Method of Integrating Factors

Recall that a first order ODE is (usually) of the form:

$$Q(x,y)y' + P(x,y) = 0.$$

We already saw how to deal with one type, the *separable* ones (the ones that can be put into the form $Q(y)y' + P(x) = 0$). Now we will see how to address another type, the *linear* ODEs. A first order ODE is *linear* if it can be put into the form

$$y' + p(x)y = g(x).$$

Note that such an ODE could be non-separable. For example,

$$y' + y = x$$

is not separable, but it is linear.

The method of integrating factors is a technique that can help you solve first order linear ODEs. Summarizing the trick:

> Multiply the ODE by a cleverly chosen function $\mu(x)$, called an integrating factor, such that the $y' + p(x)y$ part of it becomes the derivative of the product μy. Then solve by integration.

Let's apply the idea to an example (one from lecture), and then do it in general after. Consider the ODE

$$xy' + 3y = \frac{\sin(x)}{x^2}.$$

It is linear, but you can't quite tell yet. Divide by x to make it clear:

$$y' + \frac{3}{x}y = \frac{\sin(x)}{x^3}. \tag{1}$$

This is the form we want to start with: a y' plus a function of x times a y equals a function of x. Okay, now for the integrating factor trick. Let $\mu(x)$ denote the function we are about to cleverly choose. It is called an *integrating factor*. We haven't chosen it yet, I just want to show you what μ needs to achieve. Multiply the ODE by $\mu(x)$ to get

$$\mu y' + \frac{3}{x}\mu y = \frac{\sin(x)}{x^3}\mu. \tag{2}$$

The goal is to choose $\mu(x)$ so that the left hand side is $(\mu y)'$. Now

$$(\mu y)' = \mu y' + \mu' y,$$

so judging by (2), we need $\mu(x)$ to satisfy

$$\mu' = \frac{3}{x}\mu. \tag{3}$$

This is a separable first order ODE, and solving it is how we find $\mu(x)$!

Try solving it. You should get $\mu(x) = x^3$. Actually you get a one-parameter family $\mu(x) = cx^3$, but in the end all you need is a single μ that makes the trick work; that is, all you need is a single μ that satisifes (3). So any particular solution from the family would do.

Now for the payoff. Multiply our ODE (1) by $\mu(x) = x^3$:

$$x^3 y' + 3x^2 y = \sin(x).$$

The left hand side, by design, is $(x^3 y)'$. So the ODE is now

$$(x^3 y)' = \sin(x).$$

Integrate both sides with respect to x to get a 1-parameter family of solutions:

$$x^3 y = -\cos(x) + c.$$

The explicit solution would be

$$y = \frac{c - \cos(x)}{x^3}.$$

And that's the method of integrating factors.

Okay that was one example, and they all work pretty much the same way, but let's solve the problem in general to get a formula. In general a linear first order ODE looks like $y' + p(x)y = g(x)$. Multiplying by $\mu(x)$ to see how to choose it, we get

$$\mu y' + \mu p = \mu g.$$

In order to get the left hand side to be $(\mu y)'$, which is $\mu y' + \mu' y$, we need to choose μ such that

$$\mu' = \mu p.$$

This is a separable ODE:

$$\frac{d\mu}{\mu} = p(x)dx.$$

Integrating,

$$\log(\mu) = \int p(x)dx.$$

Solving for μ,

$$\mu(x) = e^{\int p(x)dx}.$$

Multiply the ODE $y' + p(x)y = g(x)$ by the integrating factor:

$$e^{\int p(x)dx} y' + p(x) e^{\int p(x)dx} y = e^{\int p(x)dx} g(x).$$

Rewrite with the left hand side being the derivative of a product:

$$\left(e^{\int p(x)dx} y\right)' = e^{\int p(x)dx} g(x).$$

Integrate both sides with respect to x:

$$e^{\int p(x)dx} y = \int \left(e^{\int p(x)dx} g(x) \right) dx.$$

And we've once and for all solved all possible linear first order ODEs!

$$y = e^{-\int p(x)dx} \int \left(e^{\int p(x)dx} g(x) \right) dx. \tag{4}$$

So it's up to you how you want to approach linear first order ODEs: you can identify the $p(x)$ and $g(x)$ and plug them into (4), or you can do out the integrating factor trick every time like I did in the example above. Remember when you're computing the integrals in (4) that the large integral results in an arbitrary constant "$+c$" to give you a family of solutions, while the small integrals $\int p(x)dx$ were just there to compute an integrating factor and therefore don't need to have an arbitrary constant. So actually, if I include the "$+c$" in the notation right now, (4) becomes

$$y = e^{-\int p(x)dx} \int \left(e^{\int p(x)dx} g(x) \right) dx + ce^{-\int p(x)dx}.$$