1a

$$\vec{y}\,' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{y}$$

The eigenvalues of $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ are $-1, -2$ with respective eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

So the general solution is

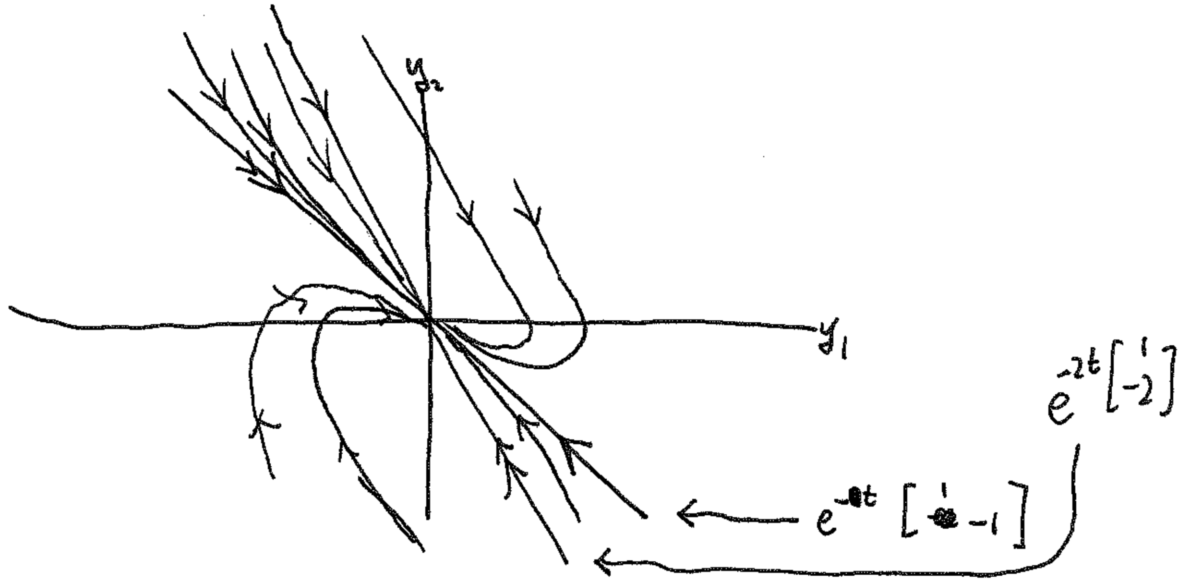
$$\boxed{\vec{y}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.}$$

In other words,

$$y_1 = c_1 e^{-t} + c_2 e^{-2t}$$
$$y_2 = -c_1 e^{-t} + -2 c_2 e^{-2t}.$$

Eigenvalues distinct, negative $\Rightarrow$ $\boxed{\text{stable node.}}$

The phase portrait is



(1b) $\vec{y}' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \vec{y}$.

eigenvalues: $\lambda_1 = 3 \qquad \lambda_2 = 2$

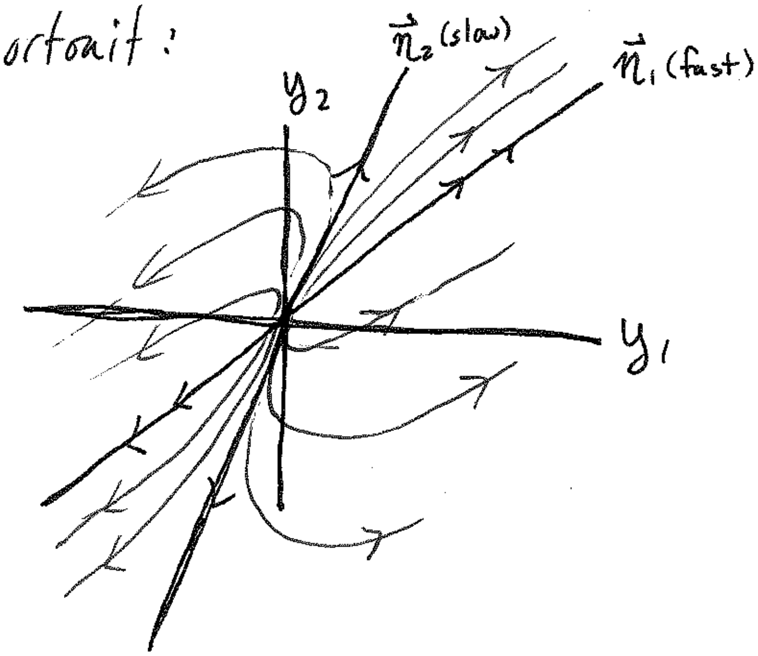
eigenvectors: $\vec{\eta}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{\eta}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Since $\lambda_1 > \lambda_2 > 0$, $\boxed{\text{unstable node.}}$

General solution:

$$y(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Phase portrait:



1c

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$ Let $\vec{y}$ denote $\begin{bmatrix} x \\ y \end{bmatrix}$.

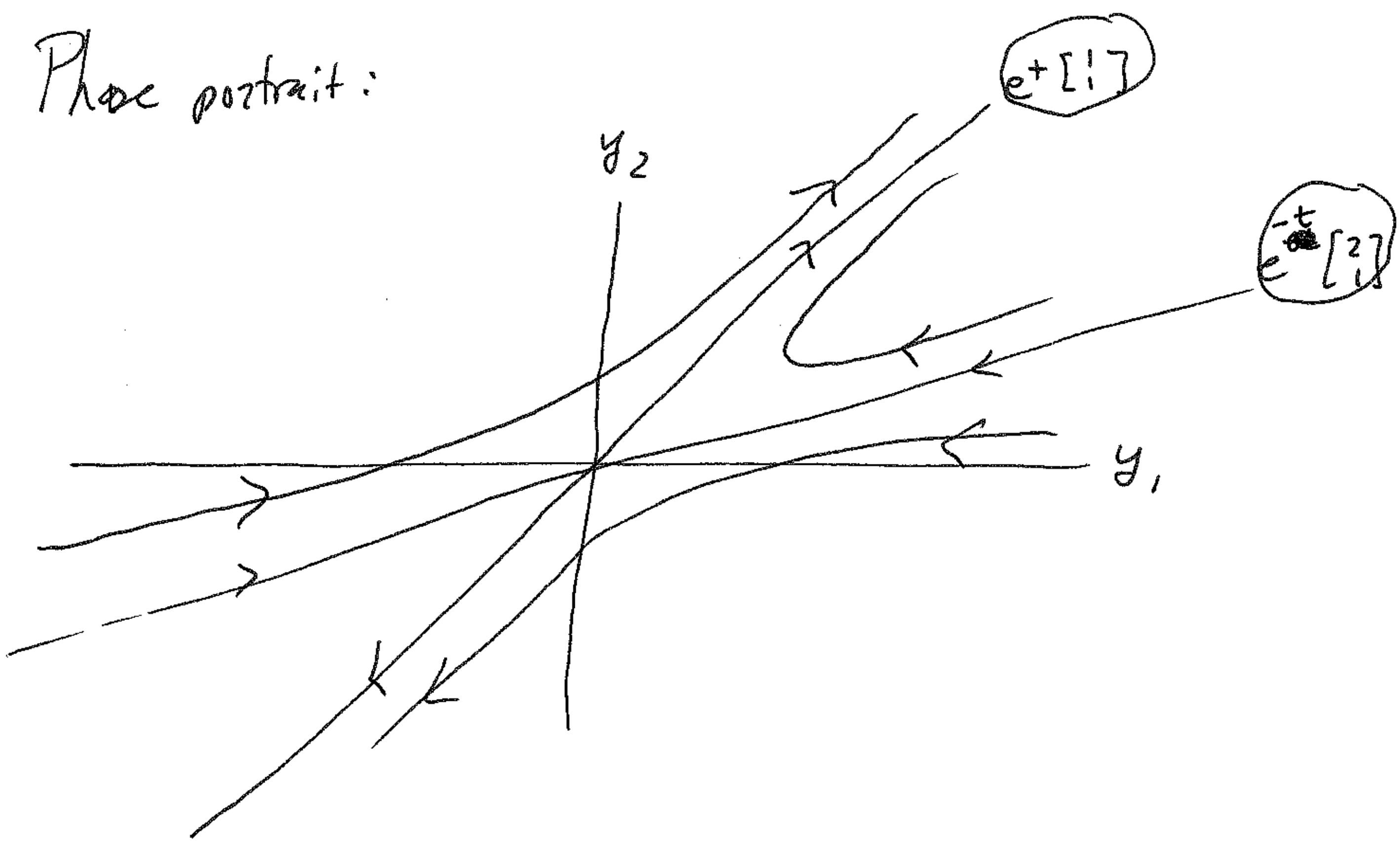
eigenvalues: $\lambda_1 = 1$ $\qquad \lambda_2 = -1$

eigenvectors: $\vec{\eta}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad \vec{\eta}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Since $\lambda_2 < 0 < \lambda_1$ the origin will be a saddle point.

solution: $\vec{y}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Phase portrait:



1d

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$ Let $\vec{y}$ denote $\begin{bmatrix} x \\ y \end{bmatrix}$.

Eigenvalues are $2 \pm i$.

$\lambda_1 = 2 + i$ $\qquad\qquad$ $\lambda_2 = 2 - i$

eigenvectors: $\vec{\eta}_1 = \begin{bmatrix} -3-i \\ 1 \end{bmatrix}$ $\qquad$ $\vec{\eta}_2 = \begin{bmatrix} -3+i \\ 1 \end{bmatrix}$

One complex solution is

$$\vec{y}(t) = e^{(2+i)t} \begin{bmatrix} -3-i \\ 1 \end{bmatrix}$$

$$= e^{2t} (\cos t + i \sin t)\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

$$= e^{2t}\left[\left(\cos t \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) + i \left(\cos t \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)\right]$$

Picking out Re and Im gives two lin. ind. real solutions.

So the general solution is

$$\boxed{\begin{aligned}\vec{y}(t) = {} & c_1 e^{2t} \left(\cos t \begin{bmatrix} -3 \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ & + c_2 e^{2t}\left(\cos t \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)\end{aligned}}$$

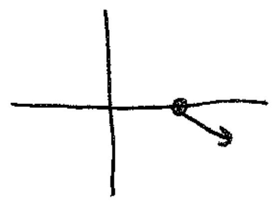
Since $Re(\lambda_1) > 0$, this is an unstable spiral.

To sketch the spiral, we should find out which way it rotates...

Compute $\vec{y}\,'$ at some test points in phase space (using the ODE system), and it should become obvious.

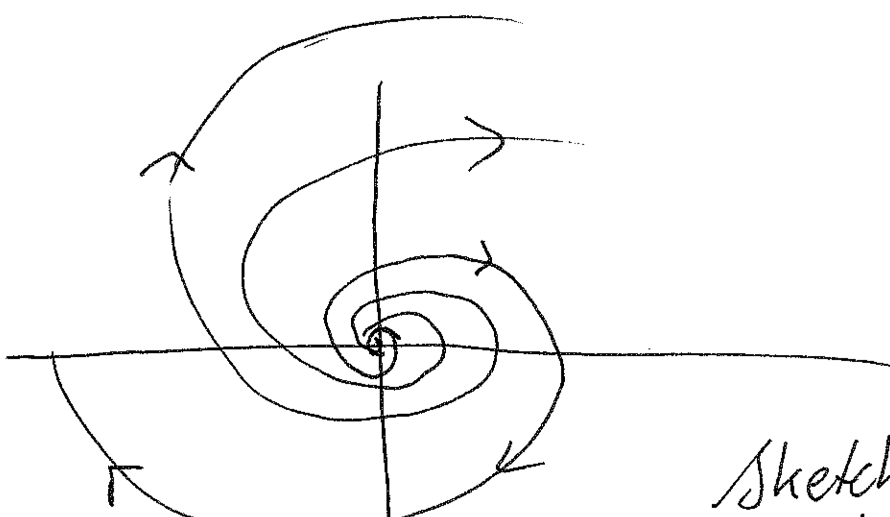
At $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ -1 \end{bmatrix}$. So



.

Seems like a clockwise outward spiral about the origin



Sketches are not exact, of course. They show qualitative features.

(2)

Define $v = y'$.

Then $v' = y''$. Sub into ODE:

$$f_2(t)\, v' + f_1(t)\, v + f_0(t)\, y = g(t)$$

Solve for v':

$$v' = \frac{g(t)}{f_2(t)} - \frac{f_1(t)}{f_2(t)} v - \frac{f_0(t)}{f_2(t)} y$$

And the other ODE in the system is

$$y' = v.$$

In matrix format, we can express the ODE system as

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} v \\ g/f_2 - f_1/f_2\, v - f_0/f_2\, y \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -f_0/f_2 & -f_1/f_2 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ g/f_2 \end{bmatrix}.$$

3

$$y'' + \omega_0^2 y = 0$$

Define $v = y'$.

ODE becomes 1^{st} order system:

$$v' + \omega_0^2 y = 0 \quad \ldots \quad v' = -\omega_0^2 y$$

$$\begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} v \\ -\omega_0^2 y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}$$

Let $\vec{y}$ denote $\begin{bmatrix} y \\ v \end{bmatrix}$.

$$\vec{y}\,' = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \vec{y}.$$

Eigenvalues are $\lambda_1 = i\omega_0$, $\lambda_2 = -i\omega_0$.

Eigenvectors: $\vec{\eta}_1 = \begin{bmatrix} 1 \\ i\omega_0 \end{bmatrix}$, $\vec{\eta}_2 = \begin{bmatrix} 1 \\ -i\omega_0 \end{bmatrix}$

One complex solution is

$$\vec{y}(t) = e^{\lambda_1 t}\vec{\eta}_1 = (\cos(\omega_0 t) + i\sin(\omega_0 t))\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ \omega_0 \end{bmatrix}\right)$$

$$= \left(\cos(\omega_0 t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(\omega_0 t)\begin{bmatrix} 0 \\ \omega_0 \end{bmatrix}\right)$$

$$+ i\left(\cos(\omega_0 t)\begin{bmatrix} 0 \\ \omega_0 \end{bmatrix} + \sin(\omega_0 t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \cos\omega_0 t \\ -\omega_0 \sin(\omega_0 t) \end{bmatrix} + i\begin{bmatrix} \sin(\omega_0 t) \\ \omega_0 \cos(\omega_0 t) \end{bmatrix}$$

Picking out Re and Im parts gives two lin. ind. real solutions:

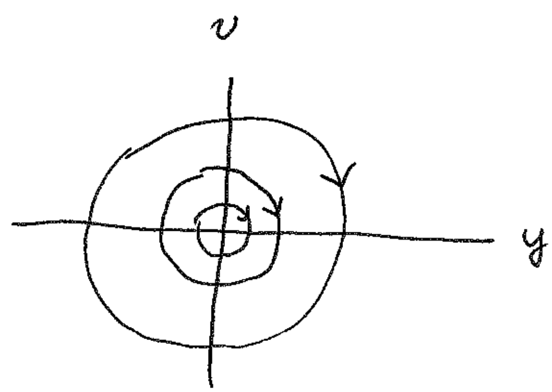
$$\vec{y}(t) = \begin{bmatrix} \cos(\omega_0 t) \\ -\omega_0 \sin(\omega_0 t) \end{bmatrix}$$

$$\vec{y}(t) = \begin{bmatrix} \sin(\omega_0 t) \\ \omega_0 \cos(\omega_0 t) \end{bmatrix}$$

General solution is built out of their linear combinations:

$$\vec{y}(t) = C_1 \begin{bmatrix} \cos(\omega_0 t) \\ -\omega_0 \sin(\omega_0 t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(\omega_0 t) \\ \omega_0 \cos(\omega_0 t) \end{bmatrix}$$

Since the eigenvalues were pure imaginary, the origin is a center. Note that when $\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{y}' = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so rotation is clockwise:



(4) $y'' + 2ry' + \omega_0^2 y = 0.$

Recall that $r>0$ for damped oscillator.

Define $v = y'$. Then $v' + 2rv + \omega_0^2 y = 0$.

So ODE system is

$$y' = v$$
$$v' = -2rv - \omega_0^2 y$$

Defining $\vec{y} = \begin{bmatrix} y \\ v \end{bmatrix}$,

$$\vec{y}\,' = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2r \end{bmatrix}$$

Let's compute eigenvalues carefully.

Characteristic polynomial is:

$$(-\lambda)(-\lambda - 2r) + \omega_0^2$$
$$= \lambda^2 + 2r\lambda + \omega_0^2$$

Its roots are $\dfrac{-2r \pm \sqrt{4r^2 - 4\omega_0^2}}{2} = -r \pm \sqrt{r^2 - \omega_0^2}$

We have

distinct real eigenvalues if $r > \omega_0$

repeated eigenvalues if $r = \omega_0$

complex eigenvalues if $r < \omega_0$.

Let's deal with each:

<u>Distinct real eigenvalues ($r > \omega_0$)</u>

$$\lambda_1 = -r + \sqrt{r^2 - \omega_0^2} \qquad \lambda_2 = -r - \sqrt{r^2 - \omega_0^2}$$

Compute eigenvectors to be:

$$\vec{\eta}_1 = \begin{bmatrix} 1 \\ -r + \sqrt{r^2 - \omega_0^2} \end{bmatrix}$$

$$\vec{\eta}_2 = \begin{bmatrix} 1 \\ -r - \sqrt{r^2 - \omega_0^2} \end{bmatrix}$$

Solution:

$$\vec{y}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = c_1 e^{(-r + \sqrt{r^2 - \omega_0^2})t} \begin{bmatrix} 1 \\ -r + \sqrt{r^2 - \omega_0^2} \end{bmatrix}$$

$$+ \; c_2 e^{(-r - \sqrt{r^2 - \omega_0^2})t} \begin{bmatrix} 1 \\ -r - \sqrt{r^2 - \omega_0^2} \end{bmatrix}$$

Since $\lambda_2 < \lambda_1 < 0$, this is a <u>stable node</u>.

This is the <u>overdamped</u> case.

<u>Repeated eigenvalues:</u> $(r = \omega_0)$

$$\lambda_* = -r$$

Compute eigenvectors carefully:

$$\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2r \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} r & 1 \\ -\omega_0^2 & -r \end{bmatrix} = \begin{bmatrix} r & 1 \\ -r^2 & -r \end{bmatrix}$$

The nullspace of this is just the span of $\begin{bmatrix} 1 \\ -r \end{bmatrix}$ ...

NOT ENOUGH EIGENVECTORS!

The picture is a <u>degenerate node</u>.

We haven't yet learned how to write the general solution from the ODE system perspective.

This is the <u>critically damped case</u>

complex eigenvalues: $(r < \omega_0)$

$$\lambda = -r \pm i\sqrt{\omega_0^2 - r^2}$$

Let's focus on $\lambda = -r + i\sqrt{\omega_0^2 - r^2}$.

An eigenvector for that is

$$\vec{\eta} = \begin{bmatrix} 1 \\ -r + i\sqrt{\omega_0^2 - r^2} \end{bmatrix}$$

So one complex solution is

$$\vec{y}(t) = e^{(-r + i\sqrt{\omega_0^2 - r^2})t} \begin{bmatrix} 1 \\ -r + i\sqrt{\omega_0^2 - r^2} \end{bmatrix}$$

$$= e^{-rt}\left[\cos(\sqrt{\omega_0^2 - r^2}\,t) + i\sin(\sqrt{\omega_0^2 - r^2}\,t)\right]\left(\begin{bmatrix} 1 \\ -r \end{bmatrix} + i\begin{bmatrix} 0 \\ \sqrt{\omega_0^2 - r^2} \end{bmatrix}\right)$$

$$= e^{-rt}\left[\cos(\sqrt{\omega_0^2 - r^2}\,t)\begin{bmatrix} 1 \\ -r \end{bmatrix} - \sin(\sqrt{\omega_0^2 - r^2}\,t)\begin{bmatrix} 0 \\ \sqrt{\omega_0^2 - r^2} \end{bmatrix} + i\left[\cos(\sqrt{\omega_0^2 - r^2}\,t)\begin{bmatrix} 0 \\ \sqrt{\omega_0^2 - r^2} \end{bmatrix} + \sin(\sqrt{\omega_0^2 - r^2}\,t)\begin{bmatrix} 1 \\ -r \end{bmatrix}\right]\right]$$

The general solution is:

$$\boxed{\begin{aligned} \vec{y}(t) = {} & c_1 e^{-rt}\begin{bmatrix} \cos(\sqrt{\omega_0^2 - r^2}\,t) \\ -r\cos(\sqrt{\omega_0^2 - r^2}\,t) - \sqrt{\omega_0^2 - r^2}\sin(\sqrt{\omega_0^2 - r^2}\,t) \end{bmatrix} \\ & + c_2 e^{-rt}\begin{bmatrix} \sin(\sqrt{\omega_0^2 - r^2}\,t) \\ \sqrt{\omega_0^2 - r^2}\cos(\sqrt{\omega_0^2 - r^2}\,t) - r\sin(\sqrt{\omega_0^2 - r^2}\,t) \end{bmatrix} \end{aligned}}$$

Since $\mathrm{Re}(\lambda) < 0$, this is ~~[illegible]~~ a stable spiral. (underdamped case)

5

$$\theta'' + \frac{g}{\ell}\sin\theta = 0$$

Define $\omega = \theta'$.

$\theta' = \omega$ is one ODE. The other is

$$\omega' + \frac{g}{\ell}\sin\theta = 0, \quad \text{so } \omega' = -\frac{g}{\ell}\sin\theta.$$

The ODE system is

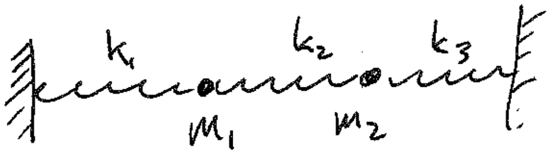
$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \underbrace{\begin{bmatrix}\omega \\ -g/\ell \, \sin\theta\end{bmatrix}}.$$

This could never be written as a matrix times $\begin{bmatrix}\theta \\ \omega\end{bmatrix}$ ...

the ODE ~~system~~ system is <u>nonlinear</u>!

(Just like the ODE that produced it).

(6)



First, what are the dynamical variables?

We need <u>coordinates</u>.

Here are some nice, though not the only, ones:

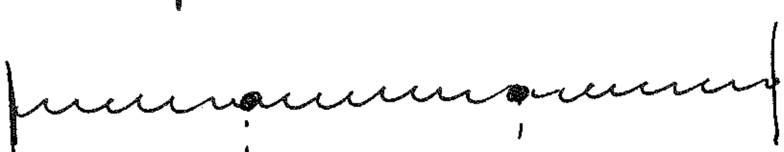
Let x_1 be the displacement of m_1 from equilibrium.

Let x_2 be the displacement of m_2 from equilibrium.

"Equilibrium" here means the state in which all springs are at their natural length; such a state exists due to the assumption in the problem. Also

Let positive x_1, x_2 be to the right in the picture.

So for example if



is equilibrium, then

is represented by $x_1 = 2$ and $x_2 = -1$.

That's the hard part, having clean coordinates.

Now, we just calculate.

The forces on m_1 are

$$\cancel{-k_1 x_1 + k_2(x_2 - x_1)}$$

~~The distance by which the left spring is compressed~~

$$+k_1(-x_1) - k_2(x_1 - x_2)$$

The distance by which the left spring is compressed from its natural length.

The distance by which the middle spring is compressed from its natural length.

Getting the signs right is kind of tricky!

The forces on m_2 are

$$-k_3 x_2 + k_2(x_1 - x_2)$$

distance by which right spring is compressed from natural length

So we have a 2^{nd} order ODE system:

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 - k_2 x_1 + k_2 x_2 = -(k_1+k_2)x_1 + k_2 x_2$$

$$m_2 \frac{d^2 x_2}{dt^2} = -k_3 x_2 + k_2 x_1 - k_2 x_2 = k_2 x_1 - (k_2+k_3) x_2$$

To write a 1^{st} order ODE system, define $v_1 = \frac{dx_1}{dt}$, $v_2 = \frac{dx_2}{dt}$.

Then

$$m_1 \frac{dv_1}{dt} = -(k_1+k_2)x_1 + k_2 x_2$$

$$m_2 \frac{dv_2}{dt} = k_2 x_1 - (k_2+k_3) x_2$$

The ODE system is then:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix}.$$

7a

$$A = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix}.$$

Characteristic polynomial:

$$\det\left(\begin{bmatrix} \lambda-\alpha & b \\ 0 & \lambda-\alpha \end{bmatrix}\right) = (\lambda-\alpha)^2$$

(as a polynomial in α ... λ was annoyingly taken as a variable)

Roots of characteristic polynomial are $\alpha = \lambda$, repeated.

The eigenvectors that go with eigenvalue λ are the nonzero vectors in the nullspace of

$$\begin{bmatrix} \lambda-\lambda & b \\ 0 & \lambda-\lambda \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix},$$ which

is just $\text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ ... it's

one-dimensional!

7b

The system is

$$y_1' = \lambda y_1 + b y_2$$

$$y_2' = \lambda y_2$$

← This can be solved directly.

We have $y_2 = c_1 e^{\lambda t}$ in any solution.

Substituting that in, the first ODE becomes

$$y_1' = \lambda y_1 + c_1 e^{\lambda t}$$

This is linear, 1st order, nonhomogeneous.

We can use the method of integrating factors:

$$y_1' - \lambda y_1 = c_1 e^{\lambda t}$$

$e^{-\lambda t}$ seems like a good integrating factor:

$$e^{-\lambda t} y_1' - \lambda e^{-\lambda t} y_1 = c_1$$

$$\frac{d}{dt}\left(e^{-\lambda t} y_1\right) = c_1$$

$$e^{-\lambda t} y_1 = c_1 t + c_2$$

$$y_1 = c_1 t e^{\lambda t} + c_2 e^{\lambda t}$$

General solution is $\boxed{\vec{y}(t) = c_1 e^{\lambda t} \begin{bmatrix} t \\ 1 \end{bmatrix} + c_2 e^{\lambda t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

8a

$$R' = aJ \qquad\qquad a, b > 0.$$
$$J' = -bR$$

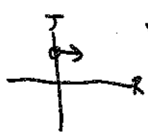
$$\begin{bmatrix} R' \\ J' \end{bmatrix} = \begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}.$$

Char. poly of $\begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix}$ is $\lambda^2 + ab = 0$.

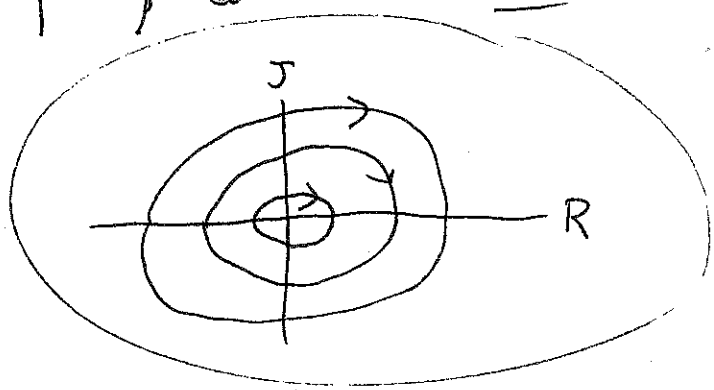
So eigenvalues are pure imaginary.

Hence phase portrait shows the origin is a <u>center</u>.

To find the direction of rotation, note that when $\begin{bmatrix} R \\ J \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} R' \\ J' \end{bmatrix} = \begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$

(so 

) ~~ccw~~ . . . <u>clock</u>wise.



<u>endless love/hate cycles !</u>

8b

$$R' = -aR + bJ \qquad a, b > 0$$
$$J' = bR - aJ$$

$$\begin{bmatrix} R' \\ J' \end{bmatrix} = \begin{bmatrix} -a & b \\ b & -a \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$$

Char. poly. is $(-a-\lambda)^2 - b^2 = \lambda^2 + 2a\lambda + (a^2-b^2)$

Roots are $\dfrac{-2a \pm \sqrt{4a^2 - 4(a^2-b^2)}}{2} = -a \pm |b| = -a \pm b$

So eigenvalues are $\lambda_1 = -a + b \qquad \lambda_2 = -a - b$.

λ_2 is negative no matter what.

If $a > b$, then λ_1 is also negative and we get a stable node.

(If the lovers are both too cautious, the relationship fizzles out eventually, no matter the initial conditions).

If $a < b$, then λ_1 is positive and we get a saddle point.

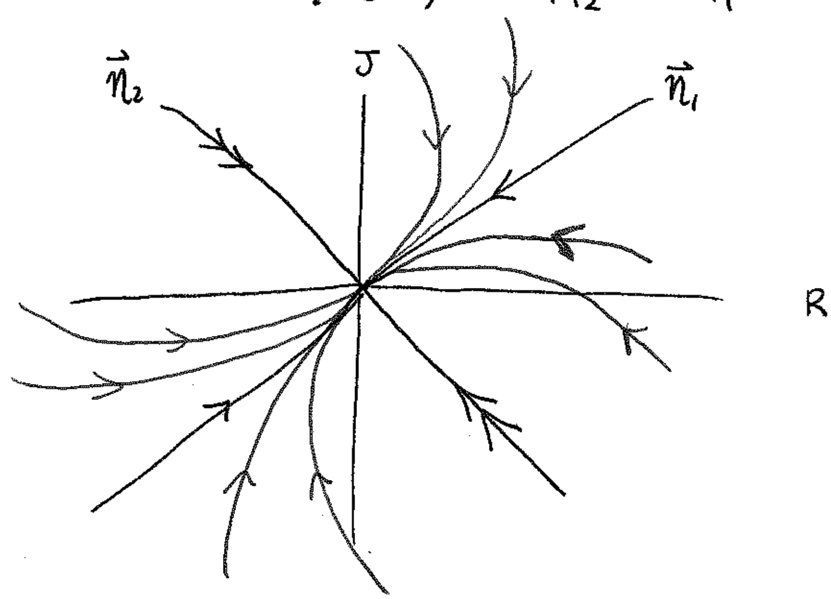
(If the lovers are both not cautious enough, the relationship is explosive — we need eigenvectors to see exactly in what way it is explosive).

Let's draw phase portraits.

We compute the eigenvectors to be

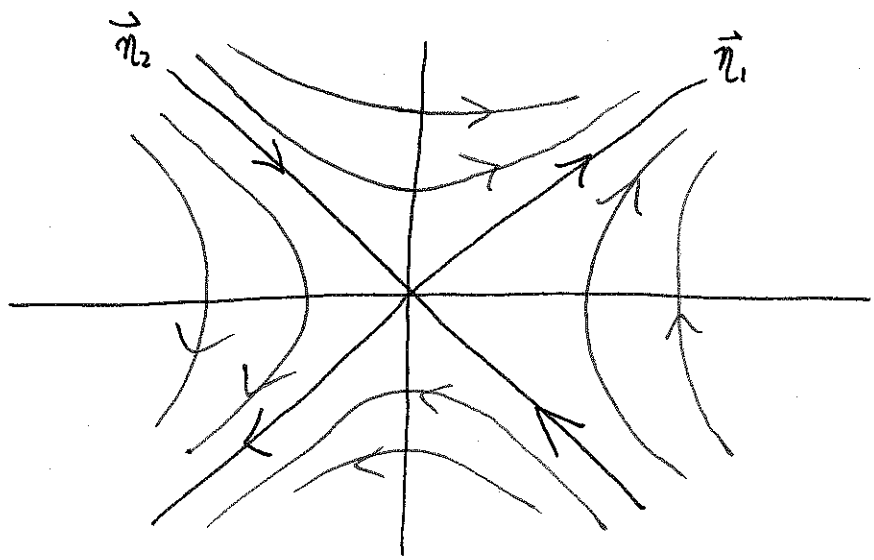
$$\vec{\eta}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{\eta}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

both too cautious; $a > b$; $\lambda_2 < \lambda_1 < 0$



relationship fizzles out no matter what

both too sensitive; $a < b$; $\lambda_2 < 0 < \lambda_1$



love fest or war, depending on initial conditions.