Lecture II-17: $N = 2$ Super-symmetric Yang-Mills theories in dimension four: Part 1

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I Introduction

Today we shall be studying the dynamics of $N = 2$ super-symmetric Yang-Mills theories in four-dimensions. The underlying manifold on which we shall study these theories will be Minkowski four-space $M^4 = \mathbf{R}^{3,1}$. We are mainly interested in the case when the gauge group G is $SU(2)$. Since low energy limits of $SU(2)$ theories can generate $U(1)$-theories, we shall begin by studying $U(1)$-Yang-Mills theory. One should note that these $U(1)$-theories are purely unrenormalizable (all terms are unrenormalizable) in dimension four. Thus, they don't make sense as fundamental theories, but they are free in the infra-red. But, by the same token, we can study their low energy limits by doing semi-classical calculations. Of course, when we get to the $SU(2)$-theory we will be dealing with an asymptotically free, renormalizable theory which has strong quantum effects. This is then a reasonable fundamental theory, but its low energy limit is hard to understand.

We shall eventually use the analysis of low energy limits of these gauge theories on Minkowski space as a tool in the analysis of low energy limits of these theories on compact Riemannian four-manifolds. Understanding these limits will give another mechanism for computing certain correlation functions associated to cycles in the four-manifold. These correlation functions can be computed in the low energy limit because one shows that they are invariant under deformation of the parameters. Computing in the high energy limit, one identifies these correlation functions with differentiable invariants, namely the Donaldson invariants of the four-manifold. In this way we obtain totally different expressions for the Donaldson invariants of a four-manifold. In

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most cases these invariants can be directly computed from a $U(1)$-monopole theory. This then produces the link between the Donaldson polynomial invariants and the Seiberg-Witten $U(1)$-monopole invariants. All this is the subject of the next several lectures. Today we study low energy limits of $U(1)$-gauge theories.

II Low Energy $U(1)$ $N = 2$ super Yang-Mills Theories

II.1 The basic fields

Consider pure $N = 2$ super Yang-Mills theory in four-dimensions with gauge group $U(1)$. This means that the fields in the Lagrangian will form a single $N = 2$ vector super-multiplet. That is to say, we will have the gauge field, and its $N = 2$ super-symmetric partners. All the fields will be massless. Let us recall from the super-homework what the $N = 2$ massless vector super-multiplet looks like in dimension 4. Massless particles are associated to representations of the 'little' subgroup $Spin(2)$ in the Lorentz group $Spin(3, 1)$ fixing a nontrivial vector $v \in R^{3,1}$ of norm zero. The 'helicity' of such a particle (i.e., its spin around its direction of motion) is the representation of its stabilizer. These are indexed so that the fundamental representation of $Spin(2)$ is labeled 1/2. The irreducible representations of the Poincaré group that comprise the $N = 2$ vector super-multiplet in dimension four are arrayed as pictured below:

helicity	-1	$-1/2$	0	$\oplus$	0	$1/2$	1
multiplicity of representation	1	2	1		1	2	1.

The action of CPT conjugation on these representations is to interchange the right and left halves of this table. This multiplet has a $U(2)_R$ symmetry, acting so as to preserve helicity. Thus, the $SU(2)_R \subset U(2)_R$ acts trivially except on those particles of helicities $\pm(1/2)$. On each of these it acts via the fundamental two-dimensional representation. The $U(1)_R$ acts trivially on the states of helicity ± 1 and acts by (two copies of) the standard representation on the states of helicity $\pm 1/2$ and by the ± 2 power of the fundamental representation on the states of helicity zero. The CPT involution normalizes the $U(1)_R$ and acts on it by conjugation. Thus, the representation of $U(1)_R$ on the other half of the multiplet is given by the negative of the above.

The fields of an $N = 2$ vector super-multiplet on $N = 1$ super-space in dimension four decompose into an $N = 1$ vector multiplet and an $N = 1$ chiral multiplet. The $N = 1$ vector super-multiplet is the gauge $N = 1$ super-multiplet $\mathcal{A}$. The field $\mathcal{A}$ is the super-connection and (after partial gauge fixing to Wess-Zumino gauge) consists of a triple

$$\mathcal{A} = (A, \lambda, D)$$

where A is the usual $U(1)$-gauge field, λ is a fermion field, and D is an auxiliary field. The curvature of $\mathcal{A}$ is a two-form on $N = 1$ super-space. The entire curvature form is determined by certain components W_α which are $(\frac{1}{2}, 0)$ spinor chiral super-fields. (Representations of $Spin(3,1) = SL_2(\mathbf{C})$ are indexed by an ordered pair (a, b) of half-integers. The pair $(1/2, 0)$ indexes the defining two-dimensional complex representation. It is chiral because the second half-integer is zero.) The only way that the super-connection $\mathcal{A}$ enters the action is through the field W_α. The other fields form an $N = 1$ super-multiplet called Φ. This is a (scalar) chiral super-multiplet. It consists of

$$\Phi = (\phi, \psi, F)$$

where ϕ is a complex-valued scalar field, ψ is a fermion, and F is an auxiliary field. The auxiliary fields D and F enter the action quadratically and without derivatives so that they can be integrated out using the equations of motion. Thus, they play no role in the analysis.

II.2 The Lagrangian

Now we are ready to write the Lagrangian for the pure $N = 2$ $U(1)$-gauge theory in four-dimensions. Though we require impose $N = 2$ super-symmetry, we choose to write the Lagrangian first as a Lagrangian for fields on $N = 1$ super-space and then study the consequences of the extra super-symmetry. The Lagrangian is:

$$\mathcal{L} = \left(\frac{i}{4\pi} \int d^4x d^2\theta \tau(\Phi) W_\alpha W^\alpha + \text{c.c.} \right) + \frac{1}{4\pi} \int_{R^{3,1|4}} d^4x d^4\theta K(\Phi, \overline{\Phi}). \tag{1}$$

As we remarked above, W is a piece of the curvature of the $N = 1$ super connection $\mathcal{A}$. It is a chiral super-field. The function $\tau(\Phi)$ is a holomorphic function on the complex plane (or more generally the complex curve) where Φ takes its values. Since Φ is a chiral super-field it follows that $\tau(\Phi)$ is also a chiral super-field. We write

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}.$$

Lastly, $K(\Phi, \overline{\Phi})$ is a Kähler potential on Φ-space.

We have encountered this Lagrangian before, with one essential difference. That difference is that here we are not assuming τ is constant. Rather we allow it to be a holomorphic function of Φ. Briefly, the reason that we consider this generalization is that this theory will arise as a low energy limit of some $SU(2)$-theory. That theory will have a complex plane of classical vacua (i.e., the space of minima of the potential energy will be $\mathbf{C}$). We find terms in the low energy limit which involve functions on this space of classical vacua (cf, sigma-models arising as low energy limits). It is in this way that the non-constant holomorphic function τ arises. The main application we have in mind for this study is to the low energy limits of $N = 2$ super-symmetric pure $SU(2)$ gauge theory. Of course, in the low energy limit Φ will be slowly varying, i.e., almost constant. To the extent that Φ is almost constant the same will be true of τ, and we find ourselves expanding around a low energy effective theory with constant τ_{eff}.

Another difference of this Lagrangian from the general $N = 1$ super-symmetric Lagrangian is that there is no super-potential term. The reason is that one can have such a term in an $N = 1$ super-symmetric theory on $N = 1$ super-space in four-dimensions, but the extra conditions imposed by requiring the theory to have $N = 2$ super-symmetry imply that this term must be zero. Another related, but more delicate, consequence of the same idea is that $N = 2$ super-symmetry imposes a relation between τ and K. To see this, let us write out the Lagrangian in components. We get

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\pi}\int_{R^{3,1}} d^4x \Big(i\left[\overline{\tau}(F^+\wedge F^+) - \tau(F^-\wedge F^-)\right] + K_{\phi\overline{\phi}}\partial\phi\overline{\partial\phi} \\ &\qquad + \mathrm{Im}\tau\overline{\lambda}i\partial\!\!\!/\lambda + K_{\phi\overline{\phi}}\overline{\psi}i\partial\!\!\!/\psi\Big) \end{aligned} \tag{2}$$

One thing which is clear from this formula, which we have seen before, is that it is not the Kähler potential $K(\Phi,\overline{\Phi})$ *per se* that enters into the Lagrangian; only the Kähler metric $K_{\phi\overline{\phi}}\partial\phi\overline{\partial\phi}$ of the potential enters.

To see the relation of τ and K we use the $SU(2)_R$-symmetry which is a consequence of $N = 2$ super-symmetry. Notice that if $K_{\phi\overline{\phi}} = \mathrm{Im}\tau$, then the pair (λ,ψ) form a multiplet for the $SU(2)_R$ which is isomorphic to the standard representation. Conversely if there is an $SU(2)_R$-symmetry, then with an appropriate holomorphic redefinition of the coordinates the (λ,ψ) form a standard multiplet for the $SU(2)_R$ action. Thus, in these coordinates the coupling terms for λ,λ and ψ,ψ must have the same strength, and hence with this choice of coordinates we have $\mathrm{Im}\tau = K_{\phi\overline{\phi}}$. One immediate consequence of this is that $\mathrm{Im}\tau > 0$. This seems like a problematic condition.

Locally it is not hard to ensure that it holds, but for a global holomorphic function on all of $\mathbf{C}$ it cannot be achieved. There are two possibilities to contemplate. One is that as τ approaches the real axis, for some reason the physics breaks down. One would then be faced with studying what happens at these 'bad' limits. Another possibility is that τ is multi-valued. After all, we already know that the transformation $\tau \to \tau + 1$ (i.e., increasing θ by 2π) does not affect the theory. So at least to that extent one must consider τ as multi-valued. Of course, this does not solve the problem we are addressing since any non-constant, entire holomorphic function τ to $\mathbf{C}/\{\tau \cong \tau+1\}$ still cannot have $\mathrm{Im}\tau > 0$. To arrange this we need a larger group of indeterminacy for the values of τ. For example, recall that it is easy to construct global holomorphic functions into the upper half-plane if we permit them to be multi-valued, say under the standard action of $SL_2(\mathbf{Z})$. As we shall see, this is exactly what happens in this case.

II.3 Indeterminacy of τ

In order to make the notation consistent with that of [Seiberg-Witten ???] we set $\phi = a$. The term in the Lagrangian derived from the Kähler potential becomes

$$\frac{i}{4\pi} \int_{R^{3,1}} d^4x(\tau - \overline{\tau}) \frac{\partial a}{\partial x^i} \frac{\partial \overline{a}}{\partial \overline{x}^i},$$

which means that the induced Kähler metric on the Φ-plane, now called U, is

$$ds^2 = \frac{i}{4\pi}(\tau da d\overline{a} - da \overline{\tau} d\overline{a}),$$

where a is a local holomorphic coordinate on the U. It is convenient to introduce another holomorphic function a_D (called *the dual of* a) locally on an open subset V of U with the property that $da_D = \tau da$. Given this, we can re-write the metric on V as

$$ds^2 = \frac{i}{4\pi}(da_D d\overline{a} - da d\overline{a}_D).$$

We denote by Ω the Kähler form of this metric:

$$\Omega = \frac{-1}{8\pi}\left(da_D \wedge d\overline{a} - da \wedge d\overline{a}_D\right).$$

We can view this more symmetrically in a and a_D by forming the complex two-space with coordinates (a, a_D). Then there is an embedding of the complex curve $f\colon V \to \mathbf{C}^2$ into this complex two-space so that the image is a solution curve for the differential equation $da_D = \tau da$.

This embedding will be the graph of either of the variables as a function of the other. Let $\omega = \frac{-1}{8\pi}(da \wedge d\overline{a}_D - da_D \wedge \overline{a})$. It is a closed, non-positive $(1,1)$-form on complex two-space. Of course, f must be such that $f^*\omega = \Omega > 0$.

The question is: How much freedom do we have in the representation of the theory in terms of the parameters a, a_D (and implicitly τ)? Said another way, In what ways can we find other Lagrangian's of this form with different a and a_D (and hence different τ) which represent the same theory? Just considering the Kähler potential term we see an obvious geometric change of coordinates that brings the Lagrangian back into the same form with different values of the parameters. Namely, we can act by an extended version $ISL_2(\mathbf{R})$ of $SL_2(\mathbf{R})$. We define

$$0 \to \mathbf{C}^2 \to ISL_2(\mathbf{R}) \to SL_2(\mathbf{R}) \to 0$$

to be an extension which is a semi-direct product with the natural action of $SL_2(\mathbf{R})$ on $\mathbf{C}^2$. An element $(M, \begin{pmatrix} c_D \\ c \end{pmatrix})$ in this group acts by

$$(M, \begin{pmatrix} c_D \\ c \end{pmatrix}) \cdot \begin{pmatrix} a_D \\ a \end{pmatrix} = M \cdot \begin{pmatrix} a_D \\ a \end{pmatrix} + \begin{pmatrix} c_D \\ c \end{pmatrix}.$$

Since $\tau = \frac{da_D}{da}$, one sees that the action of this group on τ is the usual fractional linear transformation of M on $\tau \in \mathbf{H}$:

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \cdot \tau = \frac{u\tau + t}{s\tau + r}.$$

In this way this change of variables brings the $d^4\theta$ term of the Lagrangian back into the same form with a different a, a_D, τ.

This computation with the Kähler metric leads us to ask whether the non-uniqueness in our description of the holomorphic function τ should be that we are allowed to compose τ with any $SL_2(\mathbf{R})$ fractional-linear transformation. But we must examine the other term in the Lagrangian where τ appears. That is in the term involving the gauge field (or photon). Recall from Lecture II-8 what happens for constant τ. Since $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$ and the action is given by $e^{i\mathcal{L}}$, if we consider strictly upper triangular matrices the only ones that leave invariant this part of the action are those of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

with $b \in \mathbf{Z}$. As we have seen in Lecture II-8, this action of the strictly upper triangular matrices in $SL_2(\mathbf{Z})$ extends to an action of $SL_2(\mathbf{Z})$ on representations of the theory. That is to say, if τ and τ' in $\mathbf{H}$ differ by the action of $SL_2(\mathbf{Z})$, then the pure $U(1)$-gauge theory with τ as parameter has another representation as a $U(1)$-gauge theory (with different basic fields) where the parameter is τ'. The transformation $\tau \to (-1)/\tau$ is the duality transformation in $U(1)$-gauge theory interchanging the electric and magnetic charges. The upshot of all this is that the full symmetry group of representations of the theory by Lagrangian's of the given form is

$$ISL_2(\mathbf{Z}) = \mathbf{C}^2 \ltimes SL_2(\mathbf{Z}).$$

The way this group transforms the parameter τ is through composing with the usual fraction linear transformation action of $SL_2(\mathbf{Z})$ on the upper half-plane.

We now see how to describe our low energy effective theory globally. There is a complex curve U of vacua and there is a global holomorphic function $\overline{\tau}\colon U \to \mathbf{H}/PSL_2(\mathbf{Z})$. Furthermore, for each $u \in U$ there is a neighborhood V of u, a lifting of $\overline{\tau}$ to a holomorphic function $\tau\colon V \to \mathbf{H}$, and $N = 1$ super-fields $\mathcal{A}, \Phi$ such that there is a Lagrangian in the form given in Equation 1 (or Equation 2 in components) which represents the theory in V. As we pass from one of these open subsets to another, the representation of the low energy effective theory locally by Lagrangian's changes. The function τ changes by the natural action of $SL_2(\mathbf{Z})$ on $\mathbf{H}$. The chiral super-field changes by the action of $ISL_2(\mathbf{Z})$ described above, and the gauge super-field $\mathcal{A}$ changes by the duality action of $SL_2(\mathbf{Z})$ action on the representations of pure $N = 1$ super-symmetric $U(1)$-gauge theory as described in Lecture II-8. Because there are global holomorphic functions to $\mathbf{H}/SL_2(\mathbf{Z})$, this is a completely consistent picture of a global theory in which $\mathrm{Im}\tau$ never goes to zero.

II.4 The Family of elliptic curves associated to τ

We have now established all the relevant general features of the low energy effective theories we want to study. We have an $N = 2$ super-symmetric pure $U(1)$-gauge theory with a family of vacua parameterized by a complex curve U, where the scalar field $\phi = a$ takes its values. This curve has an open covering $\{V^\alpha\}$ and on each open subset there is a pair of dual holomorphic coordinates a^α, a^α_D so that $\tau^\alpha = \frac{da^\alpha_D}{da^\alpha}$ is the parameter appearing in the Lagrangian. We assume that the different local representations of the theory differ by automorphisms in $ISL_2(\mathbf{Z})$. We have just seen that the local holomorphic function τ on U which appears in the Lagrangian is in fact a global function of U to $\mathbf{H}/SL_2(\mathbf{Z})$. Thus, for each $u \in U$ it is natural to consider $\tau(u)$ as the j-invariant of an elliptic curve $E(u)$. Of course, it is possible that at a discrete set of points $\{p_i\}_i$ in U that τ goes off to infinity, or equivalently that the elliptic curves $E(u)$ develop a node as $u \to p_i$. At these points the physics changes and there are more massless fields, fields that become massive at nearby $u \in U$.

There is a natural way to fit these elliptic curves together. Namely, there is a complex analytic surface $\mathcal{E}$ together with a proper holomorphic map $\mathcal{E} \to U$ whose fibers form the family of elliptic curves $\{E(u)\}_{u\in U}$ The simplest way to construct such a family is to take the Weierstrass form. That is to say, there are functions A, B on U such that $\mathcal{E}$ is defined by

$$Y^2 = X^3 + AX + B,$$

with an appropriate completion of the missing point at infinity in each elliptic curve. (For globally non-trivial bases U, we will have a line bundle L over U and A and B will be sections of $L^{\otimes 4}$ and $L^{\otimes 6}$, respectively.) This family of elliptic curves has a natural section (the section at infinity in the Weierstrass description), and thus, it can be viewed as a family of one-dimensional Abelian varieties. There are, of course, other families of holomorphically varying genus-one curves over the same base with the same j-invariant function but without a section. But given the information of the j-invariant function only, there is no distinguished family except the one with a section.

Suppose that we have a family of elliptic curves $\mathcal{E} \to U$ with a section σ. Furthermore, suppose that we have an open covering $\{V^\alpha\}_\alpha$ of U such that on each V^α there are a pair of dual holomorphic coordinates a^α, a_D^α such that $\tau^\alpha = \frac{da_D^\alpha}{da^\alpha}$ is a function to the upper half-plane. We suppose that these two sets of data are related as above. That is to say, $\mathcal{E}|_{V^\alpha}$ is isomorphic to the family $V^\alpha \times \mathbf{C}/(1, \tau^\alpha(v))$. This allows us to define one-cycles $E_1^\alpha(u), E_2^\alpha(u) \subset E(u)$ for all $u \in V^\alpha$ as the images of the arcs $\{u\} \times [0,1]$ and $\{u\} \times [0, \tau^\alpha(u)]$ in $\{u\} \times \mathbf{C}$. Then, there is a differential of the second kind λ^α (i.e., a meromorphic one-form with trivial residue along its polar locus) on $\mathcal{E}|_{V^\alpha}$ such that

$$\int_{E_1^\alpha(u)} \lambda^\alpha = a^\alpha(u)$$

$$\int_{E_2^\alpha(u)} \lambda^\alpha = a_D^\alpha(u).$$

In general, we can not fit these differentials λ^α of the first kind together to produce a global object. (We will see the precise conditions when this can be done soon.) But we claim that we can fit together the two-forms $d\lambda^\alpha$. Namely, we claim that we there is a unique global holomorphic two-form η on $\mathcal{E}$ such that on $\mathcal{E}|_{V^\alpha}$ we have

$$\int_{E_1^\alpha} \eta = da^\alpha$$

$$\int_{E_2^\alpha} \eta = da_D^\alpha.$$

Of course, the uniqueness assertion immediately implies that the restriction of any such global two-form η to $\mathcal{E}^\alpha$ is equal to $d\lambda^\alpha$.

Recall that $\mathcal{E}^\alpha$ is identified with $V^\alpha \times \mathbf{C}/(1, \tau^\alpha(u))$. On the universal covering space $V^\alpha \times \mathbf{C}$ we have the holomorphic one-form dz induced from the standard holomorphic coordinate on $\mathbf{C}$. This one-form does not descend to the quotient because τ is not a constant function of a^α. Nevertheless, the two-form $da^\alpha \wedge dz$ does descend to give a well-defined two-form on $\mathcal{E}^\alpha$. It is clear that the only holomorphic two-form on $\mathcal{E}|_{V^\alpha}$ which satisfies the first integral equation above is

$$\eta^\alpha = -da^\alpha \wedge dz.$$

This two-form η^α also clearly satisfies the second integral equation. Now we need to see that these local two-forms fit together to give a global form η. But this is clear from the uniqueness of the local forms.

Now let us compute $\pi_*(\eta \wedge \overline{\eta})$, where π is the projection from $\mathcal{E} \to U$. Let us restrict attention to $\mathcal{E}|_{V^\alpha}$. From our local description we have

$$\pi_*(\eta \wedge \overline{\eta}) = \int_{\text{fibers}} d\lambda^\alpha \wedge d\overline{\lambda}^\alpha.$$

It follows immediately from the formulas for the integrals of $d\lambda$ over E_1^α and E_2^α and the fact that $E_1^\alpha \cdot E_2^\alpha = 1$ in the homology of $E(u)$ that the result is

$$\pi_*(\eta \wedge \overline{\eta}) = da^\alpha \wedge d\overline{a}^\alpha \overline{\tau}^\alpha + da^\alpha \wedge d\overline{a}^\alpha \tau^\alpha = (\tau^\alpha - \overline{\tau}^\alpha) da^\alpha \wedge d\overline{a}^\alpha = 8\pi\Omega|_{V^\alpha}.$$

Since the two-form η is holomorphic, it integrates trivially on any fiber and on the section σ of the elliptic fibration. Thus, its homology class is determined by the integrals of η along two-tori which lie over circles in U.

The deRham cohomology class of η measures whether or not the automorphisms required in passing between our local descriptions in terms of various sets of (a, a_D, τ) lie in $SL_2(\mathbf{Z})$ or in the extended group $ISL_2(\mathbf{Z})$. We claim that if the deRham homology class of η is trivial, then the automorphisms we use lie in $SL_2(\mathbf{Z})$. The point is the following. We already know that the change in da, da_D is by elements in $SL_2(\mathbf{Z})$. We need to compute the changes in a, a_D. If we go around any loop γ in U with the property that E_1 comes back to itself, then the cycles $E_1(\gamma(t))$ fit together to make a torus and the integral around this torus of η is equal to $\int_\gamma da$. If this integral is trivial, then a can be analytically continued around this loop to a single valued function. Similar arguments apply to a_D. Thus, if all the periods of η are trivial, then a, a_D transform by $SL_2(\mathbf{Z})$.

In general, the deRham cohomology class of a holomorphic two-form such as η is trivial if and only if there is a global differential of the second kind λ on $\mathcal{E}$ with $d\lambda = \eta$. In the more general case, we can write the holomorphic two-form η as $d\lambda$ for some global differential λ of the third kind on $\mathcal{E}$ (i.e., λ is a meromorphic one-form on $\mathcal{E}$ with constant residue along its polar locus.) In turns out in our application to pure $SU(2)$-gauge theory, that the automorphisms between our various descriptions lie in $SL_2(\mathbf{Z})$, so that we shall not need to consider this more general case. The proof of this fact is the subject of the next section.

II.5 The BPS formalism

Our goal here is to show that if the low energy effective $U(1)$-gauge theory that we have been considering is the low energy limit of a pure $N = 2$ super-symmetric $SU(2)$ gauge-theory, then the transformations between the various local descriptions in terms of (a, a_D, τ) of the $U(1)$-theory lie in $SL_2(\mathbf{Z})$. We shall establish this by using the BPS formalism. In any $U(1)$-theory we have two conserved charges, the electric and magnetic charges:

$$Q_e = \frac{1}{4\pi}\int_{S^2} *F$$
$$Q_m = \frac{1}{4\pi}\int_{S^2} F.$$

Here, we are doing the integrals on the sphere at infinity in some time slice in Minkowski four-space. In pure $U(1)$-theory on four-space, these charges are both zero. But in the case that our $U(1)$-theory is the low energy limit of an $SU(2)$-theory these charges can be non-zero. Recall from Lecture II-9 that our $SU(2)$-theory has a $U(1)$-symmetry whose conserved charge is

$$N_e = Q_e + \frac{\theta Q_m}{2\pi}.$$

As a result, this operator has integral eigenvalues. It is called the electric charge. There is another conserved charge, the magnetic charge, given by

$$N_m = Q_m.$$

This is the first Chern class of the line bundle at infinity and hence is clearly has integral eigenvalues as well. Of course, as we saw in Lecture II-9 and II-9 these two charges are interchanged by duality. Thus, both charges come from $U(1)$-symmetries of the theory.

At least in the case of constant τ they are related to the Q_e, Q_m by

$$\begin{aligned} N_m &= Q_m \\ N_e &= Q_e + \frac{\theta Q_m}{2\pi} \end{aligned}$$

The $N = 2$ super-symmetry generators $Q^i_\alpha,\ \ i = 1, 2$ satisfy the relations of the super Poincaré group

$$\{Q^i_\alpha, Q^j_{\dot\alpha}\} = \delta^i{}_j P_{\alpha\dot\alpha}$$

where $P_{\alpha\dot\alpha}$ is infinitesimal translation in the spatial direction $\{\alpha\dot\alpha\}$. Also, we have

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon^{ij}\epsilon_{\alpha\beta} U$$

where U is multiplication by a scalar, the central charge of the theory.

In general, there is the BPS bound on the mass of any state which says

$$M \geq |U|.$$

In addition, there is a good understanding of the BPS saturated states, i.e., those for which this inequality is an equality.

In our case we wish to compute the central charge U. To do this recall that the Lagrangian (for τ constant) is:

$$\mathcal{L} = \frac{1}{4\pi}\int d^4x d^2\theta i\tau W_\alpha W^\alpha + \int d^4x d^4\theta \mathrm{Im}\tau\overline{\Phi}\Phi.$$

From this one knows (from Lecture II-9) that for this value of τ

$$U = aN_e + a_D N_m$$

where $\Phi = a + \ldots$. (On dimensional grounds, this formula holds even if τ is not constant.) The interpretation is that different realizations of the low energy theory correspond to different values of a. In these realizations U depends on the values of a, a_D by the above formula. Recall from Lecture II-9 that (N_m, N_e) transform under $SL_2(\mathbf{Z})$ as we take different representations of our theory. Since U has to be unchanged, we see that if

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \mapsto M \cdot \begin{pmatrix} a_D \\ a \end{pmatrix},$$

then

$$\begin{pmatrix} N_m & N_e \end{pmatrix} \mapsto \begin{pmatrix} N_m & N_e \end{pmatrix} \cdot M^{-1}$$

for any $M \in SL_2(\mathbf{Z})$.

The first important point to notice is that we can not compensate for the operation of adding a constant to $\begin{pmatrix} a_D \\ a \end{pmatrix}$ by a change in (N_m, N_e). This shows that the automorphisms between the various representations of the $U(1)$-theory in terms of (a, a_D, τ) lie in $SL_2(\mathbf{Z})$. Thus, in this case we can construct a differential λ of the second kind on $\mathcal{E}$ such that for each α we have

$$\int_{E_1} \lambda = a^\alpha$$

$$\int_{E_2} \lambda = a_D^\alpha.$$

on $\mathcal{E}|_{V^\alpha}$.

More generally, if the $SU(2)$-theory has extra symmetries (as might be the case if we add matter), then the form of the central charge becomes

$$U = aN_e + a_D N_m + S$$

where S is some other conserved charge. In this case it is possible to compensate for the affine translations in (a_d, a). Namely, we can have

$$\begin{pmatrix} a_D \\ a \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} M & \begin{pmatrix} c_D \\ c \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & 1 \end{pmatrix} \cdot \begin{pmatrix} a_D \\ a \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} N_m & N_e & S \end{pmatrix} \mapsto \begin{pmatrix} N_m & N_e & S \end{pmatrix} \cdot \begin{pmatrix} M^{-1} & \begin{pmatrix} * \\ * \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & 1 \end{pmatrix}.$$

In all these transformations no derivatives of τ enter. In fact, what we did was to compute the transformation laws for τ constant and just extended them in the obvious way to the case when

τ is holomorphic. The reason that we know that this mechanism works is that on dimensional grounds alone we know that no derivatives of τ can enter the formulas. While one can consider the low energy limits of these more complicated $SU(2)$-theories with matter, we shall restrict ourselves to the case when there are no extra symmetries and the transformations between the various representations of our low energy theory lie in $SL_2(\mathbf{Z})$.

We gave a derivation of the formula for the central charge above from the form of the Lagrangian. Let us give another derivation of the same formula (or at least a piece of it). Let us consider $U(1)$-gauge theory with a charged $N = 2$ vector hyper-multiplet. In the $N = 1$ language, this hyper-multiplet is a pair of chiral super-fields $(T, \tilde{T})$ of equal but opposite charges. Super-symmetry implies that the charges must be opposite, let us denote them by $\pm n_e$. In the Lagrangian we have added a super-potential

$$W = n_e \Phi T \tilde{T}.$$

Since $\Phi = a + \ldots$, we see that the mass of the pair $(T, \tilde{T})$ is aN_e. Thus, we see this term in the central charge of the theory from this point of view.

II.6 Jumping of the BPS spectrum

Let us return to our low energy $N = 2$ super-symmetric $U(1)$-gauge theory given locally by a, a_D. In such theories the BPS mass inequality is

$$M \geq |n_e a + n_m a_D|.$$

Since N_e and N_m are integers, this is a good bound in the sense that it implies that there is a positive constant which is a lower bound for M as long as the lattice in $\mathbf{C}$ generated by a and a_D is a honest lattice. Let us examine what happens as the lattice generated by a and a_D degenerates, i.e., when a and a_D become collinear in $\mathbf{C}$. At such points the triangle inequality is no long strict and this permits states to decay to other lower states. In this way BPS states can disappear (or appear), accounting for the jumping in the BPS spectrum.

Generically, we expect this to happen along a real curve in U. (Notice the lattice generated by $(1, \tau)$ is not degenerating, rather it is the lattice of values of a and a_D which is degenerating.) This phenomenon is similar to that in Vafa's lecture for $N = 2$ super-symmetric theories in dimension two. There the jumping occurred when the values of the central charge became collinear. In the example of interest, we shall obtain explicit information about the curve in U where the BPS spectrum jumps.

II.7 More about the application to $SU(2)$-gauge theory

As we have remarked in passing several times, we will study an $N=2$ super-symmetric $SU(2)$-gauge theory with no matter hyper-multiplets. We shall do this by studying the theory as an $N=1$ super-symmetric theory with Φ an adjoint-valued chiral super-field. The microscopic (or high energy) effective Lagrangian is given by:

$$\mathcal{L}_{\mathrm{mac}}=\frac{1}{4\pi}\int d^4xd^2\theta\tau_0(\Lambda_0)\mathrm{Tr}W_\alpha W^\alpha+\mathrm{c.c.}+\frac{1}{4\pi}\int d^4xd^4\theta\mathrm{Im}\tau_0(\Lambda_0)\mathrm{Tr}\overline{\Phi}\Phi$$

where W is a piece of the super-curvature of the super-connection $\mathcal{A}$ on a principal $SU(2)$-bundle over Minkowski space, and $\Phi=\phi+\theta\psi+\cdots$ with ϕ being a field with values in the complexification of the adjoint bundle. The Λ_0 refers to the ultraviolet momentum cut-off that we have chosen to regulate the $SU(2)$-theory. Here, τ_0 is a constant (in the fields) which depends on the value of Λ.

The space of classical vacua is given by configurations consisting of trivial connections and constant (covariantly constant) sections ϕ of $\mathbf{C}\otimes su(2)$ which are zeros for the potential, up to gauge equivalence. The potential for this Lagrangian is the norm-squared of the moment map

$$V=\frac{\mathrm{Im}\tau_0}{4\pi}\int d^4x\mathrm{Tr}\left([\phi,\overline{\phi}]^2\right).$$

If ϕ is a zero of the potential function, then clearly ϕ and $\overline{\phi}$ commute. Writing $\phi=\gamma+i\delta$ with $\gamma,\delta\in su(2)$, we see that the condition that ϕ and $\overline{\phi}$ commute is simply that γ and δ commute. This means that γ and δ can be simultaneously diagonalized. That is to say, up to gauge transformation,

$$\phi=\frac{1}{\sqrt{2}}\begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix}$$

for $a\in\mathbf{C}$. There is a further conjugation, namely the action of the Weyl group, which in this case is $\mathbf{Z}/2\mathbf{Z}$ and acts by sending $a\mapsto -a$. Thus, the moduli space of classical vacua is a complex plane parameterized by the complex coordinate

$$u=\mathrm{Tr}\phi^2=a^2.$$

In the classical theory, for $u \neq 0$ the $SU(2)$-gauge symmetry is broken to $U(1)$ and the low energy theory is as we have discussed today. It has a single massless multiplet, which becomes Higgs'ed and acquires a mass. The low energy effective theory at the exceptional point $u = 0$ remains an unbroken $SU(2)$-gauge theory.

Let us examine it in light of our discussion today about low energy effective $U(1)$-theories. As we have just explained the space of classical vacua is the complex plane – the u-plane. Let us examine the metric on the u-plane. All points of this plane except zero correspond to a $U(1)$-theory as described in this lecture. The metric on the space of classical vacua is induced from the natural metric on the space of trivial connections and covariantly constant scalars in the complexified Lie algebra. This is simply the ad-invariant metric on the complexified Lie algebra. Restricting to the diagonal matrices as above, we see that the induced metric is $da \wedge d\overline{a}$. Since $u = a^2$, this means that the metric on the u-plane is

$$ds^2 = \frac{du d\overline{u}}{|u|^2},$$

which has a singularity at the origin. This singularity reflects the fact that at the origin there are other masses fields so that the the low energy effective theory is not a pure $N = 1$ supersymmetric $U(1)$-gauge theory at that point.

Now we can describe the family of tori over the u-plane punctured at 0, and the holomorphic two-form η on the total space of this family. The function τ is constant, and hence the family of tori has monodromy contained in $\{\pm 1\} \subset SL_2(\mathbf{Z})$. Since the metric on the u-plane minus the origin is given by $\mathrm{Im}\tau da \wedge d\overline{a}$ we see that the parameter $a = \sqrt{u}$ is the parameter of the same name in the $U(1)$-theory (i.e., the first of the dual pair of local holomorphic coordinates on the base) and the dual parameter a_D is τa. This means that the monodromy around infinity (or equivalently around zero) sends a to $-a$ and a_D to $-a_D$. This means that the family of tori over the puncture u-plane is obtained from the product family $(\mathbf{C} - \{0\}) \times E$ over the punctured a-plane by dividing out by the involution $(a, z) \mapsto (-a, -z)$. (Here, E is the elliptic curve $\mathbf{C}/L(1, \tau)$, the quotient of the complex plane by the lattice generated by 1 and τ.) The holomorphic two-form η is the image of $da \wedge dz$ where dz is a global holomorphic one-form on E.

This description is of the classical theory. In the next lecture we shall discuss the quantum version of this theory. It turns out that things change somewhat in going to the quantum theory, but the changes are in some ways not as drastic as one might imagine. For example the space of quantum vacua will be the same u-plane, though the singular points will be different, and they will correspond to a different type of object becoming massless. Also, the monodromy will be different.