(9)

First, notice that

$$(PAP^{-1})^2 = PAP^{-1}PAP^{-1} = PAIAP^{-1} = PA^2P^{-1}$$

$$(PAP^{-1})^3 = PA^2P^{-1}PAP^{-1} = PA^2IAP^{-1} = PA^3P^{-1}$$

and so on, so

$$(PAP^{-1})^n = PA^nP^{-1}$$

for any n. Using the definition of e^A,

$$e^{PAP^{-1}} = I + PAP^{-1} + \frac{1}{2!}(PAP^{-1})^2 + \frac{1}{3!}(PAP^{-1})^3 + \cdots$$

$$= I + PAP^{-1} + \frac{1}{2!}PA^2P^{-1} + \frac{1}{3!}PA^3P^{-1} + \cdots$$

$$= P\left[I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots\right]P^{-1}$$

$$= Pe^AP^{-1}.$$

(10)

From the definition,

$$e^{tA} = I + tA + \frac{1}{2!} t^2 A^2 + \frac{1}{3!} t^3 A^3 + \cdots$$

so

$$\frac{d}{dt} e^{tA} = 0 + A + \frac{2}{2!} t A^2 + \frac{3}{3!} t^2 A^3 + \cdots$$

$$= A + \frac{1}{1!} t A^2 + \frac{1}{2!} t^2 A^3 + \cdots$$

$$= A \left(I + \frac{1}{1!} tA + \frac{1}{2!} t^2 A^2 + \cdots \right)$$

$$= A e^{tA} .$$

11 $A = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$.

eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 3$.

eignvectors are $\vec{\eta}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{\eta}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Let $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\begin{matrix} \uparrow & \uparrow \\ \vec{\eta}_1 & \vec{\eta}_2 \end{matrix}$$

Now $P^{-1} A P = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, $A = P \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} P^{-1}$.

So $e^A = e^{P\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} P^{-1}} = P e^{\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}} P^{-1} = P \begin{bmatrix} e^5 & 0 \\ 0 & e^3 \end{bmatrix} P^{-1}$.

Compute that $P^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$.

So $$e^A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^5 & 0 \\ 0 & e^3 \end{bmatrix} \left(\frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -e^5 & e^5 \\ e^3 & e^3 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^5 + e^3 & -e^5 + e^3 \\ -e^5 + e^3 & e^5 + e^3 \end{bmatrix}.$$

(12)

eigenvalues: $\pm it$. $\lambda_1 = it$, $\lambda_2 = -it$

eigenvectors: $\vec{\eta}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$, $\vec{\eta}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

Let $P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$

Now $\begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} = P \begin{bmatrix} it & 0 \\ 0 & -it \end{bmatrix} P^{-1}$.

Compute that $P^{-1} = \frac{1}{2}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$.

So $e^{\begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}} = e^{P\begin{bmatrix} it & 0 \\ 0 & -it \end{bmatrix}P^{-1}}$

$$= P\, e^{\begin{bmatrix} it & 0 \\ 0 & -it \end{bmatrix}} P^{-1}$$

$$= \frac{1}{2}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} ie^{it} & e^{it} \\ -ie^{-it} & e^{-it} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{it} + e^{it} & -i(e^{it} - e^{-it}) \\ i(e^{it} - e^{-it}) & e^{it} + e^{-it} \end{bmatrix}$$

Notice that $e^{it} + e^{-it} = 2\cos(t)$

and $e^{it} - e^{-it} = 2i\sin(t)$, so

$$e^{\begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

(13) The ODE system is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $A = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} x \\ y \end{bmatrix}$, so the system is

$$\vec{y}\,' = A\vec{y}.$$

The general solution is

$$\vec{y}(t) = e^{tA} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

We just need to compute e^{tA}.

From exercise 11, $A = P \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} P^{-1}$, where

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

So

$$\begin{aligned} e^{tA} &= e^{tP\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}P^{-1}} \\ &= P\, e^{t\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}} P^{-1} \\ &= \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{5t} + e^{3t} & -e^{5t} + e^{3t} \\ -e^{5t} + e^{3t} & e^{5t} + e^{3t} \end{bmatrix}. \end{aligned}$$

The general solution of the ODE system is

$$\boxed{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{3t} + e^{5t} & e^{3t} - e^{5t} \\ e^{3t} - e^{5t} & e^{3t} + e^{5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{t\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is the answer, but we must compute it.
In exercise 13, we found that

$$e^{t\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}} = \frac{1}{2}\begin{bmatrix} e^{3t}+e^{5t} & e^{3t}-e^{5t} \\ e^{3t}-e^{5t} & e^{3t}+e^{5t} \end{bmatrix}$$

So the particular solution we want is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{3t}+e^{5t} & e^{3t}-e^{5t} \\ e^{3t}-e^{5t} & e^{3t}+e^{5t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3e^{3t} - e^{5t} \\ 3e^{3t} + e^{5t} \end{bmatrix}.$$

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The ODE system is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} x \\ y \end{bmatrix}$, so the system is

$$\vec{y}\,' = A\vec{y}.$$

The general solution is

$$\vec{y}(t) = e^{tA} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

We already computed e^{tA} in exercise 12:

$$e^{tA} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.$$

So

$$\vec{y}(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

is the general solution.

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The answer is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We computed in exercise ~~15~~ 12 that

$$e^{t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

so the particular solution we want is:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sin(t) - 2\cos(t) \\ 2\sin(t) + \cos(t) \end{bmatrix}$$

17 Let $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$.

Let's try to carefully compute eigenvalues and eigenvectors:

$$\det \begin{bmatrix} 5-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} = \lambda^2 - 8\lambda + 16$$
$$= (\lambda - 4)^2$$

We get repeated eigenvalues, $\lambda_1 = 4$, $\lambda_2 = 4$.

Now for eigenvectors:

$$\begin{bmatrix} 5-4 & -1 \\ 1 & 3-4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Nullspace is $\text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\ldots$

Oh no, there is only one-dimensional eigenspace to go with the repeated eigenvalue $\lambda = 4$!

In other words, we don't have enough eigenvectors; we only have $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We cannot form a P such that $P^{-1}AP$ is diagonal, and therefore cannot compute e^A using the usual technique.