

The Disambiguated Temperley-Lieb Algebra

Ellie Grano

Advisor: Stephen Bigelow

UC Santa Barbara

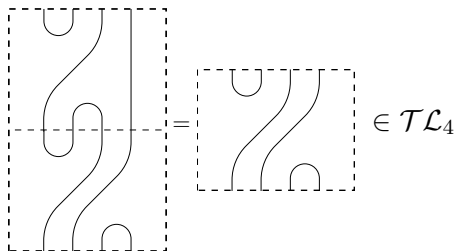
January 4, 2012

The Temperley-Lieb Algebra

- ▶ Diagrams with n non-crossing strands form a basis for \mathcal{TL}_n over \mathbb{C}
- ▶ Multiplication is vertical stacking:

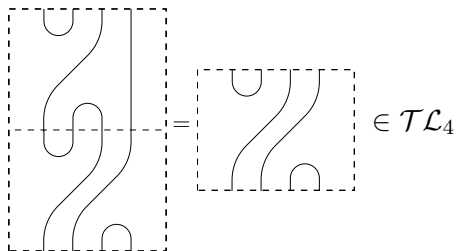
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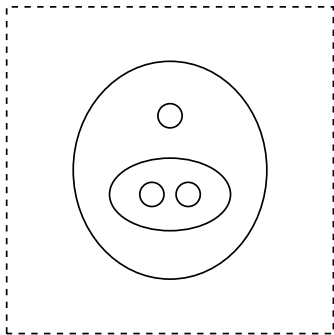
- ▶ These vector spaces assemble together into a planar algebra

The Temperley-Lieb Algebra

- ▶ The “closed diagrams” are collections of loops

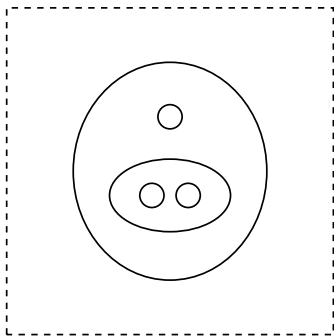
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$$\rightarrow \sum_{\text{states}} \prod_{\text{regions}} \#$$

The Disambiguated Temperley-Lieb Algebra

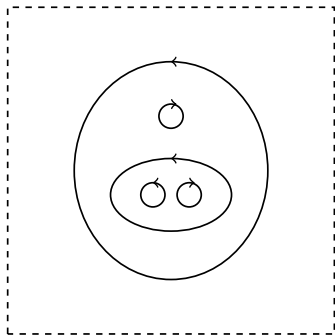
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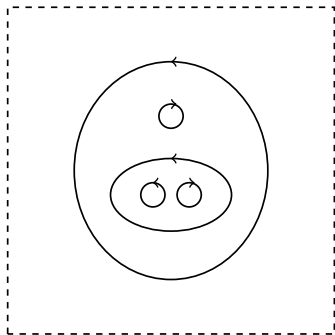
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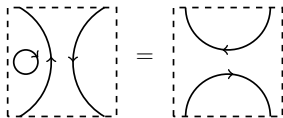


→ $\prod_{\text{regions}} \#$

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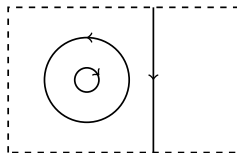


The Disambiguated Temperley-Lieb Algebra

- ▶ The pop-switch relations:



- ▶ Consequence

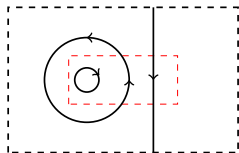


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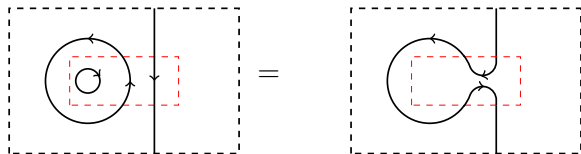


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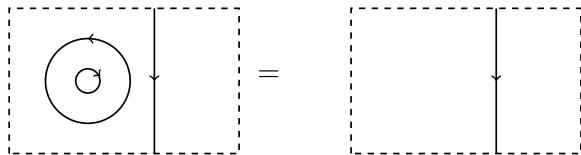


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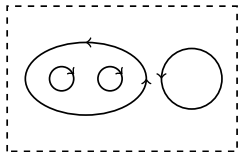


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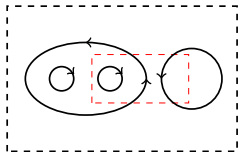
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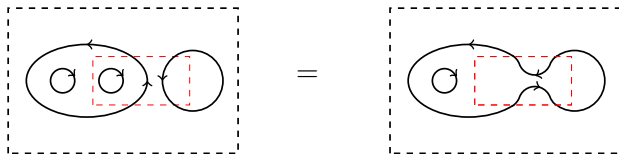
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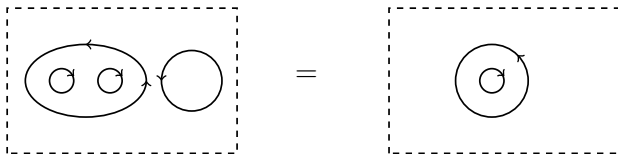
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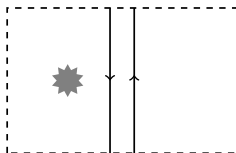
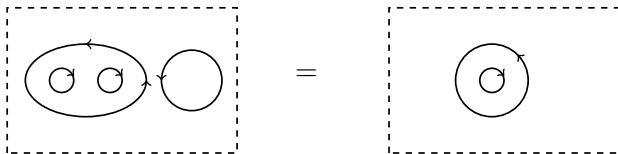
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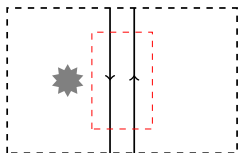
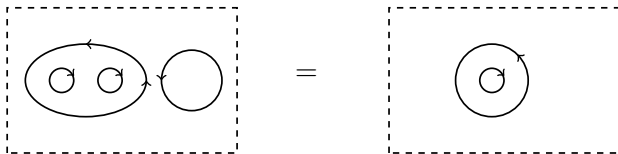
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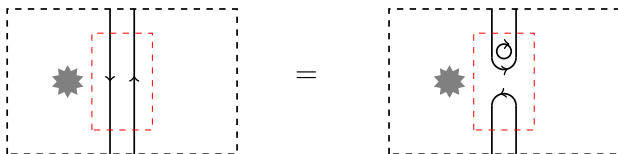
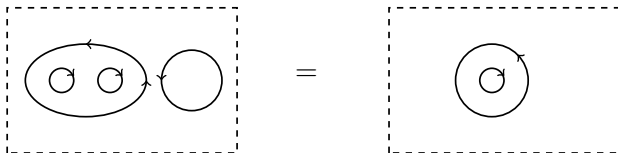
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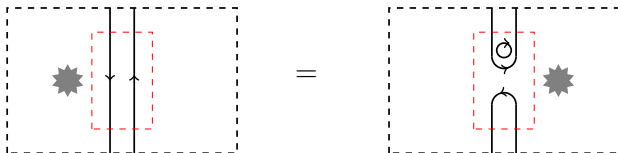
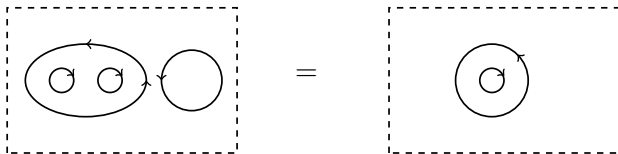
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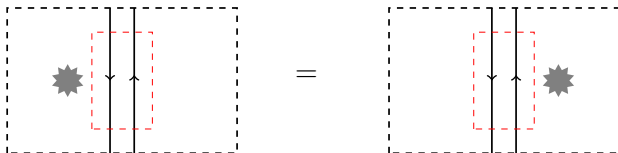
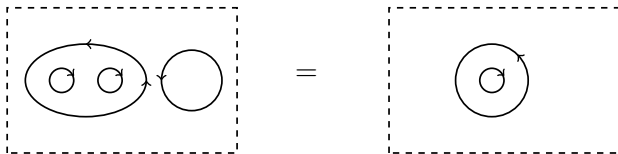
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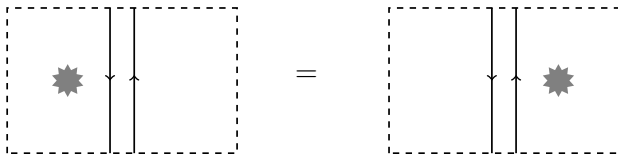
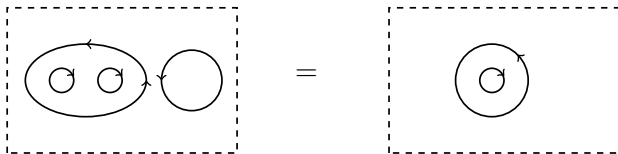
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- ▶ Notation

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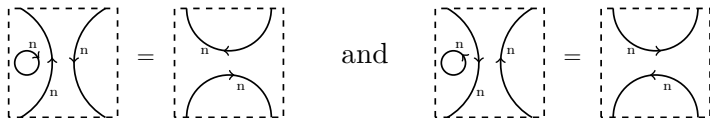
- ▶ Consequence: mult-pop-switch

The image shows two diagrammatic equations. Each equation consists of a left-hand side (LHS) and a right-hand side (RHS) separated by an equals sign, all enclosed within a dashed rectangular box. The first equation shows a LHS with a small circle on the left and two larger arcs on the right, with arrows and labels 'n' indicating flow and weight. The RHS shows two arcs, one above and one below, with arrows and labels 'n'. The second equation is similar but with the arrows on the arcs pointing in the opposite direction.

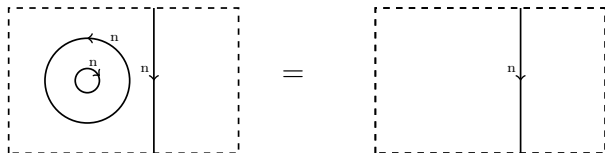
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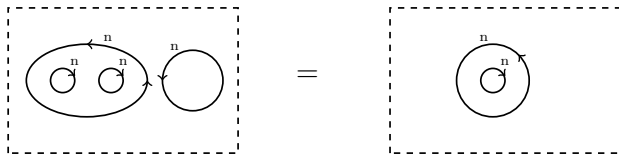


- ▶ More Consequences



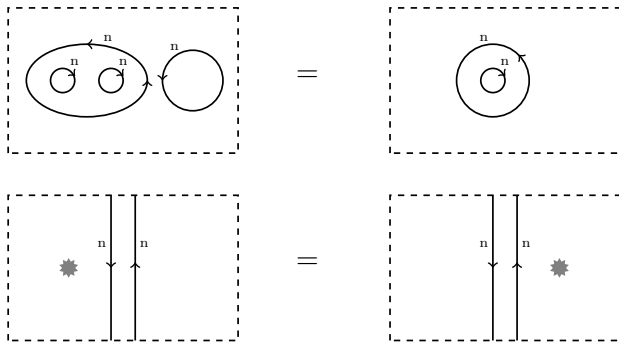
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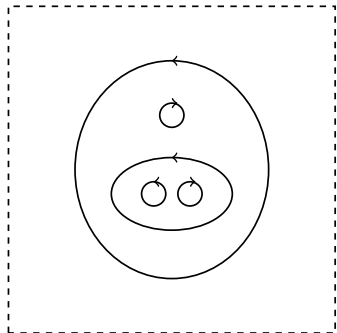
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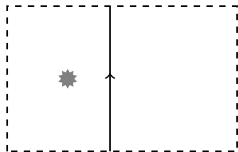


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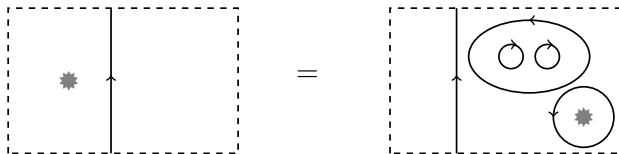
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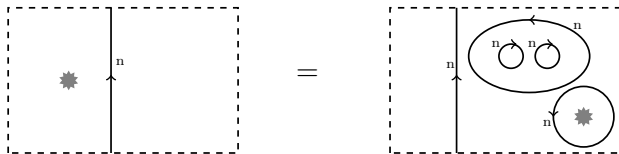
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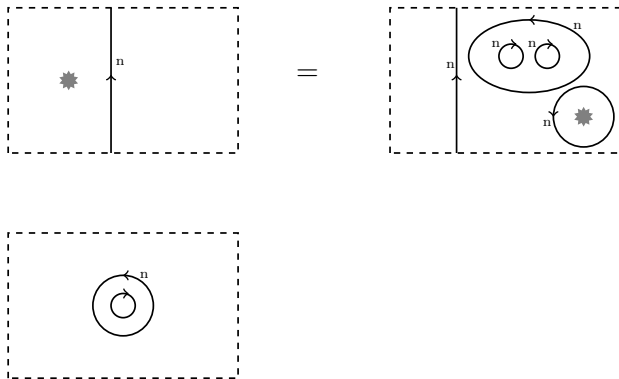
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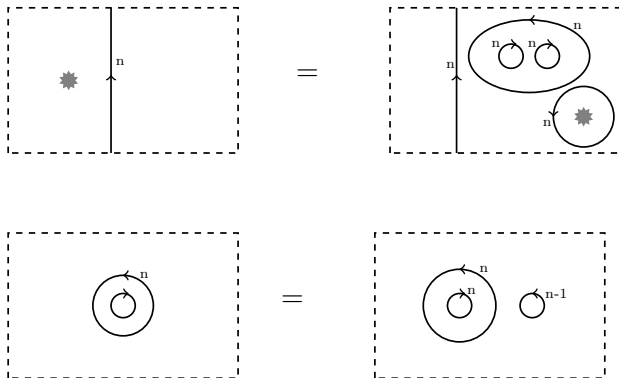
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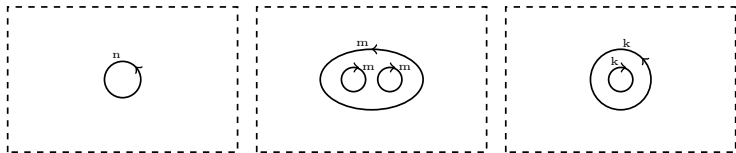
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- ▶ Basis elements are products of the following types of diagrams:

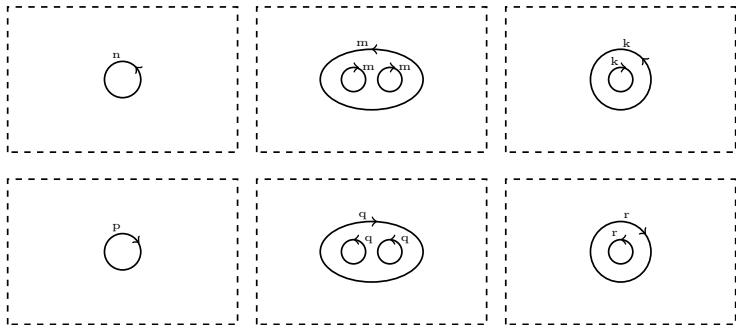
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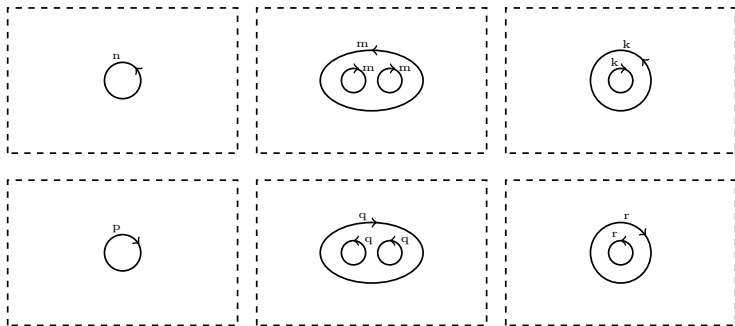
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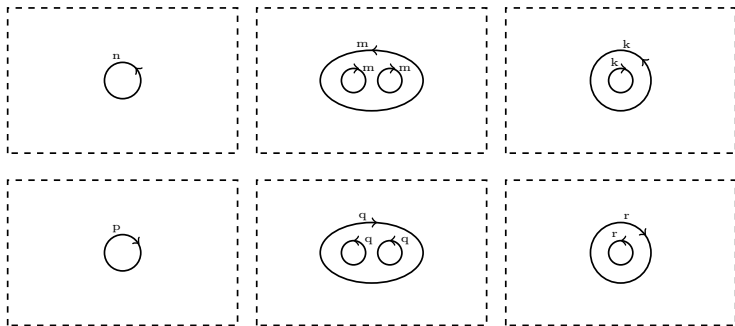
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subject to the properties discussed earlier.

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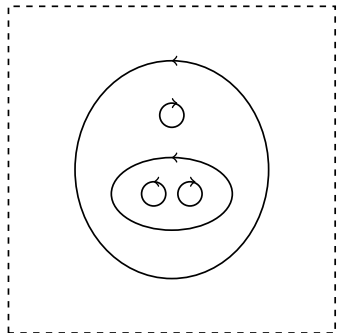


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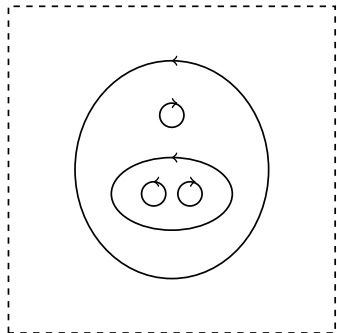
- ▶ For example $n \neq m$ and there is at most one k with $k > n$ and $k > m$.

Example

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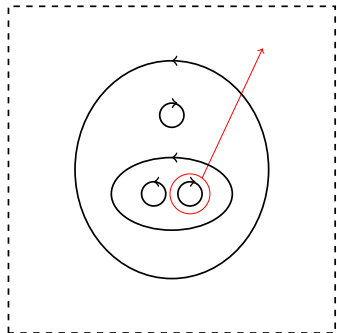


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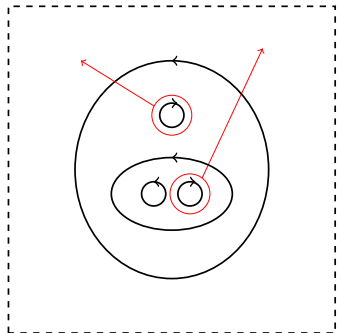


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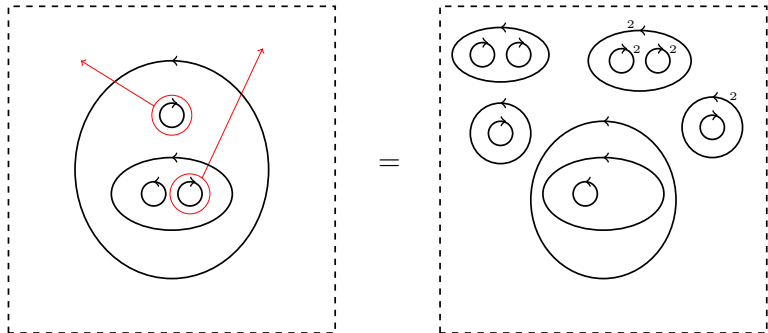
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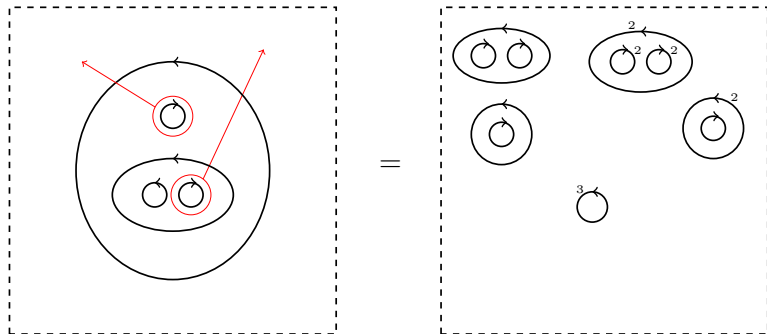
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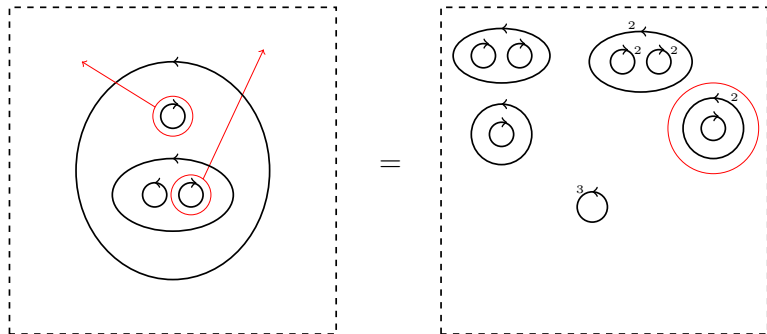
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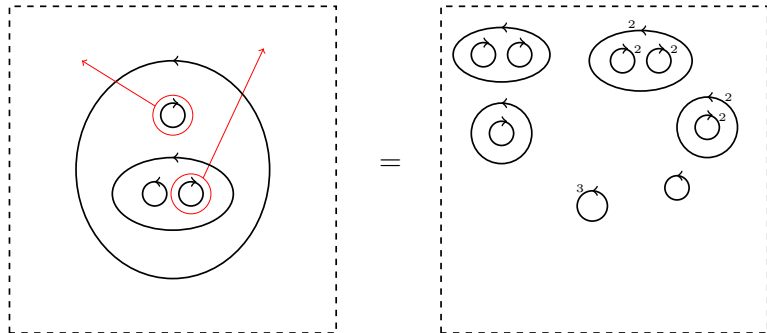
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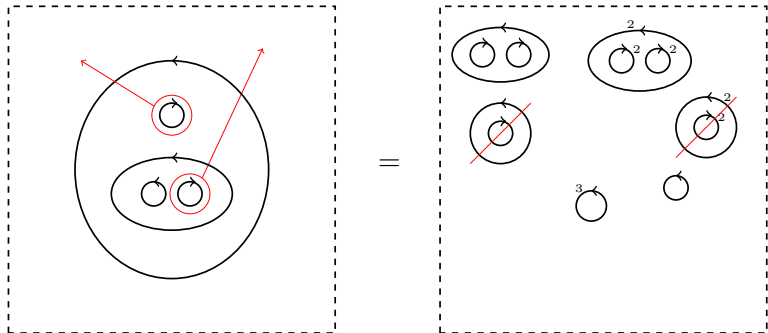
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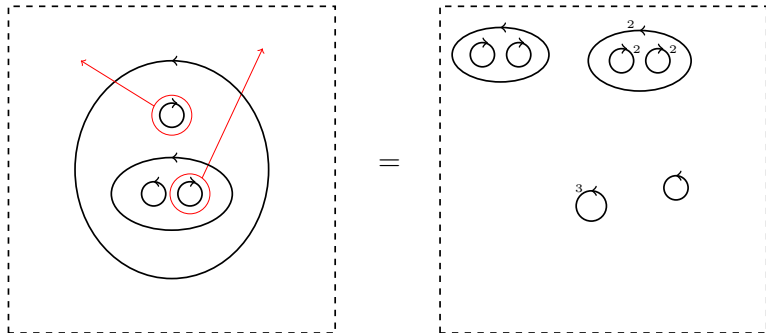
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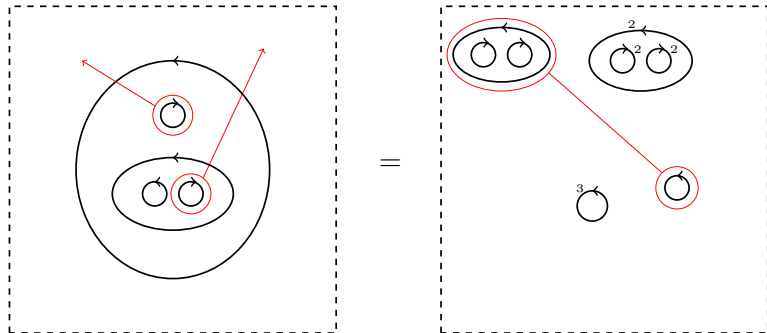
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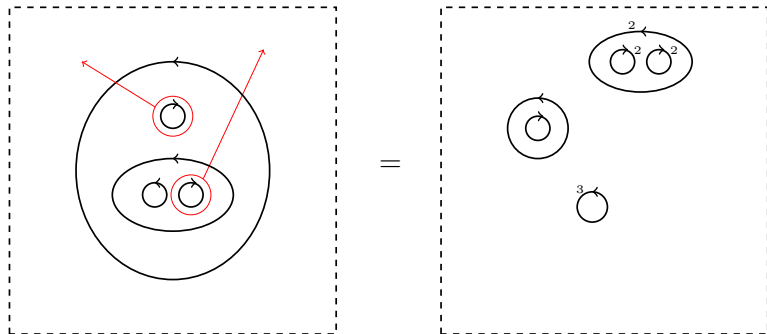
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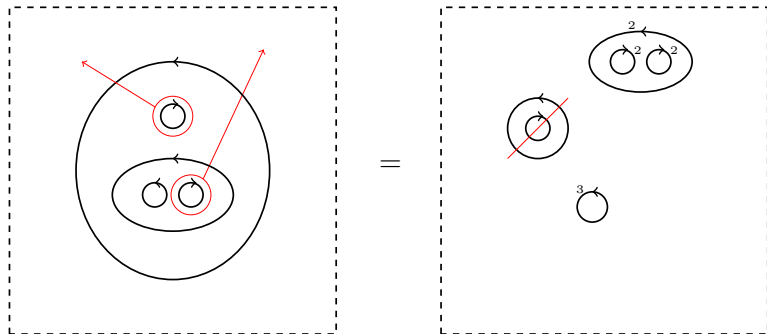
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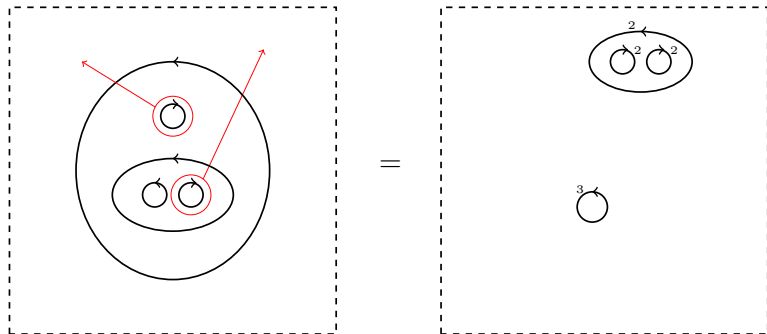
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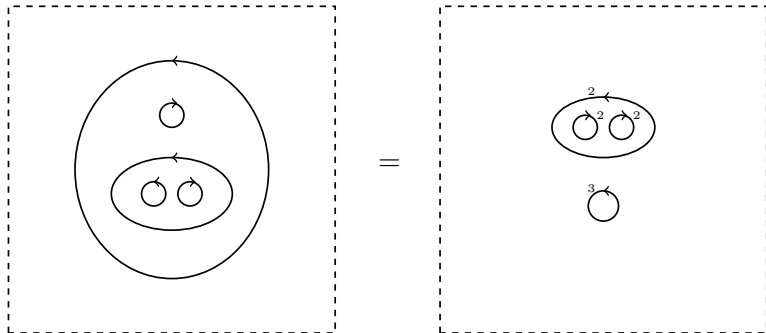
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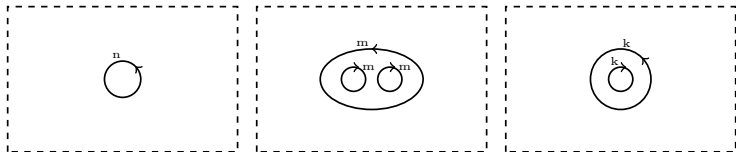
Sequences of Integers

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- ▶ Our diagrams correspond to finite sequences of integers that sum to 0, with a marked point.

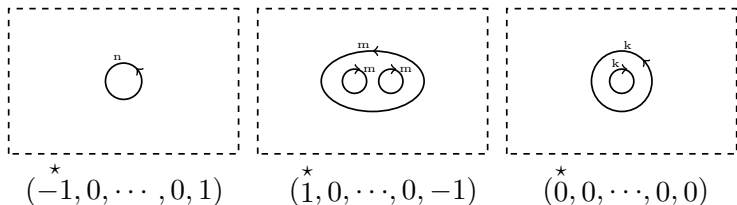
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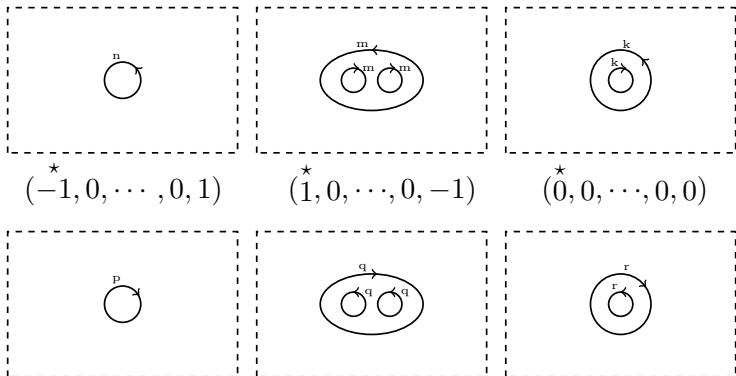
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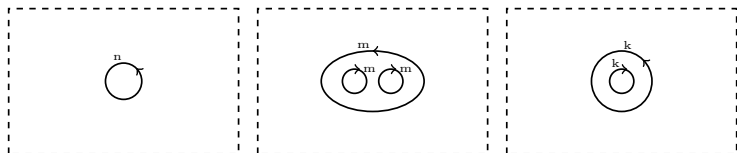
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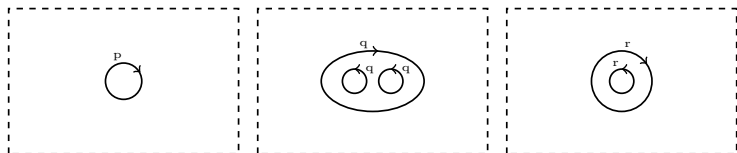
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$$(-1, 0, \dots, 0, 1)$$

$$(1, 0, \dots, 0, -1)$$

$$(0, 0, \dots, 0, 0)$$



$$(1, 0, \dots, 0, -1)$$

$$(-1, 0, \dots, 0, 1)$$

$$(0, 0, \dots, 0, 0)$$

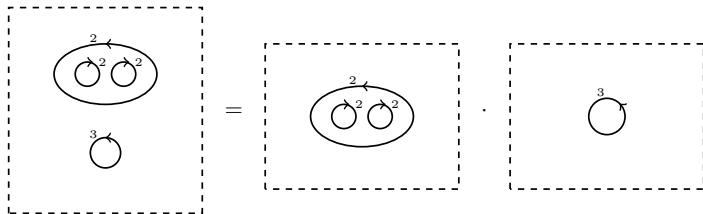
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Sequences of Integers

- Multiplication of diagrams corresponds to addition of the sequences

The diagrammatic equation illustrates the multiplication of two diagrams, which corresponds to the addition of their respective sequences. On the left, a dashed box contains two diagrams: the top one consists of two circles with arrows pointing towards each other, labeled with '2' above them, and a third circle below with an arrow pointing clockwise, labeled with '3' above it. This is followed by an equals sign. To the right of the equals sign are two dashed boxes. The first contains the top diagram from the left (two circles with arrows pointing towards each other, labeled '2'). The second contains the bottom diagram from the left (one circle with an arrow pointing clockwise, labeled '3'). Below the diagrams, the equation is written as:

$$\overset{\star}{(0, 0, -1, 1)} = \overset{\star}{(1, 0, -1)} + \overset{\star}{(-1, 0, 0, 1)}$$

Current Work

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- ▶ The \mathcal{TL} planar algebra can be thought of as sitting inside the \mathcal{DTL} planar algebra if we define

$$\boxed{\text{---}} = \boxed{\uparrow} + \boxed{\downarrow}$$

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$$p_1 \cong \left[\begin{array}{c} \uparrow \\ \downarrow \end{array} \right] \oplus \left[\begin{array}{c} \downarrow \\ \uparrow \end{array} \right]$$

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$$p_1 \cong \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \downarrow \\ \hline \end{array}$$
$$p_2 \cong \begin{array}{|c|} \hline \uparrow \uparrow \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \downarrow \downarrow \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \downarrow \uparrow \uparrow \downarrow \\ \hline \end{array}$$

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$$p_1 \cong \begin{array}{|c|} \hline \uparrow \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \downarrow \\ \hline \end{array}$$

$$p_2 \cong \begin{array}{|c|} \hline \uparrow \uparrow \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \downarrow \downarrow \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \downarrow \uparrow \uparrow \downarrow \\ \hline \end{array}$$

$$\text{Is } p_3 \cong \begin{array}{|c|} \hline \uparrow \uparrow \uparrow \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \downarrow \downarrow \downarrow \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \downarrow \downarrow \uparrow \uparrow \downarrow \downarrow \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \uparrow \uparrow \downarrow \downarrow \downarrow \uparrow \uparrow \\ \hline \end{array} ?$$

Current Work

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Conjecture

p_n is isomorphic to a direct sum of $n + 1$ diagrams, each consisting of vertical strands with a sequence of up or down orientations.

Thank you!