an algebraic calculation for the known algebraically constructed models in the IR, and (ii) directly for the Landau–Ginzburg models in the UV. The comparison of these two will then provide additional evidence that the Landau–Ginzburg theory indeed flows to the "minimal model" in the IR.

II.1 Algebraic calculation in the IR

The Hilbert space of a "minimal model" takes the form $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha \otimes \overline{\mathcal{H}}_\alpha$, with the left-movers in $\mathcal{H}_\alpha$ and the right-movers in $\overline{\mathcal{H}}_\alpha$. Thus, the elliptic genus can be written

$$F(q,\gamma) = \sum_\alpha \left(\mathrm{Tr}_{\mathcal{H}_\alpha} q^{(H-P)/2} e^{i\gamma J_L} \right) \left(\mathrm{Tr}_{\overline{\mathcal{H}}_\alpha} \overline{q}^{(H+P)/2} (-1)^{F_R} \right).$$

The right-moving contribution $\mathrm{Tr}_{\overline{\mathcal{H}}_\alpha} \overline{q}^{(H+P)/2}(-1)^{F_R}$ evaluates to 1 if $\overline{\mathcal{H}}_\alpha$ has highest weight with $H+P=0$ and 0 otherwise. This has the effect of selecting a distinguished subset from among the irreducible representations. Let us introduce a restricted sum $\sum'$ to denote the sum over that subset. Then our elliptic genus can be written

$$F(q,\gamma) = \sum_\alpha{}' \ \mathrm{Tr}_{\mathcal{H}_\alpha} q^{(H-P)/2} e^{i\gamma J_L}.$$

(Note that the eigenvalues of the charge J_L are not integers in general. Rather, the object $\exp(2\pi i J_L)$ commutes with the $N=2$ algebra and is a constant in each representation; the constant depends on the R-charges of the chiral primary fields, which are rational numbers. So $F(q,\gamma)$ is periodic in γ, but the period is not 2π.)

II.2 Direct calculation in the UV

For the Landau–Ginzburg theory itself, we can try to directly calculate in the UV. We use the fact that $F(q,\gamma)$ is independent of the area of the torus. We take the area to be very small. Roughly, in the UV the interaction is negligible and we should be able to set $W=0$. The only reason for this to fail has to do with the behavior for large fields. The bosonic part of the action is

$$\int |d\phi|^2 + \int \left|\frac{dW}{d\phi}\right|^2 .$$

In the presence of the W term, the unique minimum of the action is $\phi = 0$. If we would set W to zero, the action would be minimized for any constant ϕ. One would have to integrate over the space of constant ϕ's even in a leading approximation to the path integral, and then when ϕ becomes sufficiently large, the W term, if its coefficient is not strictly zero, is important.

The conclusion is, then, that as long as ϕ is a *function* on the torus, the W term cannot quite be ignored. If, however, ϕ were not a function but a section of a nontrivial flat bundle, then even at $W = 0$, the $|d\phi|^2$ part of the action would have a unique and nondegenerate minimum. In that situation, it would be straightforward, in the limit of small area on the torus, to set W to zero.

In the path integral evaluation of $F(q,\gamma)$ for generic γ, we get just such a situation in which ϕ is a section of a nontrivial flat bundle. The reason is that J_L acts on ϕ by $\delta\phi = \alpha\phi$. So in evaluating the elliptic genus we are making the identification

$$\phi(P+\tau) = e^{i\gamma\alpha}\phi(P)$$

(where we regard the torus as the quotient of the complex plane by the lattice generated by 1 and τ, and P is any point on the complex plane). ϕ is a section of the flat bundle on the torus specified by this gluing. If γ is generic, the zero-section is isolated and moreover nondegenerate minimum of the action even if we set $W = 0$. The nondegeneracy makes computations straightforward. The calculation of $F(q,\gamma)$ can then be carried out in detail, setting $W = 0$ and expressing F as a ratio of determinants. Explaining how this goes would take us too far afield, but the result[3] is

$$F(q,\gamma) = e^{-i\gamma k\alpha/2} \cdot \frac{1 - e^{i\gamma(k+1)\alpha}}{1 - e^{i\gamma\alpha}} \cdot \prod_{n=1}^{\infty} \frac{(1 - q^n e^{i\gamma(k+1)\alpha})(1 - q^n e^{-i\gamma(k+1)\alpha})}{(1 - q^n e^{i\gamma\alpha})(1 - q^n e^{-i\gamma\alpha})}.$$

When compared with the algebraic calculation in the IR, one can extract formulas for certain characters of the N=2 algebra. These formulas can be verified independently, giving strong

[3] E. Witten, "On the Landau–Ginzburg Description of N=2 Minimal Models," Int. J. Mod. Phys. A **9** (1994) 4783–4800; `hep-th/9304026`.

support to the claim that the Landau-Ginzburg theory flows to the $N = 2$ minimal model in the infrared.

II.3 Connection to Calabi–Yau models

Consider now the $U(1)$ gauge theory with chiral superfields Φ_j of charge α_j and P of charge -1, such that $\sum \alpha_j = 1$. (We are no longer assuming that all of the α_j's are equal, and we are using a different normalization for the generator of $U(1)$ than in lecture II-12.) Denote the gauge supermultiplet by

$$\Sigma = \sigma + \cdots,$$

let $W(\Phi_1, \ldots, \Phi_n)$ be a transverse, quasi-homogeneous polynomial as above (i.e., satisfying $W = \sum \alpha_j \Phi_j \frac{\partial W}{\partial \Phi_j}$, with $W = dW = 0$ only at the origin), and consider the Lagrangian

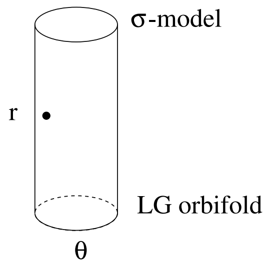
$$\mathcal{L} = \int d^2x\, d^4\theta \left(\frac{\overline{\Sigma}\Sigma}{4e^2} + \overline{\Phi}^j \Phi^j \right) + \left(\int d^2x\, d^2\theta\, PW(\Phi) + \text{c.c.} \right) + \left(\int d^2x\, d\theta^+\, d\theta^-\, t\Sigma + \text{c.c.} \right)$$

such that $t = \frac{\theta}{2\pi} - ir$.

The polynomial $PW(\Phi)$ is gauge invariant, and we can repeat our earlier story: for $r \gg 0$ we classically get a Calabi–Yau hypersurface $\{W = 0\}$ in weighted projective space $\mathbf{WCP}^{n-1}_{(\alpha_1,\ldots,\alpha_n)}$ (which may inherit some singularities from those of the weighted projective space); for $r \ll 0$ we get a Landau–Ginzburg orbifold whose superpotential is W. The quantum theory is singular only when $r = r_0$, $\theta = 0$. (r_0 is a constant computed as in lecture II-12.) The computation of the central charge in Landau–Ginzburg theory gives

$$\widehat{c} = \sum_j (1 - 2\alpha_j);$$

the computation for the σ-model gives $\widehat{c} = \dim_{\mathbf{C}} X$, and these agree since $\sum \alpha_j = 1$.



The elliptic genus should be independent of t since it is a topological invariant; this leads to a formula which relates the elliptic genus of this type of Calabi–Yau manifold to the explicit elliptic genus which can be computed via characters of minimal models on the Landau–Ginzburg side.

Although the elliptic genus is independent of t, one of the other quantities we have discussed for N=2 theories—the $\overline{Q}_t$-cohomology—varies holomorphically with t. The *instantons* in this problem—which are holomorphic curves in the σ-model—will contribute to the path integral in the form $e^{-i\theta\int\frac{F}{2\pi}}\cdot e^{-r\int d^2x\,D}$ where $D=\sum\alpha_j|\Phi_j|^2-r$, i.e., $e^{-2\pi it\int_\Sigma\Phi^*(c_1(\mathcal{L}))}$. (I have used the fact that the instanton equation of the $N=2$ model – the condition for the field to be invariant under some supersymmetry – gives $\int d^2xD=\int F/2\pi$.) We will explore the instanton sums in the second half of the lecture.