LECTURE II-9: WILSON LOOPS,'T HOOFT'S LOOPS, AND 'T HOOFT'S MODEL OF CONFINEMENT

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9.1. 'T HOOFT LOOP OPERATOR

Let us recall abelian duality in 4 dimensions, which we discussed two lectures ago. Consider a free $U(1)$ gauge theory with a θ-angle. Thus we have two dimensionless couplings e, θ which combine into a single complex coupling $\tau = \frac{2\pi i}{e^2} + \frac{\theta}{\pi}$, and the Lagrangian is

$$\mathcal{L} = \int_M \left(\frac{i\bar{\tau}}{4\pi} F_+^2 - \frac{i\tau}{4\pi} F_-^2 \right), \tag{9.1}$$

where F is the curvature of a $U(1)$ connection A and F_+, F_- are the selfdual and antiselfdual parts of the curvature. We have seen that if the spacetime M is a spin manifold then this theory is "modular invariant" as a function of τ. One modular symmetry $\tau \to \tau + 1$ is obvious, as it corresponds to shifting the θ-angle by π, which does nothing because c_1^2 for a spin manifold is even. (On a manifold that is not a spin manifold, the symmetry would be only $\tau \to \tau + 2$.) The symmetry under the second generator of the modular group, $\tau \to -1/\tau$, is more interesting and corresponds to electromagnetic duality discovered by Maxwell. More precisely, this means that the theory of a connection A with coupling constant τ is identical both classically and quantum mechanically to the same theory with coupling $-1/\tau$ and the connection B such that $dA = const * dB$. Now, like two lectures ago, we want to see what happens to operators under this duality. In particular, we want to know what happens to the Wilson loop operator.

Recall that the Wilson loop operator has the form $W_\gamma(C) = e^{i\gamma Hol_C(A)}$, where $Hol_C(A)$ denotes the integral of the connection A along a closed oriented curve C in the spacetime M. This operator is gauge-invariant and well-defined if γ is an integer, or for any real γ if the curve C is homotopically trivial in M. (More generally, there could be several components C_i with real numbers γ_i, and the condition is that $\sum_i \gamma_i C_i$ should be an integral class in $H_1(M)$.) Matrix elements of this operator are computed, as usual, by inserting the above exponential into the path integral. Similarly to what we found in similar problems in two and three dimensions, we should get that the dual description of the Wilson loop is a recipe which says that rather than insert in the path integral an object living on C, we should integrate over connections having a singularity along C.

The precise answer is the following. For any curve C the expectation value $\langle W_\gamma(C)\mathcal{O}_1...\mathcal{O}_n\rangle$ equals to $\int e^{-\mathcal{L}(B)}\mathcal{O}_1...\mathcal{O}_n DB$, where the integral is taken over

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connections on $M \setminus C$ such that the integral of the curvature of B over a small 2-sphere S in a normal 3-space to C at any point equals $2\pi\gamma$.

Let us prove this. We assume that C is a boundary. Let D be a 2-chain whose boundary is C. Recall the calculation from the lecture on Abelian duality: our fields are A – the original connection, G- the 2-form, and B – the dual connection. We have

$$\int e^{-L(A)} W_\gamma(C) DA =$$
$$\int DA\, DG\, DB e^{\frac{-i\tau}{4\pi}\int \mathcal{F}_+^2 + \frac{i\tau}{4\pi}\int \mathcal{F}_-^2} e^{\frac{i}{2\pi}\int G\wedge F_B} e^{i\gamma\int_D (F_A - G)}, \tag{9.1}$$

where $\mathcal{F} = F_A - G$, and the last factor corresponds to the insertion of the Wilson loop (recall from the abelian duality discussion that the Wilson loop classically is $e^{i\gamma\int_D F_A}$, and F_A is to be replaced with $F_A - G$ in the extended theory). Gauging A to 0, we get

$$\int e^{-L(A)} W_\gamma(C) DA =$$
$$\int DG\, DB e^{\frac{-i\tau}{4\pi}\int G_+^2 + \frac{i\tau}{4\pi}\int G_-^2} e^{\frac{i}{2\pi}\int G\wedge F_B} e^{-i\gamma\int_D G} =$$
$$\int DG\, DB e^{\frac{-i\tau}{4\pi}\int G_+^2 + \frac{i\tau}{4\pi}\int G_-^2} e^{i\int G\wedge(\frac{1}{2\pi}F_B - \gamma[D])}, \tag{9.2}$$

where $[D]$ is the delta-function of D. Now we define $\tilde{F}_B := F_B - 2\pi\gamma[D]$. Then after integrating out G, (9.2) can be written as

$$\int e^{-L(A)} W_\gamma(C) DA =$$
$$\int DB e^{\frac{i}{4\pi\tau}\int (\tilde{F}_B)_+^2 + \frac{-i}{4\pi\tau}\int (\tilde{F}_B)_-^2}, \tag{9.3}$$

Thus, the effect of the insertion $W_\gamma(C)$ is that F_B is replaced in the final answer by $\tilde{F}_B$. So B is now a connection on a line bundle with singularity along C, as discussed above.

Note that if C is not a boundary then $\langle W_\gamma(C)\rangle = 0$ (even with insertion of any number of local operators). Indeed, we have symmetry $A \to A + d\phi$, and ϕ does not have to be globally defined as a map to a circle; in fact, $d\phi$ can be any closed one-form. So if C is not a boundary then we can choose $d\phi$ in such a way that the operator $W_\gamma(C)$ will multiply by $e^{i\alpha}$ for some nonzero α. Hence its expectation value (even with inclusion of local operators, which are invariant under this transformation) vanishes. More generally, the correlator $\langle W_{\gamma_1}(C_1)...W_{\gamma_n}(C_n)\rangle$ (with any local operators) is zero if $\sum \gamma_i C_i \neq 0$ in $H_1(M)$, where M is the spacetime. As noted before, for the product of operators in question to be well-defined, we only need $\sum \gamma_i C_i$ to be an integral class.

From our construction so far, for any curve C we have two operators:

1) $W_\gamma(C) = e^{i\gamma\int_C A}$

2) $T_\gamma(C) = e^{i\gamma\int_C B}$.

The second operator, which is dual to the Wilson loop, is called the 't Hooft loop operator.

9.2. Hilbert space interpretation of the 't Hooft loop operator

Now let us consider this picture from the Hamiltonian point of view. Then the spacetime M has the form $M = M^3 \times \mathbb{R}$ with Minkowski metric. Let C, C' be two nonintersecting closed simple curves in M^3. They define operators $W_\gamma(C)$ and $T_{\gamma'}(C')$ on the Hilbert space $\mathcal{H}$. The following commutation relation for these operators is due to 't Hooft:

$$W_\gamma(C)T_{\gamma'}(C') = e^{2\pi i\gamma\gamma' l(C,C')}T_{\gamma'}(C')W_\gamma(C). \tag{9.4}$$

Let us prove this formula. Let us work in terms of the original connection A. Then the Hilbert space consists of wave functions $\Psi(A)$. In this realization, the Wilson loop operator $W_\gamma(A)$ is simply the operator of multiplication by $e^{i\gamma\int A}$.

However, the 't Hooft loop operator is a bit harder to define. To do this, consider the homomorphism $\pi_1(M^3 \setminus C', x_0) \to \mathbb{Z}$ given by the linking number with C'. Let π_1^0 be the kernel of this homomorphism and X be the $\mathbb{Z}$-cover of $M_3 \setminus C'$ corresponding to π_1^0. Let ϕ be a function $X \to U(1)$ such that the monodromy corresponding to the generator of $\mathbb{Z}$ is $e^{i\gamma}$. Any two such functions differ by a gauge transformation, but ϕ itself is not an honest gauge transformation. Then it is not difficult to check that the 't Hooft loop operator $T_{\gamma'}(C')$ is just the "illegal" gauge transformation by ϕ:

$$(T_{\gamma'}(C')\Psi)(A) = \Psi(A^\phi). \tag{9.5}$$

Note that this is well-defined since any two such ϕ's differ by an honest gauge transformation.

Now formula (9.4) is clear since $TWT^{-1} = e^{i\gamma_C\int d\phi}W$, because of the way that T transforms the connection in the definition of W.

9.3. The 2+1-dimensional analogue of the 3+1-dimensional picture

Consider the 2+1-dimensional analogue of this picture. As we saw before, in 2+1 dimensions the theory of a scalar field ϕ is dual to a gauge theory of the dual field A. The path integral in ϕ with insertion of $e^{i\phi(x)}$ is the same as path integral in A where A is a connection on $M \setminus x$ which has $\int F = 2\pi$, where the integral is over a small sphere around x. Thus, the operator $e^{i\phi}$ corresponds to a magnetic monopole in gauge theory.

Now consider the 3-dimensional cosine theory, defined by the path integral

$$\int D\phi e^{-\int(|d\phi|^2+\varepsilon(e^{i\phi}+e^{-i\phi}))}. \tag{9.6}$$

Decomposing this path integral in a power series, and passing to the dual variable A, we get the sum

$$\sum_{m,n}\varepsilon^{m+n}\int\frac{dx_1...dx_m}{m!}\int\frac{dy_1...dy_n}{n!}\int_{\mathcal{A}_{x,y}}e^{-\int F_A^2}, \tag{9.7}$$

where $\mathcal{A}_{x,y}$ is the space of connections with monopoles at x_i and antimonopoles at y_j. Thus the cosine theory maps to the theory with monopoles. We saw this more computationally when we discussed the Polyakov model two lectures ago.

9.4. The model of confinement

Now we discuss a picture of confinement developed by 't Hooft. In general we don't assume that the gauge group is abelian. Recall the definition of confinement. We have a gauge group G and with universal cover $\widehat{G}$. We assume that G is the quotient of $\widehat{G}$ which acts faithfully on all fields in the Lagrangian. We let R be a representation of $\widehat{G}$. As we discussed before, if there is a mass gap, there are two usual patterns of decay of the expectation value $\langle W_R(C)\rangle$ of the Wilson line operator corresponding to the representation R as C gets big:

Pattern 1:

$$\langle W_R(C)\rangle \sim e^{-\lambda\,\mathrm{Length}(C)} \tag{9.8}$$

Pattern 2:

$$\langle W_R(C)\rangle \sim e^{-\lambda\,\mathrm{Area}(C)} \tag{9.9}$$

(here the parameter γ is a fixed nonzero number and the area of C means the minimal area of the spanning surface). The first regime is called the Higgs regime (the length law) and the second one is called the confinement regime (the area law).

As we discussed in the previous lecture on confinement, the first regime is the case when R is a representation of G itself, and to see confinement one needs to consider the case when R is a representation of $\widehat{G}$ but not of G. Thus interesting $W_R(C)$ correspond to elements of $\pi_1(G)^*$.

Now let us consider the 't Hooft loop operator $T_\gamma(C)$. It is defined for any G by analogy with the definition in the abelian case. We fix an element $\gamma \in \pi_1(G)$. Recall that G-bundles on a two-sphere S^2 are classified by a characteristic class that takes values in $H^2(S^2, \pi_1(G))$, which is canonically isomorphic to $\pi_1(G)$. The choice of γ therefore canonically determines an isomorphism class of G-bundles on S^2. We can now define the 't Hooft operator: a path integral with insertion of $T_\gamma(C)$ is computed by integrating over connections on $M \setminus C$ which have the property that when restricted to a small sphere S that links C, the bundle has characteristic class γ.

't Hooft's idea was to consider $T_\gamma(C)$ instead of $W_R(C)$ and find conditions under which there is an area law for its expectation value. This occurs, as he showed, for certain Higgs theories. Then, 't Hooft proposed (following earlier ideas of Nambu, Mandelstam, and others) that confinement would be related to the Higgs mechanism by a duality that maps 't Hooft loop operators into Wilson loop operators. This does not explain confinement, but it reformulates the problem: to reduce the mysterious phenomenon of confinement to the much more easily understood Higgs phenomenon, one must understand the nonlinear duality that exchanges 't Hooft and Wilson loop operators.

To illustrate the area law for the 't Hooft loop in Higgs theories, we consider a familiar example: the $U(1)$ gauge theory with a charged complex scalar ϕ (of charge 1). The Lagrangian is

$$\int \Big(\frac{|F|^2}{4e^2} + |D_A\phi|^2 + V(\phi\bar{\phi})\Big), \tag{9.10}$$

where V is a (quartic) potential. We will study the 't Hooft loop $T_\gamma(C)$, where $\gamma \in \mathbb{Z}$. Thus A is a connection and ϕ is a section for a hermitian line bundle over

$M \setminus C$ such that it has first Chern class γ when restricted to a small sphere linking C. We will compute $\langle T_\gamma(C)\rangle$ for two classes of V:

1) $V = \lambda(\phi\bar{\phi} + a^2)^2$;

2) $V = \lambda(\phi\bar{\phi} - a^2)^2$.

This theory was considered in Lecture 2. Recall the results of this consideration.

Case 1. In the infrared the theory behaves like the product of the theory of a free massive field with a free gauge theory. In particular, there is no mass gap. Thus, we can calculate $\langle T_\gamma(C)\rangle$ for large C using the free theory. But in the free theory this expectation value is the same as $\langle W_\gamma(C)\rangle$ in the dual theory. It is easy to see that the expectation values of both W_γ and T_γ behave according to the length law, because of Coulomb law of charge interaction. A theory behaving in this way is said to be in the "Coloumb phase."

Case 2. In the infrared this theory has breaking of gauge symmetry and a Higgs mechanism. In particular, speaking classically, we have a circle of vacua, and at each of these vacua the low energy part of the Hamiltonian spectrum contains a massive vector and a real massive scalar. So there is a mass gap. This theory is not believed to exhibit confinement, i.e. it is believed that it exhibits the length law for the Wilson loop. This is certainly what one computes in perturbation theory.

In case 2, we will show that there is an area law for the 't Hooft loop operator, because of the Higgs mechanism. This happens for topological reasons, as explained below.

As a warmup consider a closed spacetime M and a line bundle $\mathcal{L}$ with a nontrivial c_1. Let us consider the path integral for our theory over sections of this bundle. It turns out that the action of all field configurations in this integral has to be very large: it is bounded below by a constant (which is independent of M) times the area of the minimal 2-surface which represents a cycle Poincare dual to $c_1(\mathcal{L})$.

Indeed, if ϕ is a section of $\mathcal{L}$ then ϕ has to vanish on a 2-cycle Σ which is dual to $c_1(\mathcal{L})$. If we fix Σ, we can look at the configuration of minimal action with such a zero. For this, we can (if the metric of M is scaled up) reduce to the case that $M = \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, with Σ equal to the first factor. We can assume that ϕ and A are invariant under translations of the first factor in M. In the second factor, we want ϕ to vanish at the origin and to approach the vacuum at infinity (up to gauge transformation), such that the first Chern class of the bundle, relative to the trivialization at infinity given by ϕ, equals 1. The same problem appeared in lecture 2 (in the guise of finding an instanton solution of the two-dimensional version of the same model), and we discussed qualitative properties of the solution. Anyway, let I be the action of this solution in the two-dimensional sense (that is, integrated over just the second factor in M). Going back to a global $\Sigma \subset M$ of smallest area representing the first Chern class, the minimum action field looks in the normal directions to Σ like the instanton just described; its action is approximately $I \cdot \mathrm{Area}(\Sigma)$.

Now let us come back to the 't Hooft loop in $\mathbb{R}^4$. In this case the bundle is over $M \setminus C$, where M is the spacetime. If D is a 2-chain in M whose boundary is C then D plays the role of the Σ of the previous discussion. Indeed, if $\mathcal{L}$ is a line bundle over $M \setminus C$ with Chern class 1, and ϕ its section then ϕ must vanish on a 2-surface whose boundary is C. Thus, the same argument as above shows that $\langle T_1(C)\rangle \sim e^{-\lambda\,\mathrm{area(D)}}$, where $\mathrm{area}(D)$ is the smallest area of a disk spanning C. This is the area law which we wanted to demonstrate.

This behavior is characteristic of what is called the Higgs regime, or phase.

Now let us discuss in more detail the relation of the established behavior of the 't Hooft loop operator with confinement. It is believed that if $\pi_1(G) \neq 0$ there are at least three possible phases:

1) Coulomb: no mass gap, gauge bosons in the infrared, W and T behave like in the free theory and exhibit the length law.

2) Higgs: mass gap, length law for W, area law for T.

3) Confinement: mass gap, area law for W, length law for T.

As already suggested, 't Hooft's idea was that there should be a nonabelian analogue of duality which interchanges W with T, the Higgs and the confinement regimes, and maps the Coloumb phase to itself. Thus, the area law for T in a theory implies confinement in the dual theory. This is what happens for some supersymmetric theories, e.g. the theory relevant to Donaldson theory.

In fact, 't Hooft showed that if R and γ are such that $\gamma|_R \neq 1$ then either W_R on T_γ exhibit the area law. More specifically, he proved an even stronger statement, namely that the set H of all $(c, \gamma) \in \pi_1(G)^* \times \pi_1(G)$ such that for some representation R with central character c the operator $W_R(C)T_\gamma(C)$ (suitably renormalized) does not exhibit the area law, is an isotropic subgroup of $\pi_1(G)^* \times \pi_1(G)$ with respect to the natural symplectic form. The reason for this, very roughly, is the following. If $A(C) = W_{R_1}T_{\gamma_1}(C)$ and $B(C) = W_{R_2}T_{\gamma_2}(C)$ exhibit the length law then, when acting on the vacuum, they produce only effects that are localized along C (or there would be an area law instead of a length law). So

$$\langle A(C)B(C')\rangle = \langle A(C)\rangle\langle B(C')\rangle(1 + o(1)), d \to \infty \tag{9.11}$$

where d is the distance between C and C'. This shows that if $AB = qBA$ where q is a constant then q must be equal to 1. By 't Hooft's formula (9.4) (which is clearly valid in the nonabelian case as well), this implies that H is isotropic. By further physical arguments, one shows that if there is a mass gap, then H is maximal isotropic. This leads to a more refined classification of massive phases than was stated above: associated to each massive phase is a maximal isotropic subgroup of $\pi_1(G)^* \times \pi_1(G)$. All possibilities can arise, in general.

Remark. The argument showing that loop operators A and B must commute fails if one of the operators, say A, exhibits the area law. In this case, acting on the vacuum with this operator produces an effect that is not in any way localized near C; it rather has an effect which is localized near a minimal area disk D whose boundary is C; such a disk will always intersect C' when the linking number is not zero. Formula (9.11) is now valid only if d is the distance from C' to D, which is always 0, so the formula does not tell us anything.

Remark. The area law for the 't Hooft operator in a Higgs phase has many physical and mathematical applications. For example, with some small adjustments, what we said above in analyzing the behavior of the Higgs phase with a bundle of nonzero first Chern class could serve as an explanation of the Meissner effect, the fact that a superconductor (which is described approximately by the abelian Higgs model that we examined) expels magnetic flux. Perhaps the reader has, at a science fair, seen a demonstration of a magnet floating above a superconductor; this effect has the same origin. Mathematically, C. Taubes's analysis of the Seiberg-Witten invariants of symplectic four-manifolds made use of the same facts: the localization (in a closely analogous system of equations) of the zeroes of ϕ on a surface of smallest area.