Flops, Flips, and Matrix Factorization

David R. Morrison, Duke University

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(Based on joint work with Carina Curto)

Dedicated to my good friend

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上野 健爾

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on the occasion of his 60^{th} birthday.

Flops and Flips

- A *simple flop* (resp. *simple flip*) is a birational map $Y - - \rightarrow Y^+$ which induces an isomorphism $(Y - C) \cong (Y^+ - C^+)$, where C and C^+ are smooth rational curves on the Gorenstein (resp. $\mathbb{Q}$-Gorenstein) threefolds Y and Y^+, respectively, and

$$K_Y \cdot C = K_{Y+} \cdot C^+ = 0$$

(resp. $K_Y \cdot C < 0$ and $K_{Y+} \cdot C > 0$).

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- Thanks to a theorem of Kawamata, it is known that all birational maps between Calabi–Yau threefolds can be expressed as the composition of simple flops.

- Thanks to work of Mori, Reid, Kawamata, and others, simple flips and contractions of divisors form the building blocks for birational transformations used to construct minimal models of threefolds of general type.

- Another theorem of Kawamata relates the two: essentially, every simple flip has a cover which is a simple flop.

Atiyah's Flop

- In 1958, Atiyah noticed that if he made a basechange $s = t^2$ in the versal deformation of an ordinary double point

$$xy + z^2 = s,$$

then the resulting family of surfaces admitted a *simultaneous resolution of singularities*.

- That is, factoring the equation as

$$xy = (t + z)(t - z)$$

and blowing up the (non-Cartier) divisor described by $x = t + z = 0$, one obtains a family of non-singular surfaces which, for each value of t, resolves the corresponding singular surface.

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- In fact, there are two ways of doing this, for one might have chosen to blow up the divisor $x = t - z = 0$ instead. This produces two threefolds Y and Y^+ related by a flop.

Reid's pagoda

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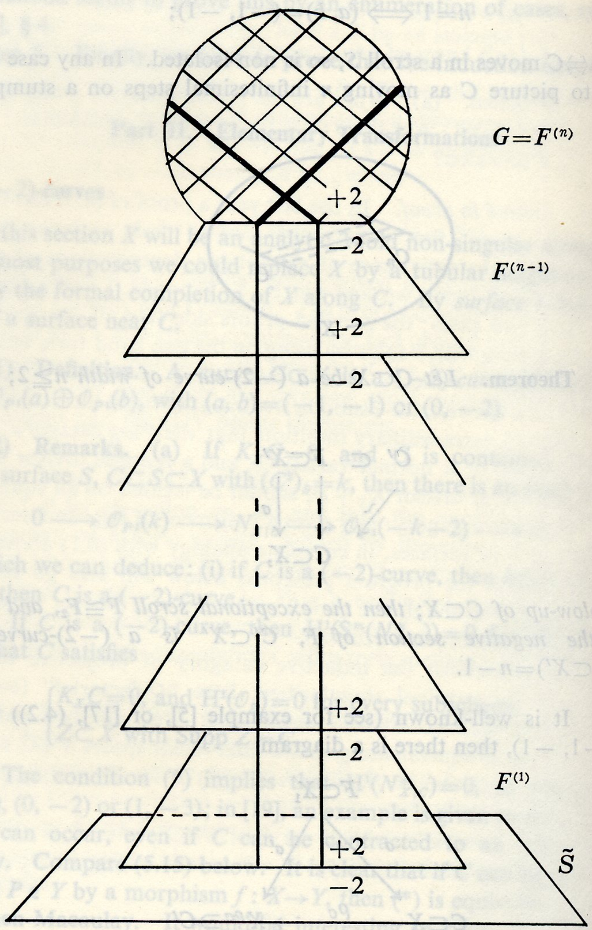
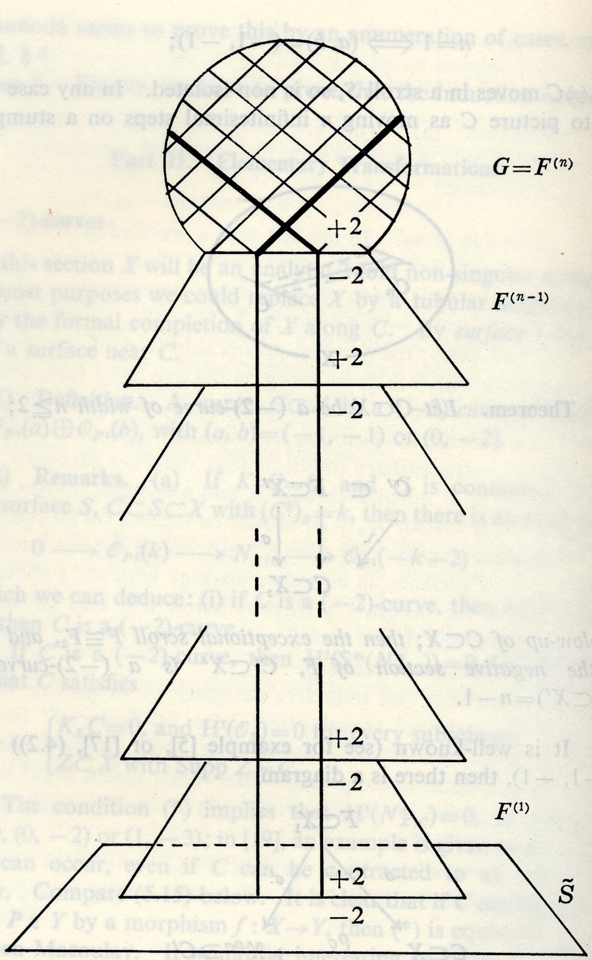
- The flop itself can be described by blowing up C and its proper transforms k times, and then blowing back down:



Pagoda (5.8)

Each of the lower layers $F^{(i)}$ is a copy of $\boldsymbol{F}_2$, meeting $F^{(i+1)}$ in the negative section, and $F^{(i-1)}$ in a disjoint section of self-intersection +2. The topmost layer G is a copy of $\boldsymbol{P}^1\times\boldsymbol{P}^1$, with normal bundle of type $(-1,-1)$, intersecting $F^{(n-1)}$ in a curve of type $(1, 1)$. The thick black lines that look like lightning conductors are fibres of the two rulings.

The base $\tilde{S}$, which is optional, is the proper transform of a surface $S\subset X$ with $C\subset S$ and $(C^2)_S=-2$.



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Each of the lower layers $F^{(i)}$ is a copy of F_2, meeting $F^{(i+1)}$ in the negative section, and $F^{(i-1)}$ in a disjoint section of self-intersection $+2$. The topmost layer G is a copy of $P^1 \times P^1$, with normal bundle of type $(-1, -1)$, intersecting $F^{(n-1)}$ in a curve of type (1, 1). The thick black lines that look like lightning conductors are fibres of the two rulings.

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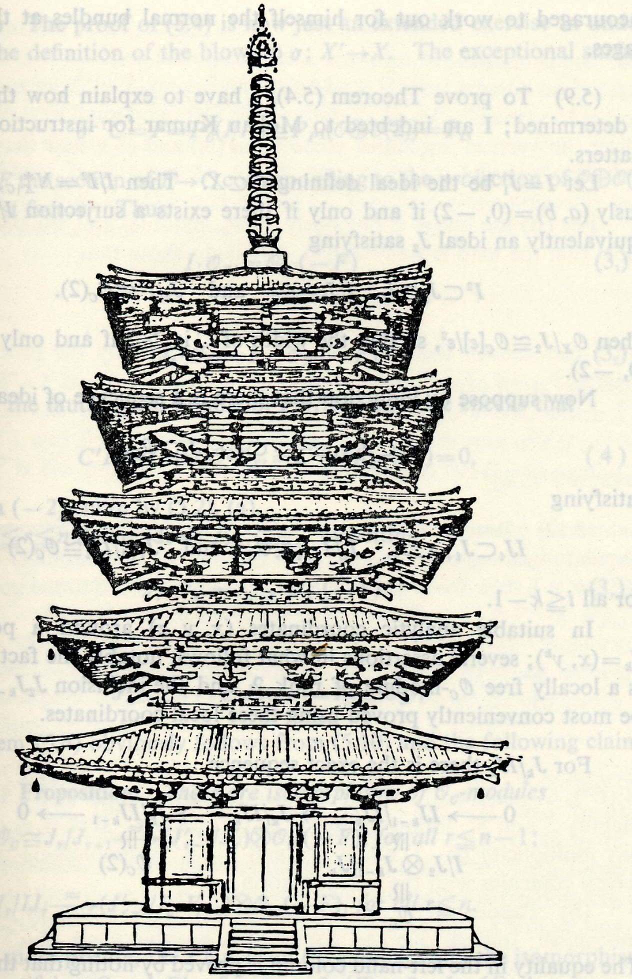

The pagoda of Hōryūji, near Nara, Japan
(from Horyuji by T. Nishioka and S. Miyakami, illustrated
by K. Hozumi, published by Soshisha, Tokyo, 1980).

(6.10) or [14] can be blown down along either of them. The blowing-down can also be done step-by-step, starting from the top, and this involves nothing more complicated than the Castelnuovo-Moishezon-Nakano criterion for contractions of geometrically ruled surfaces on analytic 3-folds. The reader who has not met this kind of thing before is

Laufer's analysis

- Laufer analyzed simple flops from a different perspective. In Atiyah's flop, the normal bundle of C in Y is

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

(and this in fact characterizes an Atiyah flop) while in Reid's examples, the normal bundle is $\mathcal{O} \oplus \mathcal{O}(-2)$ (and again, any rigid curve with such a normal bundle must be one of Reid's examples).

- Laufer showed that any rational curve on a smooth threefold which can be contracted to a Gorenstein singular point must have normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(b)$ with $(a, b) = (-1, -1)$, $(0, -2)$, or $(1, -3)$. The last possibility was extremely surprising at the time, but Laufer gave an explicit example to show that this actually happened.

- A slight generalization of Laufer's example (due to Pinkham and DRM) is

$$v_4^2 + v_2^3 - v_1 v_3^2 - v_1^3 v_2 + \lambda(v_1 v_2^2 - v_1^4) = 0$$

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$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ -v_3 & -v_4 & v_1(v_2 + \lambda v_1) & v_2(v_2 + \lambda v_1) \end{bmatrix}$$

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We shall discuss this example again later in the talk.

Simultaneous resolution

- A generalization of Atiyah's observation in a different direction was made by Brieskorn and Tyurina, who showed that the (uni)versal family over the versal deformation space, for any rational double point, admits a simultaneous resolution after basechange.

- More precisely, each rational double point has an associated Dynkin diagram Γ whose Weyl group $W = W(\Gamma)$ acts on the complexification $\mathfrak{h}_{\mathbb{C}}$ of the Cartan subalgebra $\mathfrak{h}$ of the associated Lie algebra $\mathfrak{g} = \mathfrak{g}(\Gamma)$. A model for the versal deformation space is given by

$$\mathrm{Def} = \mathfrak{h}_{\mathbb{C}}/W$$

and there is a (uni)versal family $\mathcal{X} \to \mathrm{Def}$ of deformations of the rational double point.

- The deformations of the resolution are given by a representable functor, which can be modeled by

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- In fact, there is a (uni)versal simultaneous resolution $\widehat{\mathcal{X}}$ of the family $\mathcal{X} \times_{\mathrm{Def}} \mathrm{Res}$. The construction of this resolution requires some trickiness with the algebra which will in fact be generalized and somewhat explained later in this talk.

Simultaneous partial resolution

- It is not too hard to generalize the work of Brieskorn and Tyurina to cases in which we do not wish to fully resolve the rational double point, but only to partially resolve it.

- If we let $\Gamma_0 \subset \Gamma$ be the subdiagram for the part of the singularity that is not being resolved, then we can define a functor of deformations of the partial resolution, which has a model

$$\mathrm{PRes}(\Gamma_0) = \mathfrak{h}_{\mathbb{C}}/W(\Gamma_0)$$

and there is a simultaneous partial resolution $\widehat{\mathcal{X}}(\Gamma_0)$ of the family $\mathcal{X} \times_{\mathrm{Def}} \mathrm{PRes}(\Gamma_0)$.

Back to flops

- By a lemma of Reid, given a simple flop from Y to Y^+ and the associated small contraction $Y \to X$, the general hyperplane section of X through the singular point P has a rational double point at P, and the proper transform of that surface on Y gives a partial resolution of the rational double point.

Back to flops

- By a lemma of Reid, given a simple flop from Y to Y^+ and the associated small contraction $Y \to X$, the general hyperplane section of X through the singular point P has a rational double point at P, and the proper transform of that surface on Y gives a partial resolution of the rational double point. This is the first instance of Reid's "general elephant" principle.

- Pinkham used this to give a construction for all Gorenstein threefold singularities with small resolutions (with irreducible exceptional set): they can be described as pullbacks of the (uni)versal family via a map from the disk to $\mathrm{PRes}(\Gamma_0)$ (for some $\Gamma_0 \subset \Gamma$ which is the complement of a single vertex).

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- Kollár introduced an invariant of simple flops called the *length*: it is defined to be the generic rank of the sheaf on C defined as the cokernel of $\mathcal{O}_Y \to f^*(\mathfrak{m}_P)$ where $\mathfrak{m}_P$ is the maximal ideal of the singular point P.

- It turns out that the length can be computed from the hyperplane section, and it coincides with the coefficient of the corresponding vertex in the Dynkin diagram in the linear combination of vertices which yields the longest positive root in the root system.

The generic hyperplane section

- In 1989, Katz and DRM proved that the generic hyperplane section for a flop of length ℓ was the smallest rational double point which used ℓ as a coefficient in the maximal root. The proof was computationally intensitive, and Kawamata later gave a short and direct proof of this fact.

Other general facts about flops

- Kollár and Mori gave a general construction of flops, based on the fact that the contracted variety X is a hypersurface with an equation of the form

$$x_1^2 + f(x_2, \ldots, x_n) = 0;$$

the flop is essentially induced by the automorphism $x_1 \to -x_1$.

- Much more recently, Bridgeland has constructed flops as moduli spaces for a certain category of complexes of sheaves (the complexes of sheaves on Y which are the inverse image under the flop of the structure sheaves of points on Y^+).

- Much more recently, Bridgeland has constructed flops as moduli spaces for a certain category of complexes of sheaves (the complexes of sheaves on Y which are the inverse image under the flop of the structure sheaves of points on Y^+). This work has been extended in various ways by Kawamata, Chen, and Abramovich.

Wrapping D-branes

- The physics represented by D-branes wrapping the rational curve in a resolution of Atiyah's flop has been thoroughly studied; for example, there is the large N duality proposal of Gopakumar and Vafa, and the matrix model presentation studied by Dijkgraaf and Vafa. We began this project with a desire to obtain a similarly deep understanding of the physics of D-branes wrapping the rational curve for other kinds of flops.

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- However, there is an intriguing proposal for describing certain D-branes states that was made by Kontsevich in a different context: for Landau–Ginzburg theories, Kontsevich has proposed using the category of "matrix factorizations" of the Landau–Ginzburg potential to describe D-brane states.

- Here we do not have a Landau–Ginzburg theory, but we do have a hypersurface singularity, and it makes sense to consider the matrix factorizations in that context as well.

Matrix factorizations

- Matrix factorizations are a tool introduced by Eisenbud in 1980 to study maximal Cohen–Macaulay modules (modules over R whose depth is equal to the dimension) on isolated hypersurface singularities.

- The ring R is (the localization at the origin of)

$$k[x_1, \ldots, x_n]/(f)$$

for some polynomial $f \in S = k[x_1, \ldots, x_n]$ with an isolated singular point at the origin.

- A maximal Cohen–Macaulay module M is supported on $f = 0$ and is locally free away from the origin; regarded as an S-module, there is a resolution

$$0 \to S^{\oplus k} \xrightarrow{\Psi} S^{\oplus k} \to M \to 0$$

Since $fM = 0$, $fS^{\oplus k} \subset \Psi(S^{\oplus k})$ which implies that there is a map

$$S^{\oplus k} \xrightarrow{\Phi} S^{\oplus k}$$

such that $\Phi \circ \Psi = f\,\mathrm{id}$. The pair (Φ, Ψ) is called a *matrix factorization*.

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- *Eisenbud's Theorem*

  There is a one-to-one correspondance between isomorphism classes of maximal Cohen–Macaulay modules over R with no free summands and matrix factorizations (Φ, Ψ) with no summand of the form $(1, f)$ (induced by an equivalence between appropriate categories).

The McKay correspondance

- The McKay correspondance for a rational double point can be described in terms of maximal Cohen–Macaulay modules. If $X = \mathbb{C}^2/G$ and ρ is a representation of G of rank r then

$$(\mathbb{C}^r \times_\rho \mathbb{C}^2)/G \to \mathbb{C}^2/G$$

  is generically locally free, and a maximal Cohen–Macaulay module.

- To express this more algebraically, we decompose $\mathbb{C}[s,t]$ into $\mathbb{C}[s,t]^G$ modules according to the irreducible representations ρ:

$$\mathbb{C}[s,t] = \bigoplus_{\rho \in \mathrm{Irrep}(G)} \mathbb{C}[s,t]_\rho$$

and each summand is a $\mathbb{C}[s,t]^G$ module.

- *Example.*

  Let $G = \mathbb{Z}/N\mathbb{Z}$. The algebra $\mathbb{C}[s,t]^G$ is generated by $x = s^N$, $y = t^N$ and $z = st$ subject to the relation $xy - z^N = 0$. We choose ρ_k to act as $u \longrightarrow e^{2\pi ik/N}u$. It is not hard to see that $M_k := \mathbb{C}[s,t]_{\rho_k}$ is generated over $\mathbb{C}[s,t]^G$ by s^k and t^{N-k}, with relations

$$xt^{N-k} = z^{N-k}s^k$$

$$ys^k = z^k t^{N-k}$$

- This gives a presentation of M_k as the cokernel of

$$\begin{bmatrix} x & -z^{N-k} \\ -z^k & y \end{bmatrix}$$

The corresponding matrix factorization is

$$\begin{bmatrix} y & z^{N-k} \\ z^k & x \end{bmatrix} \begin{bmatrix} x & -z^{N-k} \\ -z^k & y \end{bmatrix} = (xy - z^N) I_2$$

If we do the minimal blowup which makes the cokernel of the right-hand matrix locally free, we find that precisely one curve is blown up—the curve associated to the corresponding representation ρ_k.

Grassmann blowups

- We now describe a blowup associated to a matrix factorization. Given a matrix factorzation (Φ, Ψ), the cokernel of Ψ is supported on the hypersurface $f = 0$, and the $k \times k$ matrix Ψ will have some generic rank $k - r$ along the hypersurface (so that the cokernel has rank r).

- We do a *Grassmann blowup* which makes this locally free as follows. In the product $\mathbb{C}^n \times \mathrm{Gr}(r,k)$ we take the closure of the set

$$\{(x,v) \mid x \in X_{\mathsf{smooth}}, v = \operatorname{coker} \Psi_x\}.$$

  There are natural coordinate charts for this blowup, given by Plücker coordinates for the Grassmannian.

- The Grassmann blowup is the minimal blowup which makes the cokernel of Ψ locally free.

The explicit McKay correspondence

- Gonzalez-Sprinberg and Verdier calculated generators and relations for each $\mathbb{C}[s,t]_\rho$ (and hence a matrix factorization), and used this to calculate Chern classes of the correponding sheaves on the resolution of $\mathbb{C}^2/G$, making the McKay correspondence very concrete.

- Implicit in the literature (but I cannot find it stated explicitly) is the fact that the Grassmann blowup of the rational double point has an irreducible exceptional set: it blows up precisely the component of the exceptional divisor which is associated to the representation ρ.

- For example, taking

$$\Phi = \Psi = \begin{bmatrix} X & 0 & 0 & Y & -Z & 0 \\ 0 & X & 0 & 0 & Y & Z \\ 0 & 0 & X & -Z^2 & 0 & Y \\ Y^2 & YZ & -Z^2 & -X & 0 & 0 \\ -Z^3 & Y^2 & -YZ & 0 & -X & 0 \\ YZ^2 & Z^3 & Y^2 & 0 & 0 & -X \end{bmatrix}$$

gives the matrix factorization corresponding to the central vertex of E_6 (with equation $X^2 + Y^3 + Z^4$).

Deformations

- The versal deformation of the A_{N-1} singularity can be written as
$$xy - f_N(z)$$
for a monic polynomial f of degree N whose coefficient of z^{N-1} vanishes. (The coefficients of this polynomial generate the invariants of the Weyl group $\mathfrak{W}_{A_{N-1}}$ which coincides with the symmetric group on N letters $\mathfrak{S}_N$.)

- The partial resolution corresponding to the k^{th} vertex in the Dynkin diagram corresponds in the invariant theory to the subgroup

$$\mathfrak{S}_k \times \mathfrak{S}_{N-k} \subset \mathfrak{S}_N$$

The relationship between the invariants of these two groups is neatly summarized by writing

$$f_N(z) = g_k(z) h_{N-k}(z)$$

where g and h are monic polynomials.

- It is then clear that the matrix factorization data extends to PRes: we have

$$\begin{bmatrix} y & h_{N-k}(z) \\ g_k(z) & x \end{bmatrix} \begin{bmatrix} x & -h_{N-k}(z) \\ -g_k(z) & y \end{bmatrix} = \big(xy - f_N(z)\big) I_2$$

  This is in fact what was used above to describe Reid's pagoda cases: the two blowups are obtained from the two different matrices, and correspond to making the cokernel of the matrix locally free.

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This is in fact what was used above to describe Reid's pagoda cases: the two blowups are obtained from the two different matrices, and correspond to making the cokernel of the matrix locally free. That is, the two threefolds related by a flop are obtained by Grassmann blowups associated to $\operatorname{coker}(\Psi)$ and $\operatorname{coker}(\Phi)$.

Conjectures and Theorems

- *Conjecture 1*. For every length ℓ flop, there are two maximal Cohen–Macaulay modules M and M^+ on X of rank ℓ, such that Y (resp. Y^+) is the Grassmann blowup of M (resp. M^+).

- *Conjecture 2*. If the flop has length ℓ, then the matrix factorizations corresponding to M and M^+ are of size $2\ell \times 2\ell$, and are obtained from each other by switching the factors $(\Phi, \Psi) \to (\Psi, \Phi)$.

- *Conjecture 3.* For a partial resolution of a rational double point corresponding to a single vertex in the Dynkin diagram with coefficient ℓ in the maximal root, the versal deformation $\mathcal{X}$ over PRes has matrix factorizations of size $2\ell \times 2\ell$, such that a simultaneous partial resolution can be obtained as the Grassmann blowup of the corresponding Cohen–Macaulay module.

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- *Theorem.* The conjectures above hold for length 1 and 2.

- Note that we in fact gave (most of) the proof for length 1 in our discussion of the A_{N-1} case above.

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- *Expectation.* Bridgeland's categorical description of flops should be explicitly describable in terms of M and M^{+}.

- What about flips? We don't have a precise conjecture to make at the moment; however, we speculate that flips are obtained via equivariant matrix factorizations for some group action.

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- That is, the contracted space will be described in the form $(f = 0)/G$ and one will look for matrix factorizations (Ψ, Φ) which are G-equivariant. Notice that the group actions on the two factors Ψ and Φ are quite different, leading to the natural asymmetry one finds in a flip (as opposed to a flop).

- The conjectures above can be loosely stated as follows: the general elephant of a flop has a matrix factorization whose Grassmann blowup(s) realize the partial resolution, and that matrix factorization can be deformed to the total space.

- The conjectures above can be loosely stated as follows: the general elephant of a flop has a matrix factorization whose Grassmann blowup(s) realize the partial resolution, and that matrix factorization can be deformed to the total space. Hopefully, a similar statement of some kind is true for flips as well.

The universal flop of length 2

- We now describe a flop of length 2 which will turn out to be universal in a certain sense. Our description follows an idea of Reid although this was not the way we originally found the flop.

- We start with a quadratic equation in four variables x, y, z, t over the field $\mathbb{C}(u, v, w)$, chosen so that its discriminant is a perfect square. The one we use has matrix
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u & v & 0 \\ 0 & v & w & 0 \\ 0 & 0 & 0 & uw-v^2 \end{bmatrix}$$
whose determinant is $(uw - v^2)^2$. (Reid's matrix had $v = 0$.)

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whose determinant is $(uw - v^2)^2$. (Reid's matrix had $v = 0$.) The corresponding quadratic equation is

$$x^2 + uy^2 + 2vyz + wz^2 + (uw - v^2)t^2. \qquad (1)$$

- The quadratic in $\mathbb{C}^4$ has two rulings by $\mathbb{C}^2$; since the determinant is a perfect square, the individual rulings are already defined over $\mathbb{C}(u, v, w)$. In fact, the corresponding rank two sheaf is a maximal Cohen–Macaulay module over the hypersurface defined by equation (1) (which we now regard as a hypersurface in $\mathbb{C}^7$).

- Although the singularity of this equation is not isolated so that we cannot invoke Eisenbud's theorem, in fact this maximal Cohen–Macaulay module can be expressed in terms of a matrix factorization (Φ, Ψ).

- The one we will use has

$$\Phi = \begin{bmatrix} x - vt & y & -z & t \\ uy + 2vz & -x - vt & -ut & -z \\ wz & -wt & x - vt & y \\ uwt & wz & uy + 2vz & -x - vt \end{bmatrix}$$

and

$$\Psi = \begin{bmatrix} x + vt & y & z & t \\ uy + 2vz & -x + vt & -ut & z \\ -wz & -wt & x + vt & y \\ uwt & -wz & uy + 2vz & -x + vt \end{bmatrix}.$$

- For these matrices,

$$\Phi\Psi = \left(x^2 + uy^2 + 2vyz + wz^2 + (uw - v^2)t^2\right) I_4.$$

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- *Example.* If we set

$$x = v_4,\ y = iv_3,\ z = iv_2,\ t = v_1,$$

$$u = v_1, v = 0, w = \lambda v_1 + v_2$$

then we reproduce the Laufer–Pinkham–DRM example mentioned earlier.

- We do the Grassmann blowup corresponding to the cokernel of Ψ.

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- There are only two coordinate charts which are relevant for our computation. For the first, we introduce four new variables $A_{i,j}$ and eight equations $E_{i,j}$ by means of

$$E = \Psi \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

Note that by multiplying by Φ from the left, we see that the original equation is contained in the ideal generated by the $E_{i,j}$.

We then compute

$$zE_{1,1} + zE_{2,2} - tE_{2,1} + utE_{1,2}$$

$$= (z^2 + ut^2)(A_{1,1} + A_{2,2})$$

$$utE_{1,1} + utE_{2,2} + zE_{2,1} - uzE_{1,2}$$

$$= (z^2 + ut^2)(2v - uA_{1,2}u + A_{2,1})$$

so that the proper transform must include

$$E_1 = A_{1,1} + A_{2,2}$$

and

$$E_2 = 2v - uA_{1,2} + A_{2,1}$$

These allow us to eliminate $A_{1,1}$ and $A_{2,1}$, leaving as variables on this chart $A_{1,2}$ and $A_{2,2}$, as well as x, y, z, t, u, v, w.

We compute once again:

$$zE_{3,1} + utE_{3,2} - (xz + vzt)E_1 - yzE_2$$

$$+(utA_{1,2} - zA_{2,2})E_{2,2} + (-utA_{2,2} - uzA_{1,2} + 2vz)E_{1,2}$$
$$= -(z^2 + ut^2)(w + uA_{1,2}^2 - 2vA_{1,2} + A_{2,2}^2)$$

which implies that the proper transform must also include

$$E_3 = w + uA_{1,2}^2 - 2vA_{1,2} + A_{2,2}^2$$

In fact, the ideal is generated by $E_{1,2}$, $E_{2,2}$, E_1, E_2, and

E_3: we have

$$E_{1,1} = -E_{2,2} + zE_1 + tE_2$$

$$E_{2,1} = uE_{1,2} - utE_1 + zE_2$$

$$E_{3,1} = (uA_{1,2} - 2v)E_{1,2} + A_{2,2}E_{2,2} + (x + vt)E_1 + yE_2 - zE_3$$

$$E_{3,2} = -A_{1,2}E_{2,2} + A_{2,2}E_{1,2} - tE_3$$

$$E_{4,1} = A_{2,1}E_{2,2} + uA_{1,1}E_{1,2} + (2vz - uzA_{1,2} - utA_{2,2})E_1 +$$

$$(-zA_{2,2} + utA_{1,2})E_2 + utE_3$$

$$E_{4,2} = A_{2,2}E_{2,2} + uA_{1,2}E_{1,2} - zE_3$$

To study the second coordinate chart, we introduce four new variables $B_{i,j}$ and eight new equations $\widetilde{E}_{i,j}$ by means of

$$\widetilde{E} = \Psi \begin{bmatrix} 1 & 0 \\ B_{1,1} & B_{1,2} \\ 0 & 1 \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

Note that by multiplying by Φ from the left, we see that the original equation is contained in the ideal generated by the $\widetilde{E}_{i,j}$.

We then compute

$$y\widetilde{E}_{1,1} - t\widetilde{E}_{3,1} - wt\widetilde{E}_{1,2} - y\widetilde{E}_{3,2}$$

$$= (wt^2 + y^2)(B_{1,1} - B_{2,2})$$

and

$$wt\widetilde{E}_{1,1} + y\widetilde{E}_{3,1} + wy\widetilde{E}_{1,2} - wt\widetilde{E}_{3,2}$$
$$= (wt^2 + y^2)(B_{2,1} + wB_{1,2})$$

Thus, the proper transform must include

$$\widetilde{E}_1 = B_{1,1} - B_{2,2}$$

and

$$\widetilde{E}_2 = B_{2,1} + wB_{1,2}$$

These allow us to eliminate $B_{1,1}$ and $B_{2,1}$, leaving as variables on this chart $B_{1,2}$ and $B_{2,2}$, as well as x, y, z, t, u, v, w.

We compute once again:

$$y\widetilde{E}_{2,1}+(yB_{2,2}-wtB_{1,2})\widetilde{E}_{3,2}+(wyB_{1,2}-2vy+wtB_{2,2})\widetilde{E}_{1,2}$$

$$+(xy - vyt)\widetilde{E}_1 - yz\widetilde{E}_2 - wt\widetilde{E}_{2,2}$$
$$= (wt^2 + y^2)(u + B_{2,2}^2 + wB_{1,2}^2 - 2vB_{1,2})$$

and conclude that the proper transform must include

$$\widetilde{E}_3 = u + B^2_{2,2} + wB^2_{1,2} - 2vB_{1,2}$$

In fact, $\widetilde{E}_1$, $\widetilde{E}_2$, $\widetilde{E}_3$, $\widetilde{E}_{1,2}$ and $\widetilde{E}_{3,2}$ will generate the ideal, as we now demonstrate.

$$\widetilde{E}_{1,1} = \widetilde{E}_{3,2} + y\widetilde{E}_1 + t\widetilde{E}_2$$

$$\begin{aligned} \widetilde{E}_{2,1} = -B_{2,2}\widetilde{E}_{3,2} + (-wB_{1,2} + 2v)\widetilde{E}_{1,2} + y\widetilde{E}_3 \\ -(x - vt)\widetilde{E}_1 + z\widetilde{E}_2 \\ \widetilde{E}_{2,2} = -B_{1,2}\widetilde{E}_{3,2} + B_{2,2}\widetilde{E}_{1,2} - t\widetilde{E}_3 \end{aligned}$$

$$\widetilde{E}_{3,1} = -w\widetilde{E}_{1,2} - wt\widetilde{E}_1 + y\widetilde{E}_2$$

$$\widetilde{E}_{4,1} = wB_{1,2}\widetilde{E}_{3,2} - wB_{2,2}\widetilde{E}_{1,2} + wt\widetilde{E}_3 - wz\widetilde{E}_1 - (x - vt)\widetilde{E}_2$$

$$\widetilde{E}_{4,2} = -B_{2,2}\widetilde{E}_{3,2} + (-wB_{1,2} + 2v)\widetilde{E}_{1,2} + y\widetilde{E}_3$$

The D_n case

The versal deformation of D_n can be written in the form

$$X^2 + Y^2 Z - Z^{n-1} + 2\gamma Y - \sum_{i=1}^{n-1} \delta_{2i} Z^{n-1-i}$$

The invariant theory is based on the polynomial

$$Z^n + \sum_{i=1}^{n-1} \delta_{2i} Z^{n-i} + \gamma^2 = \prod_{j=1}^{n} (Z + t_j^2) \qquad (2)$$

The Weyl group $\mathfrak{W}_{D_n}$ is an extension of the symmetric group $\mathfrak{S}_n$ (acting to permute the t_j's) by a group $(\boldsymbol{\mu}_2)^{n-1}$. The latter is the subgroup of $(\boldsymbol{\mu}_2)^n$ (acting on the t_j's by coordinatewise multiplication) which preserves the product $t_1 \ldots t_n$. (This product coincides with γ in the polynomial.) We have $\delta_{2i} = \sigma_i(t_1^2, \ldots, t_n^2)$, with $\delta_0 = \gamma^2$.

Picking a vertex in most cases corresponds to D_{n-k} and A_{k-1}. (This holds for $0 \leq k \leq n-2$, where $k = 0$ means that we choose no vertex at all.) In the invariant theory, this means that we break the group action into $\mathfrak{S}_k \times \mathfrak{W}_{D_{n-k}}$.

The two polynomials which now capture the invariant theory are

$$f(U) = \prod_{j=1}^{k}(U - t_j) \text{ and } Zh(Z) + \eta^2 = \prod_{j=k+1}^{n}(Z + t_j^2)$$

(where η is the quantity such that $\eta = t_{k+1} \dots t_n$). To relate these to the original polynomial (2), we write

$$f(U) = Q(-U^2) + UP(-U^2)$$

and note that if $Z = -U^2$ then

$$\prod_{j=1}^{k} (Z + t_j^2) = f(U)f(-U) = Q(Z)^2 + ZP(Z)^2$$

so that

$$\prod_{j=1}^{n}(Z+t_j^2)=(Zh(Z)+\eta^2)(Q(Z)^2+ZP(Z)^2)$$

and $\gamma = \eta Q(0)$. Moreover, the relationship between $P(U)$, $Q(U)$, and the polynomials occuring in the invariant theory is captured by the existence of a polynomial in two variables $G(Z,U)$ satisfying

$$UP(Z)+Q(Z)=(Z+U^2)G(Z,U)+f(U).$$

One final manipulation: we write $Q(Z)=ZS(Z)+Q(0)$

when needed so that

$$Q(Z)^2 - Q(0)^2 = ZS(Z)[2Q(Z) - ZS(Z)].$$

Thus, the polynomial

$$F(Z) = \frac{1}{Z}\left(-\gamma^2 + \prod_{j=1}^{n}(Z + t_j^2)\right)$$

which appears in the versal deformation of D_n can be rewritten as

$$F(Z) = h(Z)[Q(Z)^2 + ZP(Z)^2] + \eta^2[2S(Z)2Q(Z) - ZS(Z)^2 + P(Z)^2]$$

We're now ready for a matrix factorization. We use the universal length two matrix factorization, with

$$x = X,\ y = Y - \eta S(Z),\ z = Q(Z),\ t = P(Z),$$

$$u = Z,\ v = 2\eta,\ w = -h(Z).$$

In the first chart, after eliminating $A_{1,1}$ and $A_{2,1}$, the proper transform is defined by the ideal $(E_3, E_{1,2}, E_{2,2})$ where

$$E_{1,2} = Y - \eta S(Z) + Q(Z)A_{1,2} + P(Z)A_{2,2}$$

allows us to eliminate Y, and

$$E_{2,2} = -X + \eta P(Z) - ZP(Z)A_{1,2} + Q(Z)A_{2,2}$$

allows us to elminate X. The remaining generator is

$$E_3 = ZA_{1,2}^2 - 2\eta A_{1,2} + A_{2,2}^2 - h(Z)$$

which is the versal deformation of a D_{n-k} singularity.

In the other chart, after eliminating $B_{1,1}$ and $B_{2,1}$, the proper transform is defined by the ideal $(\widetilde{E}_3, \widetilde{E}_{1,2}, \widetilde{E}_{3,2})$. We have

$$\widetilde{E}_{3,2} = X + \eta P(Z) + h(Z)P(Z)B_{1,2} + (Y - \eta S(Z))B_{2,2}$$

which allows us to eliminate X on this chart, but

$$\widetilde{E}_{1,2} = Q(Z) + (Y - \eta S(A))B_{1,2} + P(Z)B_{2,2}$$

does not immediately allow elimination. The third generator is

$$\widetilde{E}_3 = Z + B_{2,2}^2 - h(Z)B_{1,2}^2 - 2\eta B_{1,2}$$

To understand the geometry of this chart, we form the combination

$$\widetilde{E}_{1,2} - G(Z, B_{2,2})\widetilde{E}_3 = (Y - \eta S(Z))B_{1,2} + G(Z, B_{2,2})h(Z)B_{1,2}^2$$

$$+2\eta G(Z, B_{2,2})B_{1,2} + f(B_{2,2})$$

$$= (Y - \eta S(Z) + G(Z, B_{2,2})h(Z)B_{1,2} + 2\eta G(Z, B_{2,2}))B_{1,2}$$

$$+f(B_{2,2})$$

so introducing the variable

$$\widetilde{Y} = Y - \eta S(Z) + G(Z, B_{2,2})h(Z)B_{1,2} + 2\eta G(Z, B_{2,2})$$

we see that this forms a versal deformation of an A_{k-1} singularity

$$\widetilde{Y} B_{1,2} + f(B_{2,2})$$

since $f(U)$ is monic of degree k. Note that Z can be implicitly eliminated using $\widetilde{E}_3$.