Lecture II-18: $N = 2$ Super-symmetric Yang-Mills theories in dimension four: Part II

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I Review of material from the last lecture

We are considering pure $N = 2$ super-symmetric gauge theory (i.e., without hyper-multiplets) in four-dimensional Minkowski space. The gauge group is either $SU(2)$ or the closely related group $SO(3)$. (In fact, working as we are today over Minkowski four-space, which is contractible, we will not see the distinction between these two gauge groups.) We write the theory in the $N = 1$ super-symmetric notation. The fields which form a single $N = 2$ vector multiplet decompose under the $N = 1$ super Poincaré group in four dimensions as a vector multiplet $\mathcal{A}$ (the gauge fields) and a chiral super-field

$$\Phi = \phi + \theta\psi + \cdots$$

where ϕ is a scalar field (under the Poincaré group). It is a section of the complexification of the adjoint bundle. The microscopic Lagrangian is

$$\mathcal{L} = \int d^4x d^2\theta \tau_0(\Lambda_0)\mathrm{Tr}W_\alpha W^\alpha + \mathrm{c.c.} + \int d^4x d^2\theta \mathrm{Im}\tau_0(\Lambda_0)\mathrm{Tr}(\overline{\Phi}\Phi)$$

where W is a certain piece of the super-connection of the super-connection $\mathcal{A}$. (Here Λ_0 is a cut-off used to render finite the Feynmann path integrals. We vary τ_0 as a function of Λ_0 in order to renormalize, i.e., render the expectation values in the theory finite as $\Lambda_0 \mapsto \infty$. In order to decide how to renormalize we must fix a so-called 'subtraction point' Λ at which to fix the value of one parameter in the theory. We shall see later a convenient normalization scheme for this particular theory which involves the details of the theory (the u-plane, etc). But for now

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we leave vague the actual renormalization scheme. Once we have fixed the value of the extra parameter Λ, the coupling constant τ_0 is a function of Λ_0.) While τ depends on the cut-off, it is independent of the fields.

Our object today is to solve this theory by explicitly understanding the moduli space $\mathcal{M}$ of quantum vacua, as well as a description of the theory near each vacuum state. As we established last time, the space of classical vacua of this theory consists of gauge equivalence classes of constant matrices in $su(2)$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

modulo the action of the Weyl group, which is a group of order two switching a and $-a$. This space is analytically isomorphic to $\mathbf{C}$ and the natural local holomorphic coordinate to use on this plane is $u = \mathrm{Tr}\phi^2 = a^2$. While the analytic space of classical vacua is smooth, there is one point in the moduli space of classical vacua where the Kähler metric becomes singular. This is the origin. The singularity is a consequence of the fact that at this point there are more massless fields (in fact a super-symmetric $SU(2)$-theory is the lower energy effective theory). Furthermore, as we discussed last time, away from $u = 0$ the low energy effective theory is a pure $N = 1$ super-symmetric $U(1)$-gauge theory. Thus, away from $u = 0$, the nature of this effective theory is described by a family of tori over the punctured u-plane. In the case of the moduli space of classical vacua this is a family of tori with constant j invariant given by the choice of τ in the Lagrangian. The monodromy around infinity for this family is -1. The Kähler metric on the u-plane is

$$ds^2 = \mathrm{Im}\tau \frac{du d\overline{u}}{4|u|^2}.$$

(Notice that there is a singularity in the metric at $u = 0$ as predicted by the fact that at this vacuum there are extra massless fields classically.)

I.1 First results about the moduli space $\mathcal{M}$ of quantum vacua

Let $\mathcal{M}$ be the space of quantum vacua. Notice that, because we are assuming that our theory is $N = 2$ super-symmetric, the space of quantum vacua has a Kähler metric (from the weaker $N = 1$ super-symmetry) and over this space of vacua there is a holomorphic family of tori. The field u, which is a coordinate on the space of classical vacua, is a chiral field in the theory, and thus its expectation values in the various quantum vacua defines a holomorphic function on $\mathcal{M}$.

We denote this function by $u\colon \mathcal{M} \to \mathbf{C}$:

$$u(\Omega) = \langle \Omega | u | \Omega \rangle, \quad \text{for all } \Omega \in \mathcal{M}.$$

The part of $\mathcal{M}$ that is most closely related to the u-plane of classical vacua is the region where u is large. More precisely, suppose that we have fixed the mass parameter, Λ, in the renormalization scheme which determines the specific quantum theory we are studying. Suppose that we are studying the theory near a classical vacuum Ω with $u(\Omega) >> \Lambda^2$. (Notice that since u has dimensions of mass squared it naturally compares to Λ^2.) Because our $SU(2)$-theory is asymptotically free, and the $U(1)$-theory to which it is limiting is free in the infra-red, if $u >> \Lambda^2$, then the quantum theory is well-approximated near Ω by the classical theory. This means that the low energy quantum theory near Ω is described by a sigma model on the u-plane, perhaps perturbed by some super-potential. But $N = 2$ super-symmetry means that any such super-potential must be a constant, and hence cannot lift the vacuum degeneracy near Ω. (There is also the possibility of a Fayet-Iliopoulos θ-term to consider as well, but the $SU(2)_R$-symmetry rules these out.) What this argument shows is that for large u there is a unique quantum vacuum for each classical vacuum. That is to say, there is a constant $C(\Lambda) > 0$ and an open subset $\mathcal{U} \subset \mathcal{M}$ with the property that $u\colon \mathcal{U} \to (\{z \in \mathbf{C} | |z| > C\})$ is a holomorphic isomorphism.

In Seiberg's lectures we have seen other examples of gauge theories where near infinity in the space of classical vacua the gauge group $SU(N_c)$ is broken down to a non-abelian gauge group $SU(N_c - N_f)$. Since the gauge theory for this smaller group is not free in the infra-red, we cannot conclude that the space of quantum vacua near infinity can be identified with the subspace of critical points of a superpotential on the space of classical vacua. In fact, in these examples the moduli space of quantum vacua is a finite covering space of the space of classical vacua near infinity. To repeat, this is possible because the low energy theory is not free in the infra-red. When it is free, as in the case under discussion now, and when $N = 2$ super-symmetry rules out the existence of super-potentials, the classical and quantum moduli spaces can be identified over a neighborhood of infinity in the space of classical vacua.

One assumption that we shall make without any justification is that the moduli space $\mathcal{M}$ is complete in an appropriate sense so that it corresponds to an open complex algebraic variety.

We have found one component of $\mathcal{M}$ which has an end isomorphic under u to a neighborhood of infinity in $\mathbf{C}$. Two other questions present themselves:

- Are there other 'ends' of $\mathcal{M}$, ends that are not seen classically?
- Are there other irreducible components of $\mathcal{M}$ not seen classically?

The answer to the first question is 'No.' The point is that all ends of the quantum moduli space must be visible classically, i.e, they must be related to ends of the classical moduli space. The reason is that quantum effects are bounded and hence cannot effect a non-proper change in the passage from the space of classical vacua to the space of quantum vacua.

As to the second question, there are no examples known where there are branches of the quantum space of vacua which are not seen classically. Still in this generality, it has not been established that such components do not exist. Nevertheless, in our case we can give arguments showing that there are no other components. The first thing to note is that, by the answer to the first question, any other component of $\mathcal{M}$ must be compact. To exclude the possibility of other compact branches we use the fact that the theory has a relevant $N = 1$ super-symmetric perturbation. That is the perturbation

$$\Delta\mathcal{L} = \int d^4x d^2\theta \epsilon u + \text{c.c.}$$

where ϵ is a complex constant. As we saw in Lecture II-??, with this perturbation the resulting $SU(2)$-theory has two vacua, each with a mass gap and each with confinement. If we examine what happens to these perturbed theories in the low energy limit, then of course they look the same: they have two vacua each with a mass gap and confinement. Thus, this perturbation would have to lift the vacuum degeneracy along a compact component $\mathcal{N}$ of $\mathcal{M}$ by adding a super-potential. On the other hand, since u is a holomorphic function along this branch, it is constant. Now the super-potential term that we have added to the microscopic Lagrangian is simply ϵu which is constant along this branch. One can see that this persists in the low energy theory, so that the low energy effective super potential is also a multiple of u, and hence is constant. This means that the addition of the above super-potential cannot lift the vacuum degeneracy along this branch. Since we know the resulting low energy theory has only two vacua, this implies that the other branch is in fact not present. (This argument still allows for the possibility of isolated points (at most two), but we shall see later by counting singularities, that they do not occur either.)

We have now argued that $\mathcal{M}$ has a single branch (plus possibly at most two isolated points) and that this branch has a single neighborhood of infinity which is mapped by u isomorphically to a neighborhood of infinity in the u-plane. By our completeness assumption, we know that $\mathcal{M}$ is a possibly singular, irreducible complex curve with one puncture. The existence of the holomorphic function u with a only one pole, that being the simple pole at infinity of $\mathcal{M}$, implies that $\mathcal{M}$ is actually the complex plane and that u is a global holomorphic parameter on $\mathcal{M}$. Following the analysis of the last lecture, our goal is to solve the theory by finding the family of elliptic curves (in Weierstrass form) over $\mathcal{M}$ minus the singular points

$$y^2 = x^3 + A(u)x + B(u)$$

and a holomorphic two-form η on the total space of this family

$$\eta = f(u)du\frac{dx}{y}$$

whose periods determine the differentials da and da_D of the dual pair of local holomorphic coordinates.

I.2 The nature of infinity in $\mathcal{M}$

As we have already remarked, we obtain information near infinity in $\mathcal{M}$, i.e., where u is large using asymptotic freedom of the $SU(2)$-theory. Our goal here is to compute the monodromy at infinity of the family of elliptic curves over the moduli space of quantum vacua. We begin by studying the monodromy at infinity for the space of classical vacua. Recall that the space of classical vacua is the space of matrices

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

up to $a \mapsto -a$. The dual coordinate a_D is given by $a_D = \tau_{\text{cl.}}a$ where $\tau_{\text{cl.}}$ is a constant (denoted by τ_0 earlier) with large imaginary part. The global coordinate on the u-plane is $u = \text{Tr}(\phi^2) = a^2$, so that $a = \sqrt{u}$. From this it is clear that the monodromy at infinity in this classical family is -1.

The monodromy around infinity for the space $\mathcal{M}$ of quantum vacua is different. We give two arguments calculating it. The first is based on asymptotic freedom and the second is based on the $U(1)_R$-anomaly. To compute $\tau_{\text{eff}}(u)$ for large u we consider the 1-loop formula. With an appropriate choice of the mass parameter Λ, the formula is:

$$\mathrm{Im}\tau = \frac{4\pi}{g_{\mathrm{eff}}^2} = 4\pi\left(2b_0\ln(\sqrt{|u|/\Lambda})\right), \tag{1}$$

where b_0 is the 1-loop β-function coefficient.

One loop computations identify

$$b_0 = \frac{1}{4\pi^2}.$$

Since τ is holomorphic, knowing its imaginary part determines it uniquely. We see that for u large we have

$$\tau_{\mathrm{eff}} = \frac{2i\ln(u/\Lambda^2)}{2\pi}. \tag{2}$$

Thus, we see that monodromy around infinity shifts τ by -2 and hence shifts θ by -4π. This implies that the monodromy around infinity in $\mathcal{M}$ is given by

$$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

Said another way we are considering the system

$$\begin{aligned} u^2 &= a \\ \frac{da_D}{da} &= \tau = \frac{2i\ln(u/\Lambda^2)}{2\pi} \end{aligned} \tag{3}$$

as we turn once around infinity. Clearly, a comes back to $-a$ and τ comes back to $\tau - 2$. The above form for the monodromy then follows easily.

Now let us compute this monodromy using the anomalous $U(1)_R$ symmetry. Classically, the $U(1)_R$-charges of the component fields in the gauge super-multiplet are given as follows: the charge of A is zero, the charges of λ, ψ are 1 and the charge of a is two. This means that $u = a^2$ has charge 4. (This is related to the fact that in Donaldson theory $\mu(pt)$ is four-dimensional.) This means that under the $U(1)_R$ symmetry $e^{i\beta}$ the variable u transforms by $u \mapsto ue^{\pm 4i\beta}$. In fact, the sign is negative. On the other hand, as we know, there is a quantum anomaly breaking this $U(1)_R$ down to $\mathbf{Z}/8\mathbf{Z}$ (which is the dimension of the one-instanton moduli space on $\mathbf{R}^4$). This is an index computation which uses the fact that the Casimir of the adjoint representation of $su(2)$ is 4 times the Casimir of the defining representation. This means that under the $U(1)_R$ symmetry $e^{i\beta}$ the angle θ transforms to $\theta - 8\beta$. Thus, as we transverse a loop around infinity in the u-plane the argument of u changes by 2π, β increases by $2\pi/4 = \pi/2$ and θ decreases by 4π. This means that $\tau \mapsto \tau - 2$ under the monodromy around infinity.

Of course, these are not independent computations. Super-symmetry imposes a relation between the $U(1)_R$ anomalies and the β-function.

Actually, we can say more about the nature of τ at infinity. For $|u| >> \Lambda^2$ in perturbation theory we have a series

$$\tau(u) = \frac{2i\ln(u/\Lambda^2)}{2\pi} + \tau^{\geq 2} \tag{4}$$

where the first term is the sum of the classical value and the one-loop correction (as calculated above) and the second term is the sum of the corrections for higher loops. These higher loop corrections produce a power series in g^2 (k-loop corrections produce the power g^{2k-2}). Since there are no loop corrections to θ and since τ is a holomorphic function of u, it follows immediately that all the higher loop corrections vanish. Thus, in perturbation theory the one-loop correction is exact. There are however non-zero corrections to the perturbation expansion coming from instantons. Of course, these can not be seen in perturbation theory. Let's study their nature. We know that τ is invariant under the $U(1)_R$-symmetry. An instanton breaks the $U(1)_R$-symmetry by $8k$ where $8\pi^2 k$ is its total energy. Since u has $U(1)_R$-charge 4, the k instantons go with even u^{2k}. Of course, the dimensionless expression is Λ^{4k}/u^{2k}. Thus, the instanton action for an instanton of total energy $8\pi^2 k$ is of the form

$$f_k \exp(-8k\pi^2/g^2)e^{ki\theta} = f_k\Lambda^{4k}e^{2k\pi i\tau} = f_k\frac{\Lambda^{4k}}{u^{2k}}$$

for a dimensionless constant f_k. Thus, the full expression for τ at $|u| >> 1$ is of the form

$$\tau(u) = \frac{2i\ln(u/\Lambda^2)}{2\pi} + \sum_{k\geq 1} f_k\frac{\Lambda^{4k}}{u^{2k}}. \tag{5}$$

I.3 BPS states and singularities in $\mathcal{M}$

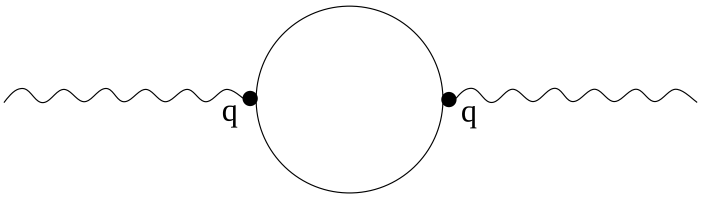
Next, we wish to understand the singularities in $\mathcal{M}$. By a *singularity* we mean a point of the moduli space where the physics changes. This may or may not be accounted for by a topological singularity in the space. As a general principle, we expect that the singularities have to do with particles becoming massless. In this case since the particles are to add singularities to the low energy limit which is a $U(1)$-theory, they must be charged under the $U(1)$. It is also quite reasonable to expect that if any particles become massless, then some BPS states (which are the states of lowest energy) become massless. In our case we have two types of BPS states near infinity in the u-plane. We have the vector multiplet (i.e., the gauge boson) which has $n_e = \pm 1$

and $n_m = 0$ and we have the magnetic monopoles with $n_m = 1$ and any value of n_e. (Of course, as we increase θ by 2π a particle with $n_m \neq 0$ has n_e which changes by ± 2, so that having one BPS state with $n_m = \pm 1$ gives rise to BPS states with all even or all odd values of n_e. Our claim here is that there are two families of these BPS states, one with even n_e and one with odd n_e.) Notice that this BPS spectrum is invariant under the monodromy at infinity since under this monodromy $n_e \mapsto -n_e + 2n_m$ and $n_m \mapsto -n_m$. On the other hand we claim that this spectrum can not be invariant under the entire monodromy of our family. The reason is that as we shall see the image of the monodromy on all of $\mathcal{M}$ is a non-abelian subgroup of SL_2, under which the above-described BPS spectrum is not invariant. This means that the BPS spectrum cannot be continuous throughout $\mathcal{M}$, that is to say there must be a jumping locus, which is a real curve in $\mathcal{M}$ along which extra BPS states become massless. As we saw in the last lecture, this happens when a/a_D becomes real.

As we have seen $\mathcal{M}$ is a complex analytic space with exactly one irreducible component. Furthermore, the holomorphic map $u\colon \mathcal{M} \to \mathbf{C}$ is proper and is an isomorphism in a neighborhood of infinity of $\mathcal{M}$. Next, we wish to analyze the singularities of $\mathcal{M}$. We are assuming that $\mathcal{M}$ is in fact a complex algebraic variety. There must be singularities in $\mathcal{M}$ since the monodromy at infinity is non-trivial. As we have argued singularities in $\mathcal{M}$ are associated with a certain super-symmetric multiplet consisting of charged BPS states becoming massless. We have two types of BPS multiplets – those where the maximal spin in the multiplet is 1 (the vector multiplet) and those where the maximal spin is 1/2, (the hyper-multiplet). A vector multiplet becoming massless corresponds to restoring a non-abelian gauge group ($SU(2)$). There are several reasons one does not expect this to happen. First of all, with unbroken $SU(2)$ one has a theory which is unstable in the infra-red. This is exactly what happens at the singularity in the classical moduli space. Since the β-function is negative, we have flowed away this theory, so we don't expect to see it occurring in the low energy limit which we are assuming to be free in the infra-red. Of course, if our gauge theory has enough hyper-multiplets becoming massless to make $\beta > 0$ then we might see such a low energy limit. But at these points we have multiplets with maximal spins both 1 and 1/2 becoming massless. So the simplest thing that we can do is to require that only a hyper-multiplet with maximal spin 1/2 become massless. Let us analyze this possibility.

Suppose that the singularity in $\mathcal{M}$ is associated to a multiplet whose maximal spin is 1/2 becoming massless. Recall that a massive $N = 2$ super-symmetric multiplet of spin at most 1/2 is automatically BPS saturated. Thus, we might expect that what is happening is that one of the BPS states that exist in theory for $|u| >> \Lambda^2$ is simply becoming massless at the singularity. Of course, more complicated phenomena could occur. For example, as we come in from infinity in $\mathcal{M}$ we could cross a jumping line of the BPS spectrum before we reach the singularity, and then at the singularity it is one of the 'new' BPS saturated states that becomes massless, rather than one of the original states that exist near $u = \infty$.

Let us assume that a hyper-multiplet becomes massless. Denote its charge by (n_e, n_m) under the electric and magnetic charge. We have already argued that this hyper-multiplet must be charged, so that $q = \text{g.c.d}(n_e, n_m)$ is a well-defined positive number. After an $SL_2(\mathbf{Z})$ change in the representation of the theory by a Lagrangian, we can arrange that $n_e = q$ and $n_m = 0$. Thus, near the singular point $u_0 \in \mathcal{M}$ our theory is a $U(1)$-theory with a charged particle which is becoming massless at $u = u_0$. (But notice that it may well be the case that $n_m \neq 0$. If that happens then the $U(1)$-gauge fields in the representation of the theory near $u = u_0$ are different from the $U(1)$-gauge fields that we naturally have at infinity.) Unlike the spin-one case, this theory is stable in the infra-red, i.e., $\beta > 0$. Since the behavior of the theory near $u = u_0$ is infra-red free, we can compute g_{eff} and θ_{eff} in this region using perturbation theory. We see that g_{eff} vanishes logarithmically and is given by the one-loop β-function. The relevant one-loop Feynmann diagram is:



Clearly, this diagram gives a multiplicative factor of q^2. It also diverges logarithmically for $u \mapsto u_0$. The result is that for u near u_0 we have:

$$\tau(u) = 2q^2 \frac{\ln(u - u_0) + \text{constant}}{2\pi i}.$$

This means that the monodromy of $\tau(u)$ in $PSL_2(\mathbf{Z})$ around this singularity is

$$\pm \begin{pmatrix} 1 & 2q^2 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

Of course, the monodromy is actually an element in $SL_2(\mathbf{Z})$ and is only determined up to sign by the monodromy in $PSL_2(\mathbf{Z})$ To pin down the sign we argue as follows. The physics near $u = u_0$ is described by a $U(1)$-gauge theory with gauge fields $(\tilde{A}, \tilde{a})$ and a charged hyper-multiplet (T, T') which is becoming massless at u_0. Thus, the Lagrangian is of the form of the gauge couplings plus a super-potential $\tilde{a}qTT'$. The scalar field $\tilde{a}$ is a local coordinate on the u-plane and in fact is related to a, a_D at infinity by an $ISL_2(\mathbf{Z})$ transformation. (The image of this transformation in $SL_2(\mathbf{Z})$ is determined by the charge (n_e, n_m) of the hyper-multiplet (T, T') with respect to the $U(1)$ at infinity in the u-plane.) Since the $U(1)$-theory exists throughout this neighborhood even at the singularity, we see that under monodromy around u_0 the field $\tilde{a}$ comes back to itself. This implies that there is an invariant vector under the $SL_2(\mathbf{Z})$-monodromy around u_0 and hence shows that the monodromy is as written in Equation 6 with a plus sign.

We can generalize this picture. Suppose that at u_0 there are several hyper-multiplets of charge q_i becoming massless. Then the local form of τ would be

$$\tau = 2\sum_i q_i^2 \frac{\ln(u - u_0) + \text{constant}}{2\pi i}$$

and the monodromy around u_0 would be

$$\begin{pmatrix} 1 & \sum_i 2q_i^2 \\ 0 & 1 \end{pmatrix}.$$

Notice that the monodromy around a singularity where a hyper-multiplet is becoming massless has trace 2 whereas the monodromy at infinity has trace -2. This means in particular, that there cannot be only one singularity in the u-plane if the singularity is of this type. (This of course also follows from the positivity of τ in the complement of the singularities.)

I.4 The number of singularities in $\mathcal{M}$

To understand the number of singularities in $\mathcal{M}$, let us begin with a microscopic $N = 2$ super-symmetric $SU(2)$-theory and break it to an $N = 1$ super-symmetric theory by adding a perturbation of the form

$$\Delta\mathcal{L} = \int d^4x d^2\theta m u + \text{c.c}, \tag{7}$$

with m a complex constant. (Recall that $u = \text{Tr}\phi^2$.) This theory has two vacua, each with a mass gap and with confinement.

Let us suppose that at a point in the space of quantum vacua we have N_h hyper-multiplets becoming massless, say (T_i, T_i') of charge q_i. So near this point the physics is described by a super potential

$$W = \tilde{a}\left(\sum_i q_i T_i T_i'\right) + mu(\tilde{a}).$$

where, as before $\tilde{a}$ is a local parameter on the u-plane and is related by an $SL_2(\mathbf{Z})$ transformation to a, a_D at infinity. Let us find the space of vacua in the macroscopic theory. That is to say we solve $dW = 0$ and divide by the $\mathbf{C}^*$-action. Clearly, the equation $dW = 0$ is equivalent to

$$\tilde{a}T_i = \tilde{a}T_i' = 0 \quad \text{for all} \;\; i.$$

$$m\frac{du}{d\tilde{a}} + \sum_I q_i T_i T_i' = 0.$$

If $N_h > 0$ this gives us a positive dimensional space of vacua. This is impossible since the microscopic theory has only two vacua. This means that (as indicated above) at each singularity we have only one hyper-multiplet becoming massless. Furthermore, since $\frac{du}{d\tilde{a}} \neq 0$ we see from the second equation that $T, T' \neq 0$. By the first equation, this means that $\tilde{a} = 0$. Since there is only one hyper-multiplet, the equation becomes

$$m\frac{du}{d\tilde{a}}|_{\tilde{a}=0} + qTT' = 0.$$

This determines the vacuum uniquely up to the action of $\mathbf{C}^*$. Thus, we see that there is a unique vacuum associated to this singularity. Having one vacuum for each singularity in the macroscopic theory, and having only two vacua in the microscopic theory, it follows that there are at most two singularities in $\mathcal{M}$. Notice also that the critical points for u on $\mathcal{M}$ also give vacua in the macroscopic theory. Thus, we see

$$N_c + N_h \leq 2$$

where N_c is the number of critical points in $\mathcal{M}$ of the analytic function u and N_h is the number of points at which a hyper-multiplet becomes massless.

As we have already remarked, the monodromy around a point where a hyper-multiplet is massless and the monodromy at infinity have opposite traces and hence are not conjugate in $SL_2(\mathbf{Z})$. At a critical point of u there is no monodromy. Thus, these facts about monodromy imply that $N_h \geq 2$. It follows that $N_h = 2$ and $N_c = 0$. That is to say $u: \mathcal{M} \to \mathbf{C}$ is everywhere a local analytic isomorphism. Since it is an isomorphism between neighborhoods of infinity, it follows that $u:\mathcal{M} \to \mathbf{C}$ is an isomorphism. (Recall that we established this fact directly from the fact that u is an isomorphism at infinity. This gives a consistency check on our model.) Even though the spaces of classical and quantum vacua are identified, there are important differences. Whereas classically, there is only one singular point in the u-plane and at this singular point the $SU(2)$-theory is restored as the low energy limit, quantum mechanically there are two singularities in the u-plane (at as yet to be identified points), and at each of these a charged spin 1/2 hyper-multiplet becomes massless. Furthermore, unlike the classical case, the monodromy in the quantum family is non-trivial in $PSL_2(\mathbf{Z})$. This implies that, unlike the classical case, the family of tori over the space of quantum vacua do not have constant j-invariant.

Notice that now that we have found two singularities in the u-plane, each contributing a vacuum when we add the perturbation in Equation 7. Since there are at most two vacua in the perturbed theory, it follows that there cannot be isolated points of the space of quantum vacua; these isolated points would also produce quantum vacua of the perturbed theory.

The $U(1)_R$-symmetry is broken down to a $\mathbf{Z}/8\mathbf{Z}$-symmetry which since u has charge four under $U(1)_R$ acts on the u-plane as a $\mathbf{Z}/2\mathbf{Z}$-symmetry. It is the symmetry $u \mapsto -u$. This implies that the two singularities in $\mathcal{M}$ are at points $\pm u_0$ for some $u_0 \neq 0$. As we have already seen dimensional analysis tells us that u goes like Λ^2, the square of the mass parameter in the theory. This means that we can make a choice of the mass parameter Λ so that the quantum theory labeled by Λ has singularities at $u = \pm\Lambda^2$. In this way we are adopting the precise definition of the mass scale parameter Λ promised in the introduction. Notice that this choice of Λ is different from the one that we have used until now. The only affect of this change in the choice of Λ simply is to introduce an additive constant into Formulas 1 through 5.

I.5 The new massless particles

The next step is to identify what is becoming massless. Let (n_e, n_m) be the charge of the hyper-multiplet becoming massless at Λ^2 when we express things in the basis in which the monodromy at infinity is upper triangular. If $n_m = 0$, then the monodromy at Λ^2 commutes with the monodromy at infinity. Since the total space of the non-singular part of the family is the three-times punctured sphere, the monodromies at Λ^2 and ∞ generate the entire monodromy image. Thus, if the monodromy at ∞ and Λ^2 are simultaneously upper triangular, then the entire monodromy image is upper triangular and hence commutative. This would mean that $\mathrm{Im}\tau$ could not be positive everywhere. Our conclusion is that $n_m \neq 0$. That is to say, the particle becoming massless at $u = \Lambda^2$ is magnetically charged. Exactly the same analysis holds for the singularity at $-\Lambda^2$. This is a very interesting phenomenon. Particles which started life when $|u| >> \Lambda^2$ as massive solitons are becoming massless as u approaches $\pm\Lambda^2$. Also, it means that our description of the theory near $\pm\Lambda^2$ as a $U(1)$-theory is a different representation from the $U(1)$-representation of the theory near infinity in the u-plane.

Now we wish to calculate the magnetic charge $\pm n_m$ of these particles. Notice that as we come in from infinity $\pm n_m$ is well-defined independent of the path we choose from infinity to $\pm\Lambda^2$ but the electric charge depends on the path – if we wind around the pair of singularities $\pm n_e$ changes by twice the magnetic charge, since the monodromy around infinity is $n_e \mapsto -n_e + 2n_m$.

The model we are discussing today has various mathematical applications, but just in terms of physics one of its most striking applications was to give a new model of confinement. We recall that confinement is the statement that there is a linear potential between external electric charges or alternatively that there is area law decay for a Wilson loop in the an appropriate representation (in the present example, the two-dimensional representation of $SU(2)$). We recall that in a Higgs phase, where a charged field gets a vacuum expectation value[1], one can explicitly calculate by a topological argument that the 't Hooft loop gets area law decay. If one could just naively exchange electricity and magnetism, then Wilson loops would be exchanged with 't Hooft loops and we would conclude that confinement will arise if a magnetically charged field gets a vacuum expectation value. This was argued heuristically in the 1970's by 't Hooft, Mandelstam, Nambu and others but with great difficulty in exhibiting actual models, since usually one does not have magnetically charged fields in the formalism and magnetically charged objects (which arise as classical solutions) are heavy in the weakly coupled regime where one understands them concretely. Thus it was generally rather hard to see how one could actually exhibit a phenomenon in which a magnetically charged field gets a vacuum expectation value.

In the present context, this happens. We have already seen that after adding the mu perturbation to the superpotential, which is expected to give confinement, the vacuum near $u = \Lambda^2$

[1] This is a somewhat imprecise description as we have seen in previous lectures, but is valid in weak coupling.

has $T, T' \neq 0$, so in fact in this vacuum (n_m being non-zero) a magnetically charged object has obtained a vacuum expectation value. Thus as expected a magnetically charged field obtained a vacuum expectation value just in the confining case. (To pursue this with greater precision, it is important that the magnetically charged field in question has n_m odd. Indeed, as the center of $SU(2)$ is $\mathbf{Z}/2\mathbf{Z}$, the explanation of confinement via a dual of the usual Higgs mechanism requires a nonzero expectation value of a field of odd n_m. We will see that n_m actually is odd, in fact ± 1, for T and T'.)

Notice something surprising has happened. The magnetic monopoles have mass at infinity on the order of $\sqrt{|u|}\ln(|u|)$ whereas there are other BPS states of masses on the order of $\sqrt{|u|}$. Thus, at infinity the monopoles are not the least massive of the massive particles. Nevertheless, they are the ones becoming massless at the singular points.

I.6 Explicit nature of the family of elliptic curves

We are now ready to describe the macroscopic theory associated to the quantum theory labeled by mass parameter Λ by describing explicitly the family of elliptic curves and the two-form η associated with the space of quantum vacua of this theory. As we have seen, for this theory the moduli space of quantum vacua is $\mathbf{C}$ with parameter $u = \mathrm{Tr}\phi^2$. The $SL_2(\mathbf{Z})$-monodromy at infinity for the family is

$$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

There are in addition two other points $\pm\Lambda^2$ in the u-plane where there are singularities. The $SL_2(\mathbf{Z})$-monodromy, $m_{\pm\Lambda^2}$ at each of these points has trace two. We write

$$m_{\Lambda^2}^{-1} = \begin{pmatrix} 1+b & -b^2/a \\ a & 1-b \end{pmatrix},$$

for integers a, b. (This is the general form of a matrix of trace two and determinant one.) Since

$$m_{-\Lambda^2} = m_{\Lambda^2}^{-1} m_\infty$$

has trace two, we conclude that

$$2 = \mathrm{Tr} \begin{pmatrix} 1+b & -b^2/a \\ a & 1-b \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

It follows by a direct computation that $a = 2$. Since $-b^2/a$ is an integer, we see that b is even. Now let us conjugate the matrix for $m_{\Lambda^2}^{-1}$ by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which commutes with the monodromy at infinity. The result is to change b by 2. Thus, after a number of these conjugations we arrive at $b = 0$. This means that in an appropriate basis the monodromy at infinity is

$$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

and the monodromy at Λ^2 is of the form

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Thus, we have shown that the monodromy representation is uniquely determined up to conjugation in $SL_2(\mathbf{Z})$. Notice also that according to Equation 6 the magnetic charge n_m of the hyper-multiplet that is becoming masses at Λ^2 is ± 1. The same of course holds at $-\Lambda^2$. In fact, because the $SL_2(\mathbf{Z})$ is uniquely determined up to conjugation it is invariant, up to conjugation, under the symmetry $u \mapsto -u$.

It is easy to identify the family of elliptic curves over the twice punctured plane with this monodromy. The base of the family can be identified with the modular curve $\mathbf{H}/\Gamma(2)$ given by dividing out the upper half-plane by the (free) action of the subgroup of $PSL_2(\mathbf{Z})$ consisting of all matrices congruent to the identity modulo two. The family is the universal family of elliptic curves with 'level two structure' over this base. Since the singularities are at $u = \pm\Lambda^2$, the Weierstrass equation for the family of elliptic curves associated to this quantum theory is

$$y^2 = (x - u)(x - \Lambda^2)(x + \Lambda^2).$$

Now let us consider the two-form η as described in the last lecture. We know that in the Weierstrass model given above η has the general form

$$\eta = f(u)du\frac{dx}{y}.$$

Implicitly in the discussions before we assumed that η was a complex symplectic form without zeros or poles at least away from the exceptional values of u. Such zeros or poles would produce non-physical singularities. If we take f to be a non-zero constant, then the Kähler form $\pi_*(\eta\wedge\overline{\eta})$ is smooth except at $\infty, \pm\Lambda^2$ where it has the correct singular behavior. Any other f would produce either poles or zeros at other points or the wrong singular behavior at one of $\infty, \pm\Lambda^2$. It follows that $\eta = Cdu\frac{dx}{y}$ for an appropriate constant which can be computed by comparing to the formula $a^2 = u$ near $u = \infty$.

Clearly,

$$\eta = d\left(C\frac{udx}{y}\right) = d\left(C\frac{2ydx}{x^2-\Lambda^4}\right).$$

The form $\lambda = 2Cydx/(x^2-\Lambda^4)$ is a differential of the second kind on the total space $\mathcal{E}$ of the family of elliptic curves. This means, as we have been asserting all along, that $[\eta] \in H^2(\mathcal{E};\mathbf{C})$ is trivial and hence that the monodromy of our family of representations of the theory is contained in $SL_2(\mathbf{Z})$, inside the bigger group $ISL_2(\mathbf{Z})$. This was expected since the theory has no conserved charges except n_e, n_m.

I.7 Description of the BPS spectrum

The line in the u-plane where a/a_D is real is a simple closed curve that passes through the points $u = \pm\Lambda^2$. The spectrum of BPS states is continuous off this circle. Approaching from $u = \infty$, we see that at $u = \Lambda^2$ we have $a = 0$ and $a_D \neq 0$, whereas at $u = -\Lambda^2$ we have $-a + a_D = 0$. (Of course, the exact values depend on the path chosen from ∞ to $\pm\Lambda^2$, but if we come in along rays from infinity toward the origin then given equations hold.) Since we can approach the two singularities from infinity without crossing the jumping locus for the BPS states, we see that the BPS states that are becoming massless at $u = \pm\Lambda^2$ are indeed among the original BPS states at infinity. That is to say, these particles are analytic continuations of the original magnetic monopoles at infinity.

Now let us examine whether, at other points on the jumping locus besides $\pm\Lambda^2$, there are particles forced to become massless. As we move along the jumping line on the open upper arc from Λ^2 to $-\Lambda^2$, we find that for every $s, 0 < s < 1$ there is a point where $sa - a_D = 0$. So if there is a BPS state at infinity has charges (n_e, n_m) with $(n_e, n_m) = t(s, -1)$ for some $t \in \mathbf{R}$, then at some point along this open arc we find that this state becomes massless. This, and the symmetric argument for the open lower arc of the jumping line, show that all the BPS states at infinity that satisfy

$$(n_e, n_m) = (p, \pm q)$$

with $0 < p < q$ become massless somewhere on the jumping curve minus $\pm\Lambda^2$. But we know by our analysis of the singularities that there are no points on these open arcs where BPS states (or any other charged states) become massless. The conclusion is that there are no BPS states at infinity satisfying the above inequality. Said another way the only BPS states at infinity are those of charges

$$(n_e, n_m) = (\pm 1, 0)$$

$$(n_e, n_m) = (p, \pm 1).$$

Of course, these are exactly the BPS states that do exist at infinity. The first are the vector multiplets and the second are the magnetic monopoles. This gives us a check of the internal consistency of the theory.

This analysis of the BPS spectrum in the quantum theory can be compared in an interesting way to the following classical computation. Let $\mathcal{M}_k$ be the moduli space of k-monopole solutions of this theory, reduced by dividing by translations. It is a hyperkähler manifold of dimension $4k-4$ that has been much studied mathematically. It can be shown that L^2-holomorphic forms (possibly valued in a flat line bundle) on $\mathcal{M}_k$ would give k-monopole bound states, for $k > 1$. From what we have seen, there should be no such bound states, so there should be no L^2-holomorphic forms on $\mathcal{M}_k$. More generally, an argument by Sen using duality of $N = 4$ super Yang-Mills (rather than the $N = 2$ theory that we have considered in this lecture) determines the full L^2-cohomology of the $\mathcal{M}_k$.