Research Statement

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1 Introduction

My research is in topology and knot theory, in particular planar algebras, a new field in quantum topology. Planar algebras were formally defined by Jones [Jon99]. Kuperberg independently defined a spider, a similar idea to Jones' planar algebra [Ku96]. My research is focused on taking a skein theoretic, combinatorial approach to developing new planar algebras. This follows the *Kuperberg Program*:

> Give a presentation for every interesting planar algebra, and prove as much as possible about the planar algebra using only its presentation.

2 Background

The most fundamental planar algebra is the Temperley-Lieb ($\mathcal{TL}$) planar algebra. Before the planar algebra came the algebra, which arose from the study of Statistical Mechanics by Temperley and Lieb [TL71]. The n^{th} Temperley-Lieb algebra ($\mathcal{TL}_n$) as a vectorspace over $\mathbb{C}$ has as a basis the diagrams with n non-crossing strands. Multiplication is defined by vertical stacking and replacing any closed loops by factors of a fixed number $\delta \in \mathbb{C}$.

The $\mathcal{TL}$ planar algebra is the assembly of these $\mathcal{TL}_n$ with infinitely many operations corresponding to the all the ways to connect these diagrams together. These operations are called "planar tangles". In general, planar algebras may contain more types of diagrams along with skein relations. Skein relations are relationships between diagrams that are equivalent except in a small, local region.

My research has been heavily involved with the closed diagrams of certain planar algebras. The closed diagrams are the elements of the 0^{th} dimensional vectorspace,i.e. those with no end points. The space of closed diagrams of a planar algebra is an associative algebra. In the $\mathcal{TL}$ planar algebra, the space of closed diagrams is isomorphic to $\mathbb{C}$.

Planar algebras have already been applied to knot theory. For example, if you define a crossing to be a specific linear combination of its two types of smoothings, then knot diagrams can be thought of as the closed diagrams of the $\mathcal{TL}$ planar algebra. This type of resolving of crossings is used to evaluate a knot diagram to get a knot polynomial. Conway used this method to compute the Alexander polynomial. The Kauffman bracket does the same for the Jones polynomial. In general, an algorithm that produces a complex number for each closed diagram is called an "evaluation algorithm".

3 Overview

In Section 4, I discuss an evaluation algorithm called the "jellyfish algorithm" discovered by Bigelow during his work on the "ADE" planar algebras [Big09]. In my dissertation, I fit his work fully into the Kuperberg program by showing this algorithm produces a well-defined output.

Early work of Penrose on the four color conjecture used tools we now recognize as planar algebras and skein relations. In Section 5, I describe a new planar algebra called the Disambiguated Temperley-Lieb ($\mathcal{DTL}$) planar algebra motivated by a more general notion of colorings. In my dissertation, I provide a basis for the closed diagrams of this planar algebra.

I am currently investigating the images of the Jones-Wenzl projections in the $\mathcal{DTL}$ planar algebra. My progress has led to the conjecture that these important elements are presented more simply in this new planar algebra. This work is described in Section 6.

In Section 7, I discuss my future research goals, including plans to use planar algebras to work on developing knot invariants. Finally, I discuss the opportunities for undergraduate research in this area in Section 8.

4 The Jellyfish Algorithm

Arising from the theory of von Neumann algebras, subfactor planar algebras are special kinds of planar algebras that are the standard invariants of subfactors. Jones classified the subfactor planar algebras with $\delta < 2$ and showed that they correspond to certain Dynkin diagrams of types A, D, and E.

Corresponding to the $\mathcal{TL}$ planar algebras at specific values of δ, the planar algebras of type A are fairly well understood. Morrison, Peters, and Snyder describe the subfactor planar algebras of type D using generators and relations [MPS08]. They develop an evaluation algorithm to show that the closed diagrams are one-dimensional, a necessary property for a subfactor planar algebra. This algorithm involves using a partial braiding to bring generators together pair by pair and using a specific relation to remove them. Much of the work is to show that their algorithm is well-defined, thus showing their planar algebra is nontrivial.

A combinatorial description for the remainder of the ADE planar algebras were given by Bigelow [Big09]. His proof used a new algorithm to evaluate closed diagrams. However he did not give a direct proof that the algorithm produces a well-defined output, instead appealing to known properties of these planar algebras. Thus it was not yet a self-contained purely combinatorial construction.

In my dissertation, I finish the combinatorial proof that the planar algebras of type ADE are nontrivial. The jellyfish algorithm has potential to be used to define and analyze new planar algebras, including those that do not come from subfactors.

5 The Disambiguated Temperley-Lieb Algebra

The work of Penrose established a link between the Four Color Conjecture (now the Four Color Theorem) and planar algebras, long before planar algebras were formally defined [Pen71]. Vaughan Jones showed that subfactor planar algebras could be interpreted as counting more general kinds of colorings of regions in a plane [Jon00].

In Jones' construction, the set of colors are represented as vertices on a graph, and furthermore each color has an associated positive real number called a weight. A legal coloring is one in which adjacent regions are always assigned colors corresponding to adjacent vertices. The evaluation of a closed diagram is then given by the formula

$$\sum_{\text{colorings}} \prod_{\text{regions R}} (\text{color in R})^{\chi(\text{R})}, \text{ where } \chi \text{ is the Euler characteristic.}$$

That is, sum over the legal colorings, and for each region take the product of the value of the color in that region to the power of the Euler characteristic of that region. In the case where we want to evaluate a diagram that is not closed, in place of Euler characteristic is the Euler measure.

For the closed diagrams of the $\mathcal{TL}$ planar algebra ($\mathcal{TL}_0$), we can use colors indexed by natural numbers. If a region is colored k then any adjacent region must be colored $k \pm 1$. The Disambiguated Temperley-Lieb ($\mathcal{DTL}$) planar algebra can be thought of as removing the ambiguity in the above "$k \pm 1$".

The $\mathcal{DTL}$ planar algebra is defined similarly to the $\mathcal{TL}$ planar algebra, but with oriented strands. Furthermore, this planar algebra does not have the relation of removing closed loops. It does have a pair of new relations, called the "pop-switch" relations:

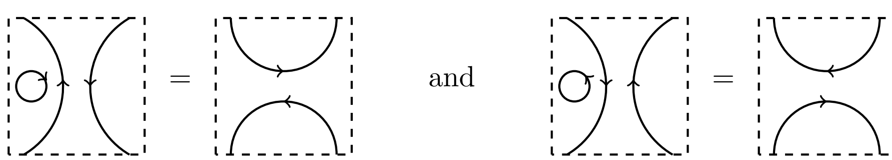

Figure 1: The pop-switch relations

Once a region in the diagram is given a color, this orientation determines the color choice for the remaining regions. In particular, if a clockwise loop is inside a region colored k, this interior of this loop must be colored $k - 1$. The pop-switch relations preserves the coloring evaluation, i.e. inserting either side of the equation into a closed diagram results in the same evaluation.

Again, we use the colors indexed by the natural numbers. Now once a region in the diagram is given a color, the orientation of the arrow separating it from its neighboring region determines the color choice for that region. In particular, if a clockwise loop is inside a region colored k, this interior of this loop must be colored $k-1$. The pop-switch relations preserves the coloring evaluation, i.e. inserting either side of the equation into a closed diagram results in the same evaluation.

In my dissertation, I describe a basis for the space of closed diagrams. I also show that this basis is in bijection with the set of all finite sequences of integers that sum to zero, and that come with a specified marked entry.

The $\mathcal{TL}$ planar algebra can be thought of as sitting inside the $\mathcal{DTL}$ planar algebra if one defines a non-oriented strand to be the sum of an oriented strand in each direction. In the $\mathcal{TL}$ planar algebra, any closed loops are replaced by factors of δ. To capture this we add the following relation to the $\mathcal{DTL}$ planar algebra.

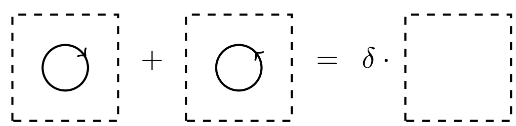

Figure 2: The bubble bursting relation

Adding this relation reduces the basis for the space of closed diagrams, which I describe in my dissertation.

6 Current Work

The Jones-Wenzl projections are very important elements in the $\mathcal{TL}$ planar algebra. They are used to form an alternative, theoretically fundamental basis for each $\mathcal{TL}_n$ algebra. While important, these elements of $\mathcal{TL}$ are hard to work with because they are complicated to write down. The n^{th} Jones-Wenzl projection ($\mathcal{JW}_n$) is a linear combination of every diagram with n nonintersecting strands. The number of these diagrams is the n^{th} Catalan number. Furthermore, one must calculate the corresponding coefficient to each of these diagrams. If the Jones-Wenzl projections were written more simply, it would be of great practical and theoretical importance.

In an attempt to achieve this, I am viewing the $\mathcal{DTL}$ planar algebra as a matrix category as in [MPS08]. This matrix category has direct sums of elements of $\mathcal{DTL}$ as well as isomorphisms between such sums. Bigelow and I have found the first and second Jones-Wenzl projections in this matrix category and have made progress on the third.

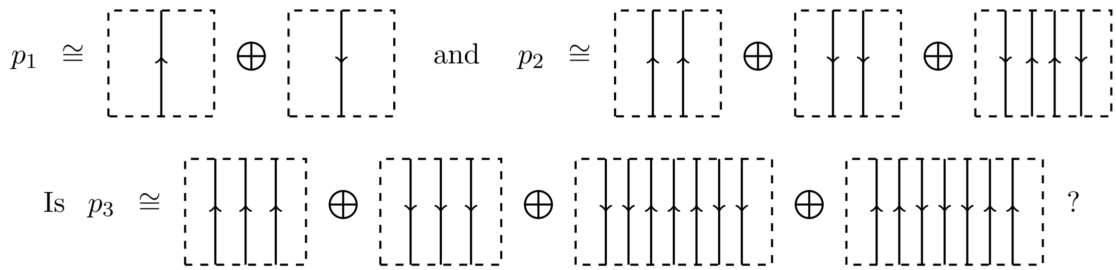

Figure 3: The Isomorphisms

Conjecture $\mathcal{JW}_n$ is isomorphic to a direct sum of $n+1$ diagrams, each consisting of vertical strands with a sequence of up or down orientations.

7 Future Work

I plan to continue to find and describe new planar algebras. As mentioned, I will use the jellyfish algorithm to define new planar algebras.

Additionally, I will use the $\mathcal{DTL}$ planar algebra to investigate the colorings of new planar algebras. In particular, Bigelow has a method for coloring Kuperberg's spider, which corresponds to the representations of $\mathcal{SL}_3$. His method considers linear combinations of 3-colored $\mathcal{DTL}$ diagrams. This is a new version of the coloring found by Evans and Pugh [EP09]. I would like to do something similar for the conjectured planar algebra corresponding to the representations of $\mathcal{SL}_4$. Morrison gives this conjecture in his thesis [Morr08].

Furthermore, I plan to use my research to the search for knot invariants. Planar algebras have always been closely related to knot invariants since Jones' fields medal winning discovery of the Jones polynomial from the $\mathcal{TL}$ algebra. My work promises to lead to new invariants and to unlock new properties of known invariants.

8 Undergraduate Research

My research is very accessible to undergraduates because of its tangible diagrams and combinatorial nature. At the same time, this research is cutting-edge. This is a perfect combination for undergraduate research.

I have a few ideas of places for students to start. For a student with some programming experience, we could follow my computation for the jellyfish algorithm corresponding to the planar algebras of type E. Discovering the conditions necessary for the proof to hold, we can create new planar algebras.

Another idea for an undergraduate project is to change the pop-switch relation in the $\mathcal{DTL}$ planar algebra to the following more general relation that still preserves the coloring evaluation.

Figure 4: New Relations

Work in planar algebras has already been completed by undergraduates. For example, during an REU with Bigelow, undergraduate students have already successfully completed research [BRY11]. So despite its connection to very deep mathematics, the material is very accessible.

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