LECTURE II-5: THE BOSE-FERMI CORRESPONDENCE AND ITS APPLICATIONS

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5.1. 2-dimensional gauge theories with fermions.

Today we will consider 2-dimensional gauge theories with fermions. We will work with Euclidean Lagrangians.

In 2 dimensions, the two spinor representations S_+, S_- are 1-dimensional and complex. They are defined by the formula $\theta \to e^{\pm i\theta/2}$, $\theta \in [0, 4\pi)$, where θ is the parameter on $Spin(2)$.

Let Σ be a Riemann surface with a specified Spin structure. Then S_+, S_- define holomorphic line bundles on Σ, which are called the Spin bundles, and denoted by the same letter. We have $S_+S_- = \mathbb{C}$, and $S_+^2 = T\Sigma, S_-^2 = T^*\Sigma$. Sections of ΠS_- (respectively, ΠS_+) will be called left (respectively, right) moving fermions, by analogy with the Minkowski picture.

We denote by $D_+ : S_- \to S_+$, $D_- : S_+ \to S_-$ the corresponding Dirac operators in Spin bundles.

We will do gauge theory with gauge group G (a compact Lie group). Let E_L, E_R be orthogonal, unimodular representations of G. Let P be a principal G-bundle on Σ, and let E_R, E_L denote the orthogonal vector bundles associated to the representations E_R, E_L. Let $\overrightarrow{\psi_+}, \overrightarrow{\psi_-}$ be sections of the bundles $E_R \otimes S_+$, $E_L \otimes S_-$.

The Lagrangian of a 2-dimensional gauge theory with fermions is

$$\mathcal{L} = \int d^2x \left(\frac{1}{4e^2} | * F_A|^2 + \frac{1}{4\pi} \overrightarrow{\psi_+}(D_-^A)\overrightarrow{\psi_+} + \frac{1}{4\pi} \overrightarrow{\psi_-}(D_+^A)\overrightarrow{\psi_-} \right), \tag{5.1}$$

where A is a connection in P, and $D_\pm^A$ are the corresponding Dirac operators.

In order for this theory to make sense quantum mechanically, the representations E_L, E_R have to satisfy an additional condition. To derive this condition, recall that the partition function of (5.1) is given by

$$Z = \int DA \int D\overrightarrow{\psi_+} D\overrightarrow{\psi_-} e^{-\mathcal{L}}. \tag{5.2}$$

(in this integral, we sum over all topological types of principal bundles). The fermion integral is easy to compute: it equals $I_A = Pf(D_-^A|_{E_R})Pf(D_+^A|_{E_L})$, where Pf denotes the Pfaffian. The expression I_A is a section of the Pfaffian line bundle $B = PF(D_-^A|_{E_R})PF(D_+^A|_{E_L})$ on the space of gauge classes of connections. In order for the A-integral to make sense, this expression should be a function, i.e. the bundle B has to be trivial. It is easy to show that this boils down to the condition

$$\mathrm{Tr}(\rho_L(t)\rho_L(t')) = \mathrm{Tr}(\rho_R(t)\rho_R(t')), \tag{5.3}$$

where $t, t' \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Gauge group G, and $\rho_{L,R} : \mathfrak{g} \to SO(E_{L,R})$ are the representation maps. If G is simple, and E_L, E_R are irreducible, this condition means that the Casimirs of E_L, E_R are the same. Equation (5.3) is called the condition of cancelations of anomalies.

Today we will consider a simple case: $G = U(1)$, and E_L, E_R are irreducible 2-dimensional real representations. In this case (5.3) says that E_L, E_R are the same: $E_L = E_R = E$. We will take $E = \mathbb{C}$ with metric $|z|^2$, and $U(1)$ acting by multiplication (but remember that tensor products $S_\pm \otimes E$ are over $\mathbb{R}$). We decompose $S_\pm \otimes E$ in a direct sum of two 2-dimensional representations. Using this decomposition, we will write $\overrightarrow{\psi}_\pm = (\psi_\pm, \bar\psi_\pm)$, where for $z \in U(1)$ one has $z(\psi, \bar\psi) = (z\psi, \bar z\bar\psi)$.

In the new notation, Lagrangian (5.1) has the form

$$\mathcal{L} = \int d^2x \left(\frac{1}{4e^2} |*F_A|^2 + \frac{1}{2\pi}\bar\psi_+ D_-^A \psi_+ + \frac{1}{2\pi}\bar\psi_- D_+^A \psi_- \right), \tag{5.4}$$

We can also add to this Lagrangian a topological term:

$$\mathcal{L}_\theta = \mathcal{L} - \frac{i\theta}{2\pi}\int F, \tag{5.5}$$

and a mass term:

$$\mathcal{L}_{\theta,m} = \mathcal{L}_\theta + \frac{m}{2\pi}\bar\psi_-\psi_+ + \frac{\bar m}{2\pi}\bar\psi_+\psi_-, \tag{5.6}$$

(here m is complex).

5.2. Chiral symmetry.

Chiral symmetry is a $U(1)$-symmetry of the classical Lagrangian (5.4) or (5.5) given by

$$\psi_+ \to \psi_+, \psi_- \to e^{i\gamma}\psi_-. \tag{5.7}$$

This symmetry is violated by the mass term. However, even if there is no mass term, this symmetry may be violated quantum-mechanically, for the following topological reason.

Let the spacetime be a closed Riemann surface. Then under the chiral symmetry, the "measure" $\mu = D\psi_- D\bar\psi_-$ transforms as $\mu \to e^{i\gamma I}\mu$, where I is the index of the operator $D_+^A : S_- \otimes_{\mathbb{C}} E \to S_+ \otimes_{\mathbb{C}} E$. "Proof": $I = dim(\{\psi_-\}) - dim(\{\bar\psi_-\})$; since $(\{\bar\psi_-\}) = (\{\psi_-\})^*$, we have $I = dim(\{\psi_-\}) - dim(\{\psi_+\}) = ind(D_+^A|_{S_+\otimes_{\mathbb{C}} E})$.

It is known that the index I equals $c_1(P) = \int \frac{F_A}{2\pi}$. Thus, since the bundle P may be nontrivial topologically, chiral symmetry is violated in the quantum theory.

This effect is called an anomaly: a classical symmetry does not hold quantum mechanically, because the measure is not invariant. The difference with spontaneous symmetry breaking, discussed in Lecture II-1, is that in the case of spontaneous symemtry breaking, the classical symmetry does exist in the quantum theory, but cannot be realized. Unlike spontaneous symmetry breaking, anomalies can occur even in 1 dimension.

Let us discuss the mechanism of chiral symmetry breaking in the language of currents. Classically, chiral symmetry is generated by the current

$$J_A = \bar{\psi}_- \psi_- \tag{5.8}$$

(this is a 1-form of type (0,1)), which is of course conserved: $dJ_A = 0$. Quantum mechanically, this differential equation may be deformed: $dJ_A = \mathcal{O}$, where $\mathcal{O}$ is some operator with values in 2-forms on the surface. It is not hard to show by listing all possible operators that there is only one operator (up to a factor) that can arise: it is the curvature operator F. Namely, it is enough to show that dJ_A is a functional of F, which can be seen by considering Feynman diagrams. Thus, $dJ_A = \alpha F$, where α is a constant, and the previous topological computation shows that $\alpha = 1/2\pi$.

Now consider Lagrangian (5.5), which depends on the theta-angle. The thing we learn from the above index computation is that the value of the correlation functions defined by the path integral with Lagrangian (5.5) depend on θ in a very trivial way. Namely, if $\mathcal{O}_i$ are any operators such that $\prod_i \mathcal{O}_i \to e^{in\gamma} \prod_i \mathcal{O}_i$ under chiral symmetry, then all nontrivial contributions to the correlator $\langle \prod \mathcal{O}_i \rangle$ are from bundles P with $c_1 = n$, so this correlator has the form $e^{in\theta} \langle \prod O_i \rangle_0$, where $\langle \prod O_i \rangle_0$ is θ-independent. For example, $\langle \psi_- \psi_+^* \rangle = Ce^{i\theta}$, where C is theta-independent.

If instead of (5.5) we consider Lagrangian (5.6) with the mass term, then this argument shows that

$$\langle \prod \mathcal{O}_i \rangle (m, \theta) = e^{in\theta} \langle \prod \mathcal{O}_i \rangle_0 (\tilde{m}), \tag{5.9}$$

where $\tilde{m} = me^{-i\theta}$. Thus, the really important parameter of the theory is $\tilde{m}$, which we will write as $m_* e^{-i\theta_*}$, $m_* \geq 0$. Our goal in this lecture to study this theory as a function of $\tilde{m}$.

5.3. Behavior of 2-dimensional gauge theory with massive fermions

Now we will describe the behavior of the theory defined by Lagrangian (5.6), and later will justify this conclusion.

First of all, we will see that for large m_* (i.e. $m_* >> e$) the theory is similar to the 2-dimensional gauge theory with massive bosons, considered in the previous lecture. In particular, it has a mass gap. It also has a unique vacuum for $\theta \neq \pi$, and two of them for $\theta = \pi$. The discrete symmetry of reversal of space orientation ($t \to t, x \to -x$), which acts by $m_* \to m_*$, $\theta_* \to -\theta_*$, is broken on the negative real axis (far away from 0), but not on positive real axis.

Next, we will show that for $\tilde{m} = 0$ the theory is in fact free, i.e. becomes a free massive theory after a change of variables. Thus the theory has a unique vacuum and a mass gap for small m_* (i.e. for $m_* << e$).

Unfortunately, we do not know for sure what happens in the region $m_* \sim e$. The most natural thing would be that the cut in the plane of m_*, representing points with symmetry breaking which starts at $-\infty$, ends at some point $-m_c$, $m_c = e/\lambda$, and λ is dimensionless. At $\tilde{m} = -m_c$, the theory should have no mass gap (since it is a point of transition from two vacua to one vacuum).

This is what is in fact believed. Furthermore, there is a conjecture that the theory at the critical value $\tilde{m} = -m_c$ is in fact conformal, and isomorphic to the theory of a free neutral fermion.

5.4. Heavy fermions.

In this section we will study the case $m_* >> e$. In this case we can regard our theory as a perturbation of a theory with $e = 0$ with perturbation parameter $\lambda = e/m_*$. At $\lambda = 0$, we have a direct product of a 2-dimenisonal pure gauge theory (which is free), and a free theory of massive fermions (to be safe here, we should introduce $B = A/e$; this makes sense, as for $e = 0$, only the trivial $U(1)$-bundle contributes to the path integral).

It turns out that the situation here is similar to the bosonic case. Namely, the small e perturbation of the free theory for $e = 0$ is singular. This means, the space of states of the deformed theory is actually smaller than that of the undeformed theory. More precisely, confinement of fermions takes place: the only allowed states (for $\theta \neq \pi$) are states of total charge 0 (here the charge of $\psi_\pm$ is 1 and of $\bar{\psi}_\pm$ is -1, and $\theta \in [0, 2\pi)$). In particular, the fermions can only exist in pairs, quadruples, and so on, and a single fermion cannot exist.

Like in the bosonic case, the theory has a mass gap by deformation argument (the fact that the deformation is singular does not invalidate this argument, since the Hilbert space does not increase but only becomes smaller). More precisely, we have one realization for $\theta \neq \pi$, and two of them for $\theta = \pi$ (as in the pure gauge theory), and any realization has a mass gap.

5.5. Bose-Fermi correspondence.

Before studying the case of light fermions, we will consider Bose-Fermi correpondence, which will be useful in studying the case of small m_*.

Consider two 2-dimensional free field theories

1. The fermionic theory defined by the Lagrangian

$$\mathcal{L}_f = \frac{1}{2\pi} \int d^2x (\bar{\psi}_- D_+ \bar{\psi}_- + \bar{\psi}_+ D_- \bar{\psi}_+). \tag{5.10}$$

2. The bosonic theory defined by the Lagrangian

$$\mathcal{L}_b = \frac{1}{4\pi R^2} \int d^2x |d\phi|^2, \tag{5.11}$$

These two theories are conformal (both classically and quantum-mechanically), and have Virasoro central charge 1. So we can suspect there may be some connection between them. And indeed, such a connection exists, and it is called the Bose-Fermi correspondence.

Remark. Conformal field theories with small central charge are very scarce. For instance, for $c < 1$ they are completely classified by Friedan, Qiu, Shenker (Ref ???). The answer is that there is no continuous parameters, and the theory is almost completely determined by c, which can take only a discrete sequence of values. For $c = 1$, such a classification is unavailable, but very few examples are known, and all of them have a construction in terms of a free Bose field.

Let us now take a look at the bosonic theory (5.11). As we remember from Lecture II-1, in order for this theory to make sense, ϕ has to be angle-valued, i.e. take values in the circle $\mathbb{R}/2\pi\mathbb{Z}$. The constant R in the Lagrangian has the meaning of the radius of this circle.

Remark. We always model the target circle as $\mathbb{R}/2\pi\mathbb{Z}$, but consider various Riemannian metrics on it, which are parametrized by values of the radius R.

Recall from Lecture II-1 that the Hilbert space of this theory (in its unique realization) is of the form $\mathcal{H}_b = \oplus_{k\in\mathbb{Z}}(F_b \otimes F_b^*)_k$, where F_b is the bosonic Fock space. The operator algebra A_b of the theory is generated by the derivatives of ϕ (but not ϕ itself), and $:e^{in\phi}:$ (for brevity in the future we will drop the colons). Operator product expansion is given by formula (3.9) of Lecture 3 in the fall, where $D(x-y) = -R^2 \ln|x-y|$. The action of A_b in $\mathcal{H}_b$: Derivatives of ϕ do not change k, and $:e^{in\phi}:$ maps $(F_b \otimes F_b^*)_k$ to $(F_b \otimes F_b^*)_{k+n}$.

Now consider the theory (5.12). The Hilbert space of this theory is $\mathcal{H}_f = F_f \otimes F_f^*$, where F_f is generated from the vacuum by holomorphic operators $\psi_+, \bar\psi_+$, and F_f^* is generated from the vacuum by $\psi_-, \bar\psi_-$. The operator algebra A_f is generated by $\psi_+, \bar\psi_+, \psi_-, \bar\psi_-$, with the standard OPE of the free theory.

Consider more closely the operator $:e^{in\phi}:$ for $n \in \mathbb{Z}$. As we know (Lecture 3 in the fall, Lecture II-1), this operator has holomorphic dimension $n^2R^2/4$ and antiholomorphic dimension $n^2R^2/4$ (the total of $n^2R^2/2$, as we saw in Lecture 3 in the fall).

Classically, the operator $e^{in\phi}$ locally factors as a product of a holomorphic one and an antiholomorphic one: $e^{in\phi} = e^{in\phi_+}e^{in\phi_-}$, where $\phi = \phi_+ + \phi_-$, and $\partial_-\phi_+ = \partial_+\phi_- = 0$. (Of course, here ϕ_+, ϕ_- are defined only up to adding a constant). There is no analogs of $e^{in\phi_\pm}$ in our operator algebra A_b. However, imagine for a second that the operators $e^{in\phi_\pm}$ make sense. Then we will find from the OPE for $e^{i\phi}$ that $e^{i\phi_+}$ has holomorphic dimension $R^2/4$ and antiholomorphic dimension 0, and

$$\langle e^{i\phi_+(x_1)}...e^{i\phi_+(x_n)}e^{-i\phi_+(y_1)}...e^{-i\phi_+(y_n)}\rangle = \frac{\prod_{1\le i<j\le n}(x_i-x_j)^{R^2/2}\prod_{1\le i<j\le n}(y_i-y_j)^{R^2/2}}{\prod_{1\le i,j\le n}(x_i-y_j)^{R^2/2}}. \tag{5.12}$$

where x_i, y_j are viewed as complex numbers. From this formula it is clear that in order for $e^{i\phi_+}$ to make any sense, we need $R^2/2 \in \mathbb{Z}$, so that the function on the R.H.S. of (5.12) is single-valued. Similarly, in order for $e^{in\phi_+}$ to make sense, $n^2R^2/2$ has to be an integer.

Let us consider the simplest case where $e^{in\phi_+}$ can make sense, i.e. $R = \sqrt{2}$. In this case, we have

(5.13)
$$\langle e^{i\phi_+(x_1)}...e^{i\phi_+(x_n)}e^{-i\phi_+(y_1)}...e^{-i\phi_+(y_n)}\rangle = \frac{\prod_{1\le i<j\le n}(x_i-x_j)\prod_{1\le i<j\le n}(y_i-y_j)}{\prod_{1\le i,j\le n}(x_i-y_j)}.$$

It is clear from (5.13) that $e^{in\phi_+}$ behaves like a fermion when n is odd, and like a boson when n is even. This makes us hope that it is at $R = \sqrt{2}$ that our theory is related to the theory of fermions.

Let us see why this is indeed the case. Let us compute the fermionic correlation function $\langle\psi_+(x_1)...\psi_+(x_n)\bar\psi_+(y_1)...\bar\psi_+(y_n)\rangle$. Using Wick's formula, it is easy to find that

$$\langle\psi_+(x_1)...\psi_+(x_n)\bar\psi_+(y_1)...\bar\psi_+(y_n)\rangle = \det(\frac{1}{x_i-y_j}). \tag{5.14}$$

The coincidence of the right hand sides of (5.13),(5.14) is a famous combinatorial identity, which follows from comparison of zeros and poles, and asymptotics at infinity.

Remark. In fact, the explicit form of (5.13),(5.14) is not relevant to the proof of the fact that they are equal. What is relevant is only the structure of zeros and poles, and the asymptotics at infinity, which in both cases are obvious from the OPE.

Let $\hat A_b$ be the operator algebra generated by A_b and $e^{in\phi_\pm}$. Equalities (5.13),(5.14) show that we have a homomorphism $\xi : A_f \to \hat A_b$ defined by $\xi(\psi_\pm) = e^{\pm i\phi_\pm}$, $\xi(\bar\psi_\pm) = e^{\mp i\phi_\pm}$, which preserves expectation values.

Using the OPE, it is easy to find

(5.15)

$$\xi(:\bar\psi_+\psi_+:) = \lim_{\varepsilon\to 0}\xi(\bar\psi_+(x+\varepsilon)\psi_+(x)-\frac{1}{\varepsilon}) = \lim_{\varepsilon\to 0}(e^{-i\phi_+}(x+\varepsilon)e^{i\phi_+}(x)-\frac{1}{\varepsilon}) = -i\partial_+\phi.$$

Likewise, $\xi(:\bar\psi_-\psi_-:) = i\partial_-\phi$. Also, $\xi(\psi_+\bar\psi_-) = e^{i\phi}$, $\xi(\psi_-\bar\psi_+) = e^{-i\phi}$. In fact, it is not difficult to see that this homomorphism is an isomorphism.

At the level of Hilbert spaces, ξ induces a bigraded isomorphism $F_f \to F_b\otimes l_2(\mathbb{Z})$ (bidegree=(quantum scaling dimension, charge)), where the charge of $e^{i\phi_+}$ and the charge of ψ_+ equal 1. Writing the corresponding character formula, we obtain

$$\frac{\sum_{n\in\mathbb{Z}} q^{n^2/2}z^n}{\prod_{n\ge 1}(1-q^n)} = \prod_{n\ge 1}(1+q^{n-1/2}z)(1+q^{n-1/2}z^{-1}), \tag{5.16}$$

which is the famous Jacobi triple product identity.

The correpondence ξ is called the Bose-Fermi correspondence.

Remark 1. If the radius of the circle is not $\sqrt{2}$, but 1, then the operator $e^{i\phi_+}$ is meaningless, as its 2-point function would be $(x-y)^{-1/2}$, which is not single-valued. However, the operator $e^{2i\phi}$ is defined, and its 2-point function is $\frac{1}{(x-y)^2}$. This indicates that $e^{2i\phi}$ behaves like a current of some symmetry. And indeed, it turns out that the corresponding model is equivalent to the $\widehat{SU(2)}$-WZW model with Kac-Moody central charge 1, so it has an $SU(2)$-symmetry. In fact, the Fourier components of the operators $e^{\pm 2i\phi_+}$ and $\partial_+\phi$ generate the left-moving $\widehat{SU(2)}$, and the Fourier components of the operators $e^{\pm 2i\phi_-}$ and $\partial_-\phi$ generate the right-moving $\widehat{SU(2)}$.

Remark 2. In fact, the Bose-Fermi correspondence is true not only locally (at the level of operators), but also globally (at the level of path integral). Namely, for any Riemann surface Σ one has the identity of partition functions

$$\int D\phi\ e^{-\frac{1}{8\pi}\int|d\phi|^2} = \sum_\varepsilon\int D\psi_+D\bar\psi_+D\psi_-D\bar\psi_-e^{-\mathcal{L}_f(\psi)}, \tag{5.17}$$

where $\mathcal{L}_f$ is the Lagrangian given by (5.10), and ε runs over spin structures on Σ (the same spin structure is taken for ψ_+ and ψ_-). There is a similar identity for correlation functions of operators, if the correspondence between operators is made as explained above.

5.6. Bose-Fermi correspondence for nonlinear theories.

We have established a correspondence between two free theories – the theory of a boson and the theory of fermions. A remarkable fact is that this correspondence generalizes to the case when the Lagrangian of one or both of the theories is not free. Consider examples of such situations.

1. Recall that under our correspondence $\mathcal{L}_f \to \mathcal{L}_b(\sqrt{2})$, where $\mathcal{L}_b(R)$ is given by (5.11). Since $\xi(\bar{\psi}_\pm\psi_\pm) = \mp i\partial_\pm\phi$, we find

$$\mathcal{L}_f + \frac{1}{2\pi}\int d^2x(g\bar{\psi}_+\psi_+\bar{\psi}_-\psi_-) \to \mathcal{L}_b(\sqrt{2}(1+g)^{-1/2}). \tag{5.18}$$

This shows that, to our surprise, the theory with the Lagrangian $\mathcal{L}_f+\frac{1}{2\pi}\int d^2x(g\bar{\psi}_+\psi_+\bar{\psi}_-\psi_-)$ is free. In particular, its β-function is zero. This is obvious when the theory is described in Bose variables, but not obvious in Fermi variables.

2. On the other hand, consider the theory of free massive fermions, with the Lagrangian $\mathcal{L}_f + \frac{1}{2\pi}\int d^2x(m\bar{\psi}_-\psi_+ + \bar{m}\bar{\psi}_+\psi_-)$. Using the fact that $\xi(\bar{\psi}_\pm\psi_\mp) = -e^{\mp i\phi}$, we get that under ξ, this free Lagrangian goes to

$$\mathcal{L}_b(\sqrt{2}) - \frac{1}{2\pi}\int d^2x(me^{i\phi} + \bar{m}e^{-i\phi}). \tag{5.19}$$

So we get another surprising fact that the nonlinear theory defined by (5.19) is in fact free.

Remark. It may appear that the second term in (5.19) has a wrong scaling dimension. This is not the case, because the operator $e^{\pm i\phi}$ has anomalous dimension 1. More precisely, (5.19) does not fix a theory but fixes a family of theories depending on a scale μ of momenta, which is introduced when the operators $e^{i\phi}$ are renormalized. This scale enters in front of the corresponding term in the Lagrangian and cancels the discrepancy in dimensions.

Using the symmetry $\phi \to \phi + \theta$ we can reduce (5.19) to the case of real m. In this case, (5.19) looks like

$$\mathcal{L}_b(\sqrt{2}) - \frac{m}{\pi}\int d^2x \cos\phi. \tag{5.20}$$

Thus, the classical equation of motion is $\Delta\phi = -4m\sin\phi$. This equation is called the sine-Gordon equation, and it is a well-known completely integrable soliton equation.

Now consider The Lagrangian

$$\mathcal{L}_b(R) - \frac{m}{\pi}\int d^2x \cos\phi. \tag{5.21}$$

This Lagrangian is proportional to (5.20), with $m' = mR^2/2$, so classically the two Lagrangians are equivalent. However, quantum mechanically, this is not the case, as the scale of Lagrangian is now relevant. In fact, the theory now essentially depends on R. If $R=\sqrt{2}$, the theory is free, but for a general R it is not. The map ξ shows that for a general R the theory is equivalent to the fermionic theory with the Lagrangian

$$\mathcal{L}_f+\int d^2x(m\bar{\psi}_-\psi_+ + m\bar{\psi}_+\psi_- + g\bar{\psi}_+\psi_+\bar{\psi}_-\psi_-), \tag{5.22}$$

where $R=\sqrt{2}(1+g)^{-1/2}$.

As we mentioned, the theory described by the Lagrangian (5.21) is not free for $R\neq\sqrt{2}$. However, it is solvable, in the sense that its S-matrix can be computed explicitly. This computation and the result are similar to the computation of Lecture II-3, for the sigma-model into the sphere. Solvability for this theory for large R (i.e. in the classical limit) is related to the complete integrability of the sin-gordon equation at the classical level.

Now we want to apply the Bose-Fermi correspondence to gauge theory. Consider the Lagrangian $\mathcal{L}_{\theta,m}$ given by (5.6). Define

$$\mathcal{L}^f_{\theta,m}=\mathcal{L}_{\theta,m}+\frac{1}{2\pi}\int d^2x g\bar{\psi}_+\psi_+\bar{\psi}_-\psi_-. \tag{5.23}$$

Let us rewrite it in Bose variables. Then we will get the Lagrangian

$$\mathcal{L}^b_{\theta,m}=$$

$$\int d^2x\left(\frac{1+g}{8\pi}|d\phi|^2-\frac{m}{2\pi}e^{i\phi}-\frac{\bar{m}}{2\pi}e^{-i\phi}+A_+(\frac{i\partial_-\phi}{2\pi})+A_-(\frac{-i\partial_+\phi}{2\pi})+\frac{|*F|^2}{4e^2}-\frac{i\theta}{2\pi}F\right). \tag{5.24}$$

For simplicity we assume that ϕ is a homotopically trivial map. (It is easy to generalize everything to the homotopically nontrivial case). Then the integral $\int(A_-\partial_+\phi-A_+\partial_-\phi)$ can be taken by parts, and it equals $\int\phi(\partial_-A_+-\partial_+A_-)$, where ϕ is now understood as a lifting of the original ϕ to a map $\Sigma\to\mathbb{R}$. The expression $\partial_-A_+-\partial_+A_-$ equals to the curvature F, so (5.24) is simplified:

$$\mathcal{L}^b_{\theta,m}=$$

$$\int d^2x\left(\frac{1+g}{8\pi}|d\phi|^2+\frac{m}{2\pi}e^{i\phi}+\frac{\bar{m}}{2\pi}e^{-i\phi}+\frac{|*F|^2}{4e^2}-\frac{i(\phi+\theta)}{2\pi}F\right). \tag{5.25}$$

From this equation it is clear that when $m=0$, the theory is free, and there is no essential θ-dependence. This is the first thing we promised to show. Now, for $m\neq 0$, by changing ϕ to $\phi+\theta$ we find that theta can be absorbed in m.

Let us now see what happens for $m\neq 0$. It follows from Lecture II-4 that for an external field $\phi(x)$

$$\int DA\; e^{\frac{i}{2\pi}\int\phi F_A-\frac{1}{4e^2}\int|*F_A|^2}=Ce^{-\frac{e^2}{2\pi}\int d^2x\,\min_n(n-\frac{\phi(x)}{2\pi})^2}, \tag{5.26}$$

Therefore, the effective Lagrangian for ϕ for the Lagrangian (5.25) is

$$\mathcal{L}_{eff}^{\theta_*,m_*} = \int d^2x \left(\frac{1+g}{8\pi}|d\phi|^2 - \frac{m_*}{\pi}\cos(\phi-\theta_*) + \frac{e^2}{2\pi}\min_n(n-\frac{\phi}{2\pi})^2 \right). \tag{5.27}$$

From this formula, it is seen, that the theory has a mass gap for small m_*: the potential

$$U(\phi) = -\frac{m_*}{\pi}\cos(\phi-\theta_*) + \frac{e^2}{2\pi}\min_n(n-\frac{\phi}{2\pi})^2 \tag{5.28}$$

has a unique global minimum (modulo 2π) with positive second derivative. This is the case for all m_* if $\theta \neq \pi$. However, if $\theta = \pi$ and m_* grows from 0 to ∞, the global minimum at $\phi = 0$ keeps flattening, and at some point splits in two symmetric minima, when the second derivative becomes zero. This should indicate that starting at some finite value of e/m_* there should be symmetry breaking. Of course, this proves nothing, because (5.28) is not the quantum effective potential for our system (it is the potential only classically). However, we hope that the true effective potential behaves similarly, and the picture is qualitatively the same.

For more on this model, see S.Coleman's article in Annals of Physics, vol. 101, p. 239-267 (1976).