Lecture II-14, part II: Quantum Cohomology of Kähler Manifolds

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I Introduction

We return now to twisted $N = 2$ supersymmetric σ-models in dimension two with target a Kähler manifold X. Recall that there were four possibilities for twisting depending on which supersymmetry $Q \in \{\overline{Q}_+ + \overline{Q}_-, Q_+ + \overline{Q}_-, \overline{Q}_+ + Q_-, Q_+ + \overline{Q}_-\}$ is preserved. Whichever possibility one selects, the cohomology of Q as a vector space can be identified with the cohomology of X. But there is more structure; if we restrict to the case that the Riemann surface is a fixed $\S^2$, then the additional structure is a ring structure on the cohomology of Q. The resulting rings are called the (a, a), (c, a), (a, c), and (c, c) chiral rings.

With one of the choices, the classical version of the chiral ring turns out to be the usual ring structure on cohomology, but quantum mechanically there are corrections, instanton corrections, to this classical answer. For this reason the chiral ring of the σ-model is called the quantum cohomology of the Kähler manifold X. In this half of the lecture, we will discuss quantum cohomology generally and then make some explicit computations in the case $X = \mathbf{C}P^{n-1}$ and the case X is a Fano hypersurface. These computations will be done both mathematically, counting holomorphic rational curves with certain properties, and will be done from the physics point of view by using the renormalization group flow (discussed in Lecture II-12 and the first

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half of Lecture II-14) from these σ-models to Landau-Ginsburg theories with extra vacua.

II The space of 0-energy states

We wish to compute some of these chiral rings, but before we deal directly with the operators, we compute the 0-energy states in the Hamiltonian framework. The reason for doing this computation is that as we showed in the last lecture there is a map

$$\psi\colon \text{chiral ring} \to \{0\text{−energy states}\}$$

given by inserting a local operator at a point on the hemisphere. In favorable circumstances this map is an isomorphism.

In the Hamiltonian framework the bosonic space underlying the space of fields is

$$\mathcal{W} = \mathrm{Maps}(S^1, X).$$

The minimum energy configurations are the constant maps and hence the space of these is a copy of X. Since there are fermions in two copies of the spin bundle, when we quantize the resulting Hilbert space is the tensor product of two copies of the spin bundle over X, i.e., the space of differential forms on X. The Hamiltonian is the Laplacian and its space of ground states is then the space L^2-harmonic forms on X which is identified with $H^*_{L^2}(X)$. Of course, there are normal directions to the copy of X in $\mathcal{W}$ to consider, but supersymmetry implies that the quadratic form in the normal directions is non-degenerate, and hence these directions produce a tensor products of harmonic oscillators and have a unique ground state. Thus, in the end we find that the ground states for the entire theory are identified with the harmonic forms on X. We shall see that the computation of the chiral rings gives the same answer (additively), so that in the cases we consider today the map ψ is indeed an isomorphism. This is true generally when the classical theory has the property that the energy grows as the fields go to infinity.

III Generalities on the chiral ring

In order to define a chiral ring we need a global supersymmetry. As we saw in Lecture II-13 the way to obtain these is to twist the usual form of the $N = 2$ supersymmetric theory. Today we shall only consider twists making $\overline{Q}_+ + Q_-$ global. This is what we referred to in Lecture II-13 as Case A. It is a twist that can be performed for any Kähler manifold – no assumption of vanishing first Chern class is necessary. We shall consider local operators which are functions of the basic bosonic field $\phi : S^1 \to X$ and fermions. Recall that one of the effects of twisting is to make the fermions sections of bundles of integral rather than half-integral spin. That is to say, they are differential forms instead of spinors. It will suffice in describing the Q-cohomology of operators, to consider only fermions of zero spin. The reason for this is that all of the Q-cohomology classes of operators have representatives that are constructed using only those fermions. To be more precise, the Q-cohomology classes have representatives that depend only on the bosonic fields and spin zero fermions and not their derivatives. [1] Thus, our local operators will be of the form

$$\mathcal{O} = \mathcal{O}(\phi, \psi_L, \psi_R)$$

where in the untwisted version of the theory, ψ_L is a section of $K^{-1/2} \otimes \phi^* T^{0,1}X$ and ψ_R is a section of $K^{1/2} \otimes T^{1,0}X$. After we twist, these fermion fields lie in $\phi^* T^{0,1}X$ and $\phi^* T^{1,0}X$, since the twisting cancels out the line bundle over Σ. In particular, it is possible to form the sum $\psi = \psi_L + \psi_R$ and consider operators $\mathcal{O} = \mathcal{O}(\phi, \psi)$.

III.1 Local Operators

For each differential form λ on X there is an operator $\mathcal{O}_\lambda(x) = \mathcal{O}_\lambda(\phi(x), \psi(x))$. As we remarked in the Lecture II-13, under the identifications of operators and forms, Q is identified with the usual exterior derivative:

$$\{Q, \mathcal{O}_\lambda\} = \mathcal{O}_{d\lambda}.$$

When, as in the case of these $N = 2$ supersymmetric σ-models, the operator-to-state correspondence is an isomorphism we can compute the ring structure of the chiral ring by computing correlation coefficients. Our goal here is to compute in more classical topological and geometric terms the correlation coefficient

$$\langle \mathcal{O}_{\lambda_1}(x_1) \cdots \mathcal{O}_{\lambda_s}(x_s) \rangle$$

[1] This is the operator analog of the fact that in the Hamiltonian description, the zero energy states are constructed from differential forms on the target space with all oscillators in their ground state. Operators that contain derivatives of fields would correspond to states in which some oscillators are excited.

for closed forms λ_i. As we have already remarked, the correlation coefficient is unchanged if we replace the λ_i by cohomologous forms. Also, to compute this correlation function, which means to compute the path integral

$$\int \mathcal{D}\phi \mathcal{D}\psi e^{2\pi i \mathcal{L}} \prod_i \mathcal{O}_{\lambda_i}(x_i)$$

over the space of all fields, it suffices to compute the integral over the subspace of Q-invariant fields. We have already identified the bosonic part of the subspace of Q-fixed points: it is the space of holomorphic maps $\Sigma \to X$. For each i choose a cycle H_i Poincaré dual to $[\lambda_i]$. Then we can choose a representative for λ_i with support in an arbitrarily small neighborhood of H_i. To compute the correlation function one will consider the moduli space of holomorphic maps from the Riemann surface passing through all of the H_i at the points x_i. In the 'best' case when there are only finitely many holomorphic maps passing through the cycles, one simply counts each of these maps with a sign. The sign comes from evaluating the bosonic and fermionic determinants in an expansion around the classical solution. Then the result is weighted by the exponential of $2\pi i$ times the value of the action at this component.

Mathematically, there is a dimension count which must be satisfied for the differential form being integrated to be top dimensional on the moduli space of holomorphic curves, and hence for the answer to have a chance of being non-zero. From the physics point of view one sees the same dimension restriction coming from the anomaly of the R-symmetry. The measure of integration in the path integral has an anomaly under this symmetry, and one must use a set of operators whose product has precisely the cancelling anomaly for the path integral to have a chance to be non-zero. Not surprisingly, these two conditions are the same: namely, that the sum of the codimensions of the H_i in X must be equal to the dimension of the instanton moduli space of non-constant holomorphic maps from Σ to X.

If one wants to see just the classical cohomology ring, rather than the quantum cohomology, then one should consider just the constant maps to X. Then the dimension condition is

$$\sum_{i=1}^{s} \mathrm{degree}\lambda_i = \mathrm{dim}X$$

and the classical answer is simply

$$\int_X \lambda_1 \wedge \cdots \wedge \lambda_s .$$

That is to say, the classical chiral ring is the cohomology ring of X. This answer gets corrected quantum mechanically by instanton corrections given by the integrals we were describing over the moduli space of non-constant holomorphic maps. Thus, the chiral ring is a quantum correction to the usual cohomology ring of X. For this reason the chiral ring is usually called the quantum cohomology ring of X.

IV More Details on the Ring Structure

Having given the general definition of quantum cohomology as the chiral ring and indicated the sorts of mathematics that go into computing it, it is now time to be more concrete and compute an example. We take the case of $\Sigma = S^2$ mapping into a compact Kähler manifold X. Let us begin with a correlation function with only two operators:

$$\langle \mathcal{O}_{\lambda_1} \mathcal{O}_{\lambda_2} \rangle,$$

with the degree of λ_1 plus the degree of λ_2 equal to the dimension of X. We of course know that the space of constant maps gives a contribution to this correlation function equal to

$$\int_X \lambda_1 \wedge \lambda_2.$$

In this case there is no quantum correction and this is also the answer in the chiral ring. The reason that there is no quantum correction is that the space of non-constant holomorphic maps of $S^2 \to X$ passing through a point of H_1 and H_2 has a free $\mathbf{C}^*$-action on it. For a component of this space to possibly give a non-zero correction to the correlation coefficient, it must be the case that its formal dimension is zero. But if it has zero dimension and has a free $\mathbf{C}^*$-action on it, then it must be empty. (Even for a possible nonzero component of this space, the free $\mathbf{C}^*$ action ensures that the contribution is zero.) Thus, we have shown that under the identification of the chiral ring with the cohomology of X, the two point correlation functions, which compute the inner product on the vector space underlying the chiral ring, give the usual inner product on cohomology or equivalently give the intersection form on homology.

Now let us consider the ring structure. Fix a basis $\{\lambda_i\}$ for $H^*(X)$. Let the inner product in this basis be given by η_{ij}:

$$\eta_{ij} = \langle \mathcal{O}_{\lambda_i} \mathcal{O}_{\lambda_j} \rangle.$$

Then the ring structure is determined by structure constants c_{ij}^k defined by

$$\mathcal{O}_{\lambda_i} \mathcal{O}_{\lambda_j} = c_{ij}^k \mathcal{O}_{\lambda_k}.$$

Let ω_{ijk} be the three point function

$$\langle \mathcal{O}_{\lambda_i} \mathcal{O}_{\lambda_j} \mathcal{O}_{\lambda_k} \rangle.$$

Since this three point function is also the inner product of the product of the first two local operators with the third, we see

$$\omega_{ijk} = \sum_r c_{ij}^r \eta_{rk}.$$

Since η_{ij} is non-degenerate, this implies that the two and three point correlation functions determine the ring structure of the chiral ring.

V Calculations for $\mathbf{C}P^{n-1}$

We will now consider an example. We do the computation for $\mathbf{C}P^{n-1}$ by using a gauged linear σ-model with linear bosonic fields $\phi_1, \ldots, \phi_n, p$ as in Lecture II-12. To get all of $\mathbf{C}P^{n-1}$ we set the superpotential W equal to zero. We get a family of theories depending on a complex parameter $t = -ir + \frac{\theta}{2\pi}$, and as we have seen, for $\operatorname{Im} t \ll 0$ these are non-linear σ-models on X with some Kähler metric depending on t and Kähler class roughly proportional to $\operatorname{Im} t$. The only observable is $\mathcal{O}_H$ where H is the hyperplane section. We call this operator σ. Classically, in $\mathbf{C}P^{n-1}$ we have one relation, namely $\sigma^n = 0$. Let us see what happens quantum mechanically. We need to compute the three point function

$$\langle \sigma^a \sigma^b \sigma^c \rangle$$

where $a + b + c = n - 1 + dn = \dim \mathcal{M}_d$, the dimension of the moduli space of rational curves of degree d. Clearly, it suffices to consider the case when $a, b, c < n$. This implies that the only possibilities for d are $d = 0, 1$. Of course, $d = 0$ is the moduli space of constant maps and its contribution is the classical ring structure. For $d = 1$, we are considering straight lines in $\mathbf{C}P^{n-1}$. To calculate

$$\langle\sigma^{n-1}(x)\sigma^{n-1}(y)\sigma(z)\rangle$$

we must consider the space of straight lines through two points meeting a fixed hyperplane. There is exactly one such line. Since the instanton action is $e^{-2\pi it}$, this lead to the relation

$$\langle\sigma^{n-1}(x)\sigma^{n-1}(y)\sigma(z)\rangle = e^{-2\pi it}.$$

Unraveling the ring structure from this three point function yields

$$\sigma^n = e^{-2\pi it}$$

in quantum cohomology.

We can make the same calculation physically. We are considering the σ-model $\mathbf{C}P^1 \rightarrow \mathbf{C}P^{n-1}$. At the end of Lecture II-12 we found that in the infra-red this flows to n vacua with a mass gap (since this is a case in which $c_1(X) > 0$, the flow is from the σ-model to the Landau-Ginsburg model). The field σ has different expectation values in each vacuum. Since the fields are massive, we simply set them equal to their expectation values. These vacuum expectation values are $\sigma = \mu e^{2\pi i(-t+k)/n}$, $k = 0, \ldots, n-1$. These give the idempotents of the quantum cohomology ring, from which we deduce the same answer as before,[2] namely that the ring is generated by σ modulo the relation $\sigma^n = e^{-2\pi it}$.

[2]Except for a factor of μ which has to do with a mismatch between our physical and mathematical conventions. In lecture 12, σ was defined as a field of dimension one and no effort was made to compare to any topological normalization. In our mathematical discussion today, σ was normalized topologically as a field related to a hyperplane section of projective space, and in particular has dimension zero. To compare the two definitions, one should, in the setting of lecture 12, integrate out the σ field in the region $r \gg 0$ in favor of the massless fields of the sigma model. In this process, σ will turn into an operator of the low energy theory that can be derived from a two-form on $\mathbf{CP}^{n-1}$. The cohomology class of this operator will be a multiple C of the hyperplane section, and C is the factor by which the physical normalization of lecture 12 disagrees with the topological normalization that gives $\sigma^n = e^{-2\pi it}$.

VI Calculations for Fano Hypersurfaces

Now let us generalize these computations to the case of a hypersurface of degree $d < n$ in $\mathbf{C}P^{n-1}$, so-called Fano hypersurfaces.

According to [Collino-Jinzenji] the answer, computed mathematically, is that the quantum cohomology ring is generated by σ with one relation:

$$\sigma^{d-1}\left(\sigma^{n-d} - e^{-2\pi i t}\right) = 0. \tag{1}$$

Let us think about how this computation fits with the fact that again this σ-model flows in the infra-red to a Landau-Ginsburg model plus extra massive vacua. The roots of the equation $\sigma^{d-1}(\sigma^{n-d} - e^{-2\pi i t}) = 0$ are of two types: a root at $\sigma = 0$ of multiplicity $d-1$, and nondegenerate roots (that is, roots of multiplicity one) at nonzero sigma. The roots of the equation correspond to the vacua of the quantum field theory. Nondegenerate roots correspond to massive vacua, which give idempotents of the quantum cohomology ring. In analyzing the vacuum structure of this theory in lecture 12, we found $n - d$ massive vacua at nonzero sigma. These give the factor $(\sigma^{n-d} - e^{-2\pi i t})$ in the equation for the quantum cohomology. The other factor σ^{d-1} is the contribution of the Landau-Ginzburg vacua at the origin. This factor is present if $d > 1$ (for $d = 1$ we are discussing a projective space of codimension one, and there is no Landau-Ginzburg vacuum), and corresponds to a root at the origin of multiplicity greater than one if $d > 2$ (for $d = 2$, the central charge of the Landau-Ginzburg theory, which in general is $\widehat{c} = n(1 - 2/d)$, vanishes; in this case the Landau-Ginzburg theory is massive and infrared trivial). For general d, the multiplicity of the $\sigma = 0$ root of the quantum cohomology can be computed by evaluating $\mathrm{Tr}(-1)^F$ in the Landau-Ginzburg theory at $\sigma = 0$.