The Chain Rule:

Recall: For one variable

$$\begin{tikzcd} \mathbb{R} \arrow[r, "h"] \arrow[rr, bend right, "g"'] & \mathbb{R} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$g = f \circ h$

$$\frac{dg}{dx} = \frac{df}{dh} \cdot \frac{dh}{dx}$$

ex: $f(x) = e^x \qquad h(x) = \sin x$

$$g(x) = f(h(x)) = e^{\sin x}$$

$$\frac{dg}{dx}(0) = \frac{df}{dh}(h(0)) \cdot \frac{dh}{dx}(0)$$

$$= e^{(\sin(0))} \cdot \cos(0) = 1$$

For more than one variable

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "h = (h_1, h_2, h_3)"] \arrow[rr, bend right, "g"'] & \mathbb{R}^3 \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$g = f \circ h$

Here, $\displaystyle \frac{\partial g}{\partial x} = \frac{\partial f}{\partial h_1} \cdot \frac{\partial h_1}{\partial x} + \frac{\partial f}{\partial h_2} \cdot \frac{\partial h_2}{\partial x} + \frac{\partial f}{\partial h_3} \cdot \frac{\partial h_3}{\partial x}$

Example: Define $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by

$$h(x,y) = (h_1(x,y), h_2(x,y), h_3(x,y))$$

where $h_1(x,y) = x^2 y$

$$h_2(x,y) = y + 2$$

$$h_3(x,y) = y^3 - xy$$

Given $\nabla f(1,3,0) = (5, 1, -2)$

determine $\nabla g(1,1)$ where $g = f \circ h$

Solution: First compute the partials of the h_i

$$\frac{\partial h_1}{\partial x} = 2xy \qquad \frac{\partial h_2}{\partial x} = 0 \qquad \frac{\partial h_3}{\partial x} = -y$$

$$\frac{\partial h_1}{\partial y} = x^2 \qquad \frac{\partial h_2}{\partial y} = 1 \qquad \frac{\partial h_3}{\partial y} = 3y^2 - x$$

Now, $\frac{\partial g}{\partial x}(1,1) = \frac{\partial f}{\partial h_1}(1,3,0) \cdot \frac{\partial h_1}{\partial x}(1,1) + \frac{\partial f}{\partial h_2}(1,3,0) \cdot \frac{\partial h_2}{\partial x}(1,1)$

$$+ \frac{\partial f}{\partial h_3}(1,3,0) \cdot \frac{\partial h_3}{\partial x}(1,1)$$

$$= 5 \cdot 2 + 1 \cdot 0 + (-2) \cdot (-1)$$

$$= 12$$

$$\frac{\partial g}{\partial y}(1,1) = \frac{\partial f}{\partial h_1}(1,3,0)\cdot\frac{\partial h_1}{\partial y}(1,1) + \frac{\partial f}{\partial h_2}(1,3,0)\cdot\frac{\partial h_2}{\partial y}(1,1)$$

$$+ \frac{\partial f}{\partial h_3}(1,3,0)\cdot\frac{\partial h_3}{\partial y}(1,1)$$

$$= 5\cdot 1 + 1\cdot 1 + (-2)\cdot(2) = 2$$

Thus $\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$

So $\nabla g(1,1) = (12, 2)$.