LECTURE II-16: BRST QUANTIZATION OF GAUGE THEORIES

Edward Witten

Notes by Pavel Etingof and David Kazhdan

In this lecture we will discuss quantization of gauge theory by using BRST cohomology. This approach is an improvement of the original Faddeev-Popov approach. An advantage of the BRST approach as opposed to the Faddeev-Popov method is that BRST makes explicit the independence of quantization of the choice of the gauge fixing procedure.

A similar approach can be used in gravity (see D'Hoker's lectures).

16.1. The general setup.

We start with a general setup, and then consider examples. In the general setup, we have a compact gauge group G with Lie algebra $\mathfrak{g}$, and the group $\hat{G}$, which is the group of gauge transformations of a principal G-bundle E over a spacetime M. Formally, we want to compute the path integral

$$\frac{1}{Vol(\hat{G})}\int DAD\phi e^{-L(A,\phi)}, \tag{16.1}$$

where $L(A,\phi)$ is a gauge invariant Lagrangian with a gauge field A and matter fields ϕ.

The difficulty with a perturbative treatment of this path integral is that its kinetic term F_A^2 for the gauge field is degenerate along orbits of $\hat{G}$. One way to deal with this difficulty is to replace the integrand in (16.1) by some expression that integrates to 1 on orbits of $\hat{G}$ – then (16.1) would equal to the integral of this expression (at least if everything were finite dimensional). For example, this expression could be the delta-function of some gauge, i.e. of some submanifold in the space of connections and matter fields which is a cross-section for the $\hat{G}$-action (this procedure is called gauge fixing). As we know (see Kazhdan's lectures on gauge theory and Faddeev's lectures), this introduces a determinant under the integral (the Faddeev-Popov determinant). The determinant is a nonlocal expression, so in order to work only with local expressions, one should replace this determinant with a Gaussian integral over the space of fields times two copies of the odd space of sections of the coadjoint bundle of E. Thus, one has to introduce additional fermionic fields $c, \bar{c}$ with values in $ad(E)$. These fermions are called ghosts, since they do not correspond to any physical particles and violate spin-statistics. After the introduction of ghosts the path integral can be treated as usual, e.g. by perturbation theory techniques.

16.2. The BRST differential.

Of course, any gauge fixing procedure by definition destroys gauge invariance. Therefore, in order to obtain a sensible quantization, we must make sure that in the final result the gauge symmetry is restored. In particular, we must explain what replaces the gauge symmetry in the ghost setting of the previous section.

It turns out that what replaces the gauge symmetry is a certain odd derivation of the algebra of local functionals, which we will now construct.

We will first consider the classical theory. Let us look what fields our theory has after introduction of ghosts. The basic fields are the connection A, the matter fields ϕ, and the ghosts $c, \bar{c}$, which are sections of the adjoint bundle to E. We will add an auxiliary scalar bosonic field h, whose significance will be seen below.

Let R be the algebra of local functionals. We want to define an odd derivation $\delta : R \to R$ such that $\delta^2 = 0$ (the BRST differential).

Recall that the algebra of local functionals R is the quotient $\tilde{R}/I$, where $\tilde{R}$ is the algebra of local expressions in the fields and I is the differential ideal generated by field equations.

We first define a derivation $\delta : \tilde{R} \to \tilde{R}$, and then make sure that the field equations are respected.

Define δ on generators by

$$\delta c = \frac{1}{2}[c,c], \delta\phi = \delta_c\phi, \delta A = -d_A c, \delta\bar{c} = h, \delta h = 0, \tag{16.2}$$

where $\delta_c\phi$ means the variation of ϕ along the infinitesimal gauge transformation c. It is easy to check that $\delta^2 = 0$.

Recall from Kazhdan's and D'Hoker's lectures that the Lagrangian with ghosts for our theory is

$$\tilde{L} = L(A,\phi) - \delta(\bar{c}(h/2 + \Lambda)) = L(A,\phi) - (h\Lambda + \frac{h^2}{2} - \bar{c}\delta\Lambda). \tag{16.3}$$

where $\Lambda = \Lambda(A,\phi)$ is a non-gauge invariant local expression (the gauge fixing function). This Lagrangian is of course equivalent to

$$\hat{L} = L(A,\phi) + (\frac{1}{2}\Lambda^2 + \bar{c}\delta\Lambda).$$

by eliminating h.

We want the Lagrangian to be invariant under the derivation δ. Since $L(A,\phi)$ is already invariant (because it is gauge invariant), it is enough to check that the expression $h\Lambda + \frac{h^2}{2} - \bar{c}\delta\Lambda$ is closed under δ, which follows from $\delta^2 = 0$.

Since δ preserves the Lagrangian, it preserves the set of its critical points and hence indeed defines a derivation of the algebra of local functionals.

As an example consider the case when E is the trivial bundle (and there is no matter fields). One of the possible gauge fixing conditions is the Feynman gauge condition $d^*A = 0$. So we take $\Lambda = d^*A$. Then one has $\delta\Lambda = -d^*d_Ac$, so after elimination of h the Lagrangian for pure gauge theory is

$$L = \frac{1}{4e^2}\int F^2 + ((d^*A)^2 - \bar{c}d^*d_Ac). \tag{16.4}$$

We see that this Lagrangian is nondegenerate, so one can do perturbation theory with it as usual. The derivation δ on R in this case is defined by

$$\delta c = \frac{1}{2}[c,c], \delta A = -d_Ac, \delta\bar{c} = -d^*A. \tag{16.5}$$

16.3. The properties of the BRST derivation.

Thus, we have constructed a derivation δ. The main properties of δ are:

1. $\delta^2 = 0$.

2. δ is defined on $\tilde{R}$ apriori, without the use of the Lagrangian $L(A, \phi)$ and the gauge fixing term Λ. It preserves the Lagrangian with ghosts $\tilde{L}$ and therefore descends to R.

Now let us turn to quantum theory. In this case local functionals are replaced by local operators. It can be shown that there exists a renormalization procedure under which δ can be defined as above, and properties 1 and 2 hold. This is discussed below.

However, in order to use the BRST method for quantization, we will need another, purely quantum, property of δ. Namely, denote by $\mathcal{L}_{eff}$ the effective Lagrangian, i.e. the Lagrangian for which the classical theory is equivalent to the quantum theory for $\mathcal{L}$. This Lagrangian is of course nonlocal. The third property of δ that we need is

3. δ preserves the effective action $\mathcal{L}_{eff}$. That is, $\delta\mathcal{L}_{eff} = 0$.

This property may fail, and if it fails then one says that the theory is anomalous.

Remark. Although $\mathcal{L}_{eff}$ is nonlocal, it can be shown that $\delta\mathcal{L}_{eff}$ is always the integral of a local expression. It can be shown that the obstruction to making $\delta\mathcal{L}_{eff}$ zero by adding an auxiliary term to the Lagrangian (in a way that does not change the physics) is given by a 1-loop calculation. Thus, anomalies arise in the 1-loop order of perturbation theory, and don't have higher order corrections. We will see this at the end of the lecture.

16.4. Operators in gauge theory and BRST cohomology.

Assume properties 1-3 hold. Consider the path integral Z given by Lagrangian $\tilde{L}$:

$$Z = \int DADcD\bar{c}DhD\phi e^{-\tilde{L}} \tag{16.6}$$

(possibly with some gauge-invariant insertions). Properties 1-3 imply that Z is independent on the gauge fixing condition Λ. Indeed, for any local expression X we have

$$\int DADcD\bar{c}DhD\phi e^{-\tilde{L}}\delta X = \delta \int DADcD\bar{c}DhD\phi e^{-\tilde{L}} X = 0, \tag{16.7}$$

which implies the independence of Z on Λ.

Remark. In (16.7), we used that δ preserves the measure of integration $DADcD\bar{c}DhD\phi$. It is easy to see that this is equivalent to Property 3 (absence of anomalies).

The statement that Z is independent of Λ holds for operators (insertions) which are annihilated by δ; for example, for any gauge-invariant insertions into Z, depending only on A, ϕ. On the other hand, if $\mathcal{O} = \delta\mathcal{O}'$ then the integral of $\mathcal{O}$ is zero by (16.7). Thus, the space of "physical" quantum operators in our theory is the cohomology of δ. This cohomology is called the BRST cohomology.

The BRST cohomology comes with a natural $\mathbb{Z}$-grading. Namely, we have a grading in the space of local operators, in which gauge and matter fields have degree 0, c has degree 1 and $\bar{c}$ degree -1. This degree is called the ghost number. It is easy to see that δ raises the degree by 1. This allows to introduce a grading in cohomology: we denote by H^q_δ the cohomology in degree q.

The properties of this cohomology, which usually hold in this situation are:

1. H^q_δ vanishes for $q<0$.

2. H^0_δ is the space of gauge invariant local operators depending only on A and ϕ.

This shows that δ plays the role of the gauge symmetry which was broken when ghosts were introduced. Thus, we have established a setting for gauge theory which works well in perturbation theory and in which the gauge symmetry does not die but rather appears in the form of δ.

16.5. Renormalization and BRST differential.

Now let us discuss the renormalizability and renormalization group equation in the BRST approach. We will restrict ourselves to 4 dimensions and pure nonabelian gauge theory.

Recall that in order for a theory to be renormalizable, all interactions have to have nonnegative dimension. To find out whether it is so for the Lagrangian with ghosts, we will compute the dimensions of fields (assuming that δ preserves the scaling dimensions). It is easy to see that the dimensions are as follows: $[c]=0, [\bar{c}]=2, [A]=1$. This shows that all interactions in the Lagrangian with ghosts are renormalizable.

Remark. In this theory, dimensions of c and $\bar{c}$ are not uniquely determined; the only thing that is determined canonically is $[\bar{c}c]$, which equals 2. This does not lead to a contradiction, since all operators of nonzero ghost number have zero expectation value, and so their scaling dimension has no intrinsic meaning. This is why we needed to make an additional assumption that δ preserves dimensions to fix precise values of the dimensions. If we had assumed that δ raises dimension by k we would get a different answer, which would be equally good for our purposes.

Now let us look for critical couplings which will be renormalized. In the setting without ghosts, the usual thing to do is to write down renormalizable (non-gauge invariant) operators of dimension 4, which correspond to critical couplings: $[A,A]^2, [A,A]dA, (dA)^2$, and then argue that there is only one gauge invariant combination of these operators, so that the only coupling which is to be renormalized is the charge e. However, in the setting with ghosts, we also have to include operators of degree 4 involving ghosts: $\bar{c}cA^2, \bar{c}d^*dc, \ldots$. The gauge invariance condition is now replaced by the condition that δ is a symmetry, so we need to renormalize only delta-invariant interactions. This cuts down the number of operators to be renormalized, but still leaves us with two renormalizable couplings: the charge e and the gauge fixing parameter a, corresponding to the scaling of the gauge fixing term $\frac{\Lambda^2}{2}+\bar{c}\delta\Lambda$. Thus the renormalization group vector field looks like

$$W=\mu\frac{\partial}{\partial\mu}+\beta(e)\frac{\partial}{\partial e}+\tilde{\beta}(e,a)\frac{\partial}{\partial a}. \tag{16.7}$$

Here β is the beta-function of the theory, and $\tilde{\beta}$ is the ghost beta-function. The beta-function of the theory depends only on e and is physically meaningful; for example, the negativity of its leading term insures asymptotic freedom. However, the ghost beta-function $\tilde{\beta}$ has no physical meaning: it only matters for renormalization of operators and correlators containing c and $\bar{c}$, which don't make sense physically.

16.6. The Hamiltonian approach.

So far we have considered BRST method from the Lagrangian point of view. Now let us consider the connection of the BRST method with the Hamiltonian formalism.

Since ghosts violate spin-statistics (being scalar fermions), the "Hilbert space" of the theory with ghosts cannot be an actual Hilbert space. Namely, it is possible to construct a certain space $\tilde{\mathcal{H}}$ with a Hermitian form, which is analogous to the Hilbert space in actual physical theories, but the form will not be positive definite. However, on this space we have local quantum operators, obtained by quantization of classical operators in the usual way. In particular, we have the global charges – the Hamiltonian H as well as the BRST charge Q, obtained from ghosts as explained in D'Hokers lecture. We also have a grading of $\tilde{\mathcal{H}}$ by ghost number, obtained naturally from the quantization procedure.

The operator Q has the property $[Q, \mathcal{O}] = \delta\mathcal{O}$ for any operator $\mathcal{O}$ (not necessarily δ-closed) in the theory with ghosts. Also, $Q\Omega = 0$, where Ω is the vacuum, and the ghost number of Q is 1.

The operator Q has properties analogous to those of δ:

1. $Q^2 = 0$. This can be confirmed by a direct computation.

2. Q is defined apriori, without the use of L and Λ (by an explicit formula as in D'Hoker's lectures). In particular, if $\tilde{\mathcal{H}}$ is an irreducible representation of the operator algebra, then Q is completely determined by δ and the properties $[Q, \mathcal{O}] = \delta\mathcal{O}$, $Q\Omega = 0$.

3. If there is no anomalies, the element Q commutes with the Hamiltonian and with all gauge invariant local operators which involve no ghosts.

Remark. As in the Lagrangian setting, here the explicit expression for Q is independent on Λ only if one uses the operator h corresponding to the auxiliary field in the Lagrangian. This operator can be expressed via other operators in the theory, in a way which depends on Λ: $h = -\Lambda$. If one makes this substitution, the obtained formula for Q will involve Λ. Thus, property 2 should be understood as follows: there exists a formula for Q in terms of the fields (including h!) which is independent on $\mathcal{L}$ and Λ but depends only on field configuration.

Let H_Q^q be the cohomology of Q on $\tilde{\mathcal{H}}$, graded by ghost number.

The properties of Q which usually hold are

1. H_Q^q vanishes if $q < 0$.

2. The Hermitian form is degenerate on the kernel of Q in $\tilde{\mathcal{H}}^0$ (operators of ghost number zero); the kernel of this form is the image of Q. The induced form on H_Q^0 is positive definite.

The space H_Q^0 plays the role of the physical Hilbert space of the theory, so we denote it by H_{phys}.

In the space H_{phys}, we have an action of the Hamiltonian H and "physical" local operators $\mathcal{O} \in H_\delta^0$. These operators no longer involve ghosts and correspond to actual observables of the theory.

Let us now compare the BRST and the "traditional" approaches to quantization of gauge theory. For simplicity, we consider pure gauge theory. Traditionally, a scheme of quantization would be as follows. Suppose that the space part of the spacetime is compact. In this case we have seen that classically the space of solutions to the equations of motion can be realized as $T^*\mathcal{A}$, where $\mathcal{A}$ is the space of connections on a space cycle modulo gauge transformations. Therefore we would define the Hilbert space as $L^2(\mathcal{A})$ (with respect to some measure). We call this Hilbert space the traditional Hilbert space.

We claim that these approaches give the same result, i.e. H_{phys} is isomorphic to H_{trad} as a representation of the operator algebra.

First of all, H_{phys} does not depend on the gauge fixing term Λ, and the Hamiltonian and the quantum operators in H_{phys} don't depend on it either. This follows from the fact that when Λ is varied, operators in the pseudo-Hilbert space $\tilde{\mathcal{H}}$ are changed by adding a δ-exact expression, so their action on δ-closed vectors is changed by a δ-exact expression.

To identify H_{phys} with H_{trad} we can use a convenient gauge fixing term Λ. It is enough to do it for one such term, but we will do it for two – just for fun.

Set $\Lambda = ud^*A + vA_0$, where A_0 is a time component of the connection (this uses the splitting of spacetime into space and time). Then we get a sensible theory unless both $u = 0$ and $v = 0$. Even $u = 0, v \neq 0$ gives a nice theory – this gauge fixing term is called "temporal gauge".

We first consider the case $u = 0$. Then the Lagrangian is

$$\hat{L} = \int (\frac{1}{4e^2}F^2 + \frac{1}{2}v^2A_0^2 - v\bar{c}\frac{Dc}{Dt}),$$

where D denotes covariant derivative. Replacing $\bar{c}$ with $-v\bar{c}$, and tending v to infinity (using the fact that nothing depends on v), we see that the path integral is localized to the hyperplane $A_0 = 0$, and in the limit we get a Lagrangian

$$\hat{L} = \int (\frac{1}{4e^2}F^2 + \bar{c}\frac{dc}{dt}), \tag{16.8}$$

Since A_0 is now zero, we get usual quantum mechanics where dynamical variables are a spacial connection, its time derivative, and the ghosts. Thus, $\tilde{\mathcal{H}}$ has the form $\mathcal{O}(\tilde{\mathcal{A}}) \otimes \Lambda\hat{\mathfrak{g}}^*$, where $\tilde{\mathcal{A}}$ is the space of connections on the space cycle, $\mathcal{O}(\tilde{\mathcal{A}})$ is the space of functions on $\tilde{\mathcal{A}}$, $\tilde{\mathfrak{g}}$ is the Lie algebra of the group of gauge transformations on the space cycle, and $\Lambda\tilde{\mathfrak{g}}^*$ is the space of functions of $c \in \Pi\tilde{\mathfrak{g}}$. Moreover, the Hamiltonian for the ghosts vanishes since there is no nontrivial evolution on the space of classical solutions (the Euler-Lagrange equations for ghosts are simply $\frac{dc}{dt} = \frac{d\bar{c}}{dt} = 0$). Thus the Hamiltonian in our theory is the usual gauge theory Hamiltonian

$$H = \frac{1}{2e^2}\int d^3x F_A^2 + \frac{e^2}{2}\nabla_A^2, A \in \tilde{\mathcal{A}} \tag{16.9}$$

acting on the first component of the tensor product.

It is easy to see that the space $\tilde{\mathcal{H}}$ with the grading by ghost number is nothing but the space of the standard complex of the Lie algebra $\tilde{\mathfrak{g}}$ with coefficients in the module $\mathcal{O}(\tilde{\mathcal{A}})$. Moreover, from the explicit formula for Q one gets that in this case Q is exactly the differential in the standard complex. Thus, the physical Hilbert space H_{phys} which is by definition the 0-th cohomology of Q, is the 0-th the cohomology of $\tilde{\mathfrak{g}}$ with coefficients in $\mathcal{O}(\tilde{\mathcal{A}})$. This is just the space of $\tilde{\mathfrak{g}}$-invariants in $\mathcal{O}(\tilde{\mathcal{A}})$, i.e. the space of functions on $\tilde{\mathcal{A}}/G$, which is by definition the traditional Hilbert space H_{trad}.

This shows that BRST cohomology is an infinite dimensional generalization of Lie algebra cohomology.

Now consider another gauge obtained by setting $v = 0, u = 1$: $\Lambda = d^*A$. Let us try to see the isomorphism between H_{phys} and H_{trad} using this gauge. We have,

$$\hat{L} = \frac{1}{4e^2}\int F^2 + (\frac{1}{2}(d^*A)^2 - \bar{c}D^*d_A c). \tag{16.10}$$

In this case the equations of motion for ghosts are nontrivial and of second order, so the Hilbert space $\tilde{\mathcal{H}}$ consists of functions of $A_s, c, \bar{c}, A_0$, where A_s is a connection on the space cycle. In this case, one finds

$$Q = Q_{\text{Lie algebra cohomology}} + Q', \tag{16.11}$$

where $Q' = \int \pi_{\bar{c}} \pi_{A_0}$, and $\pi_{\bar{c}}$, π_{A_0} are the momentum operators for $\bar{c}$, A_0.

It is easy to check directly that the two summands in (16.11) anticommute, and that $(Q')^2 = 0$. It can also be checked that Q' is acyclic except in 0-th degree, where it has a 1-dimensional cohomology. Thus, by Kunneth formula, we again get $H_{phys} = H_{trad}$.

16.7. Anomalies.

Now let us recall conditions 1,2,3 which were necessary for the BRST construction, and analyze when they are satisfied. These conditions are

1. Lagrangian: $\delta^2 = 0$.
Hamiltonian: $Q^2 = 0$.
2. Lagrangian: δ is independent on Λ and $\tilde{\mathcal{L}}$.
Hamiltonian: Q is independent of $\Lambda, \tilde{\mathcal{L}}$.
3. Lagrangian: $\delta \mathcal{L}_{eff} = 0$.
Hamiltonian: $[Q, H] = 0$.

As we mentioned, properties 1 and 2 can always be attained.

However, as we also mentioned, Property 3 may fail if anomalies are present. So let us consider anomalies more closely.

Consider a 4-dimensional gauge theory with a chiral spinor ψ with values in a complex representation ρ of the gauge group G and antichiral spinor $\bar{\psi}$ with values in the dual representation $\bar{\rho}$. The basic gauge-invariant Lagrangian for such fields is

$$\mathcal{L} = \frac{1}{4e^2} \int F^2 + \int \bar{\psi} D_A \psi. \tag{16.12}$$

In quantum theory we are interested in the path integral $\int e^{-\mathcal{L}}$. Integrating out ψ in this path integral, we get

$$\int \det(D_A) e^{-L(A)} DA, \tag{16.13}$$

where $D_A : \Gamma(S_+) \to \Gamma(S_-)$ is the covariant Dirac operator and $L(A)$ the Lagrangian of the pure gauge theory.

Integral (16.13) may not have a gauge invariant regularization. What is worse, it may not even have a non-gauge-invariant regularization for which the gauge invariance is restored as the cutoff goes to infinity. In this case the gauge theory we are considering does not make sense quantum mechanically, even in perturbation theory, because gauge symmetry cannot be restored. This phenomenon is called an anomaly.

A geometric reason for an anomaly is that although the operator D_A is gauge invariant, its determinant $\det(D_A)$, in general, fails to be gauge invariant. In other words, this determinant is not a function on the space of gauge classes of connections but rather a section of some line bundle over this space, called the determinant line bundle; this bundle comes with a canonical connection. If this canonical connection does not trivialize the bundle, this "function" cannot be sensibly integrated.

It is useful to distinguish two types of anomalies.

1. Local anomaly. The canonical connection has a nonzero curvature. In this case for suitable spacetime manifolds this curvature may represent a nontrivial second cohomology class, so that the determinant bundle is not trivial topologically.

2. Global anomaly. The canonical connection is flat but has a nontrivial monodromy (and possibly the bundle is not trivial).

Thus, both local and global anomalies can produce topological anomalies, but only the first one can be seen in perturbation theory (by computing of the curvature).

Here we will consider only local anomalies.

Remark 1. To analyze when we can expect local anomalies, one may consider the situation from the topological point of view. We assume that our spacetime M is compact and orientable (e.g. S^d), with a specified point ∞, and will consider bundles, connections, and gauge transformations which are trivial at infinity. In this case the space of gauge classes of connections can be regarded as a classifying space $B\hat{G}$ for the group of gauge transformations $\hat{G}$. Nontrivial line bundles on $B\hat{G}$ are classified by $H^2(B\hat{G})$.

Now, if M is compact and orientable, we have the transgression map $\tau : H_2(B\hat{G}) \to H_{d+2}(BG)$ defined as follows: given a two-dimensional homology class, pick a surface S in $B\hat{G}$ which represents it, and take the corresponding principal $\hat{G}$-bundle on S. Its transition functions can be considered as transition functions of a G-bundle on the 6-dimensional manifold $S \times M$, which defines an element $\tau([S])$ of $H_{d+2}(BG)$. Consider the dual map $\tau^* : H^{d+2}(BG) \to H^2(B\hat{G})$. It can be shown that the Chern class of our line bundle is $\tau^*(C)$, where C is a fixed $d+2$-dimensional characteristic class which does not depend on M, and is computed locally from the curvature. This class is exactly the local anomaly.

Thus for $d = 4$ local anomalies live in $H^6(BG)$, or $(S^3\mathfrak{g})^{\mathfrak{g}}$, where $\mathfrak{g}$ is the Lie algebra of G.

For example, in the standard model the gauge group is $SU(3) \times SU(2) \times U(1)$, and thus the space of anomalies $(S^3\mathfrak{g})^{\mathfrak{g}}$ is 4-dimensional: it equals to the sum of four subspaces $S^3_{inv}(su(3)), S^3_{inv}(u(1)), S^2_{inv}(su(2)) \otimes u(1), S^2_{inv}(su(3)) \otimes u(1)$, which are 1-dimensional (here "inv" denotes that we are taking invariant symmetric polynomials).

This discussion illustrates why anomalies don't arise in the case when all fermions are in a real representation of the gauge group. Indeed, in this case, the determinant bundle is real, and thus its Chern class must be zero.

Remark 2. Although the local anomaly can be considered from the above topological point of view, one should remember that it has a purely local nature, and has nothing to do with the macrostructure of the spacetime. If there is a local anomaly, the quantum theory will not make sense on any spacetime, even on $\mathbb{R}^d$. The problem is that even if the determinant bundle is topologically trivial, it will not have a flat connection defined in a local way: otherwise this flat connection would have been good for any simply connected spacetime, and no topological anomaly would arise. Thus, path integral (16.13) is not sensible even on $\mathbb{R}^d$.

Remark 3. In the standard model, the gauge group is $SU(3)\times SU(2)\times U(1)$. In particular, there is a possibility for local anomalies, and they do appear in reality.

However, one can check that the anomalies coming from the different matter fields of the standard model miraculously cancel, in all four components of the space of anomalies. An explanation of this is that the representation of the gauge group in the standard model extends (after adding some insignificant summands) to a spinor representation of $Spin_{10}$, for which $H^6(BG)$ vanishes.

Let us show how to analyse anomaly in perturbation theory. Our goal is to explain why, after possibly enlarging the space of fields, properties 1. and 2. of section 16.3 can always be assumed to hold (that is, $\delta^2 = 0$ and δ is defined independently of the choice of a particular Lagrangian) but one cannot assume that the effective Lagrangian is delta-invariant.

First, let us just try to make sense of integral (16.3) perturbatively. When we write down Feynman diagrams, we will find divergences in the 1-loop order which we cannot remove in a gauge invariant fashion. To fix the 1-loop order, we will regularize the path integral by adding another, very heavy matter field χ such that its determinant bundle is inverse to that for ψ. In favorable cases, our original theory should be recovered from this theory in the limit when the mass m of χ goes to infinity. In other cases, the procedure will exhibit why there is an anomaly.

To satisfy this condition, the matter field χ can be taken to be a bosonic field (χ_+, χ_-) with values in $(S_+ \oplus S_-) \otimes \rho$. In this case the complex conjugate field $\bar{\chi}$ is with values in $(S_+ \oplus S_-) \otimes \bar{\rho}$, where S_+, S_- are the spin representations of the Poincare (recall that both S_+ and S_- are self-dual and self-complex-conjugate in Euclidean signature). It is of course needless to say that these fields violate spin-statistics and therefore, like ghosts, don't make physical sense.

The natural Lagrangian term for the fields $\chi_\pm$ would be

$$L'(A, \chi_\pm) = \int d^4x((\bar{\chi}_+, D_A\chi_-) + (\bar{\chi}_-, D_A\chi_+) + m(\bar{\chi}_+, \chi_+) + m(\bar{\chi}_-, \chi_-)) \tag{16.14}$$

(Here the Dirac operator is skewselfadjoint).

Remark. The χ's are called Pauli-Villars regulator fields.

If we add expression (16.14) to the Lagrangian, we will get the squared absloute value of the determinant rather than the determinant itself, and will not fix the anomaly. Thus, we modify (16.14) in a way that breaks the gauge invariance: we let A_0 be a fixed connection and set

(16.15)
$$L''(A, \chi_\pm) = \int d^4x((\bar{\chi}_+, D_A\chi_-) + (\bar{\chi}_-, D_{A_0}\chi_+) + m(\bar{\chi}_+, \chi_+) + m(\bar{\chi}_-, \chi_-))$$

Now consider the theory with the Lagrangian $\mathcal{L} + L''$. Integrating out the χ fields, we will get a factor $\det(D_A D_{A_0} - m^2)^{-1}$. For $m = 0$ this factor is gauge invariant up to a multiplicative constant, and cancels the determinant in the numerator, which is caused by anomaly. This shows that in this theory, we don't have a topological anomaly for any finite m (i.e. the appropriate determinant bundle is trivial). However, for $m > 0$, the gauge invariance fails. So we have to study the limit $m \to \infty$ (which is supposed to recover our original theory) and see whether the gauge symmetry reappears.

Differentiating the determinant ratio $\det(D_A D_{A_0} - m^2)/\det(D_A)$ in the direction of a gauge trasformation $t \in \hat{\mathfrak{g}}$, we obtain (using the path integral interpretation) that it is equal to $m\langle\int[(\bar{\chi}_+, t\chi_+) + (\bar{\chi}_-, t\chi_-)]\rangle$, where $\langle X\rangle$ denotes the expectation value of X. This expectation value has a decomposition in powers of $1/m$.

To see whether the failure of gauge invariance persists for $m \to \infty$, let us consider the two-point function of the curvature operator F. It is easy to see that the leading contribution (in $1/m$) to the derivative of this function in the direction of t is from the 1-loop diagram with a χ loop having the t operator inside and two outgoing F-edges. This contribution is of the 0-th order in $1/m$, and has the form $\sum d_{abc} t^a F^b F^c$, where d_{abc} is some tensor. So if $d_{abc} \neq 0$, the gauge-noninvariance remains in the limit.

Remark. In case the original fermions were in a real representation, d_{abc} is zero and the regularization in (16.14) is completely satisfactory. The problem arises when the original representation is complex. Then a regularization as in (16.14) doesn't work unless one gives up gauge invariance.

Remark. When $d_{abc} \neq 0$, one can choose a regularization scheme to remove all loop contributions to the non-gauge invariance except 1 loop.

Now let us consider anomalies from the BRST veiwpoint. If a local anomaly is present, we will have $U = \delta \mathcal{L}_{eff}(A) \neq 0$ (here $\mathcal{L}_{eff}(A)$ is the effective Lagrangian for A, with the ghosts integrated out). However, since the anomaly is local U must be the integral of a local expression of A and c which is linear in c. It is also clear that $\delta U = 0$. Furthermore, one can show that U involves only A and its first derivatives (and no matter fields).

On the other hand, if U is δN where N is the integral of some local expression of A then we can arrange $\delta \mathcal{L}_{eff} = 0$ by redefining the Lagrangian as $\mathcal{L} \to \mathcal{L} + N$. Thus, anomalies lie in the cohomology of δ on local functionals of degree ≤ 1 of A and c (linear in c) modulo complete derivatives.

Now let us show that such cocycles are in fact related to invariant symmetric tensors on the Lie algebra $\mathfrak{g}$ (or equivalently, the cohomology of $\mathfrak{g}$).

Let C be a G-invariant element in $S^{n+1}\mathfrak{g}$. To this element there corresponds a $2n+2$-dimensional characteristic class of G-bundles, namely $C(F^{n+1})$, where F is the curvature. The Chern-Simons form $CS_C(A)$ corresponding to C is the local $2n+1$ form such that $\frac{\tilde{\delta} C(F^{n+1})}{\tilde{\delta} A} = CS_C(A) \wedge \tilde{\delta} A$ modulo differentials of local forms (here $\tilde{\delta}$ denotes the variation to distinguish from the BRST differential δ).

The main property of the Chern-Simons form is the following. Although this form is not gauge invariant, its Lie derivative along an infinitesimal gauge transformation is a differential of a local form.

Now let M^{2n} be our spacetime. Let A be a connection on M^{2n}. We want to define a functional of the form $U(A) = \int W(A)$, where $W(A)$ is a $2n$-form on M^{2n} which is local in A but not gauge invariant, and such that $\delta W = 0$.

Let X^{2n+1} is a smooth manifold with boundary equal to M^{2n}. Choose an extension of the connection A to X^{2n+1} in any way (for simplicity we assume that there is no topological obstruction to the choices of X and the extension of A; this assumption is in fact inessential). Now set $V(A) = \int_X CS_C(A)$. This functional depends on the extension of A to X. However, by the main property of CS, the functional $\delta V(A) = W(A, c)$ (where δ is the BRST differential) does not depend on the extension and therefore is a local functional in A and c linear in c. One can show that it represents a nontrivial cohomology class in the local δ-cohomology. Thus, we get an injective map $S^{n+1}(\mathfrak{g})^{\mathfrak{g}} \to H_\delta^{1,local}$. For 4-dimensional theories $n = 2$ and the cocycles come from $(S^3\mathfrak{g})^{\mathfrak{g}} = H^6(BG)$ as we expected.