MATH 34A REVIEW FOR MIDTERM 2, WINTER 2012

ANSWERS

1. Lines

(1) Find the equation of the line passing through (2,-1) and (-2,9).
$y = \frac{-5}{2}x + 4$.

(2) Find the equation of the line which meets the x-axis at $x = 10$ and the y-axis at $y = -7$.
$y = \frac{7}{10}x - 7$.

(3) A line has slope $\frac{1}{2}$ and goes through the point $(2, 5)$. What is the y -coordinate of the point on this line where $x = 6$?
The line is given by the equation $y = \frac{1}{2}x + 4$. The y -coordinate of the point on this line where $x = 6$ is $y = 7$.

2. Logs and exponentials

(1) Given that $\log(5) \approx 0.7$, $\log(6) \approx 0.78$, and $\log(2) \approx 0.3$, approximate $\log(25)$, $\log(20)$, $\log(300)$, and $\log(16)$.
$\log(25) = \log(5^2) = 2\log(5) \approx 2(0.7) = \boxed{1.4}$.
  OR $\log(25) = \log(100/4) = \log(100) - \log(2^2) = 2 - 2\log(2) \approx 2 - 2(0.3) = \boxed{1.4}$.
$\log(20) = \log(5 \cdot 2^2) = \log(5) + 2\log(2) \approx (0.7) + 2(0.3) = \boxed{1.3}$.
  OR $\log(20) = \log(2 \cdot 10) = \log(2) + 1 \approx (0.3) + 1 = \boxed{1.3}$.
$\log(300) = \log(100) + \log(3) = 2 + \log(6/2) = 2 + \log(6) - \log(2) \approx 2 + 0.78 - 0.3 = \boxed{2.48}$.
  OR $\log(300) = \log(10 \cdot 30) = 1 + \log(5 \cdot 6) = 1 + \log(5) + \log(6) \approx 1 + 0.7 + 0.78 = \boxed{2.48}$.
$\log(16) = \log(2^4) = 4\log(2) \approx 4(0.3) = \boxed{1.2}$.

(2) Express 2^x as a power of 10.
$\boxed{2^x = 10^{x\log(2)}}$

(3) Solve for x.
  (a) $\frac{1}{3^x} = 4^y$
  $\boxed{x = -y \cdot \frac{\log(4)}{\log(3)}}$
  (b) log(log(x))=2
  $\boxed{x = 10^{10^2}}$.
  (c) $3^x = 6^{x+2}$. $\boxed{x = \frac{2\log(6)}{\log(3) - \log(6)}}$
  (d) $2/4^x = 1/3$
  $\boxed{x = \frac{\log(6)}{\log(4)}}$

(e) $\log(x-7)+\log(x) = 1$

$\log(x^2 - 7x) = 1$

$x^2 - 7x = 10$

$x^2 - 7x - 10 = 0$

$(x-5)(x-2) = 0$

$\boxed{x = 2 \text{ or } 5}$

3. Applications of Logs: Halflife / Doubling Time, Compounding Interest

(1) A population of rabbits doubles every year and there were 1 million rabbits at the start of 1941.

(a) How many rabbits are there in 1946?

$\boxed{\text{32 million}}$

(b) In what year will there be 10 million rabbits?

$\boxed{1941+\frac{1}{\log(2)}}$

(2) A certain bacteria doubles every 7 hours. How long will it take for there to be 5 times as much?

$\boxed{\text{In } 7\frac{\log(5)}{\log(2)} \text{ hours}}$

(3) An isotope called W has a half-life of 4 years. If there is initially 140 grams, how much is there after 21 years?

$\boxed{\text{There are } 140(\frac{1}{2})^{21/4} \text{ grams of W left}}$.

(4) Suppose the function $f(t) = 6^{-3t}$ gives the number of grams of an isotope after t years. What is the half-life of the isotope?

$\boxed{\frac{\log(2)}{3\log(6)} \text{ years}}$

(5) E. Coli bacteria are growing in a hamburger exponentially. Initially there are 100,000 bacteria. After 30 minutes there are 150,000. How many are there after an hour?

Let $f(t) = 100000(2)^{t/k}$ be the amount of bacteria after t hours. Use that $f(\frac{1}{2}) = 150000$ and solve for k. You get that $k = \frac{\log(2)}{2\log(1.5)}$. So $f(t) = 100000(2)^{t(2\log(1.5)/\log(2))}$. Then after 1 hour there are $\boxed{f(1) = 100000(2)^{(2\log(1.5)/\log(2))} \text{ bacteria}}$.

OR notice that $150000 = \frac{3}{2} \cdot 100000$. So the time it takes for there to be $\frac{3}{2}$ as much is $\frac{1}{2}$ an hour. Then use the formula $f(t) = 100000(\frac{3}{2})^{t/k}$ where $k = \frac{1}{2}$ is the time it takes for there to be $\frac{3}{2}$ as much. Then after 1 hour there are $\boxed{f(1) = 100000(\frac{3}{2})^2 \text{ bacteria}}$.

(6) Suppose a bank pays 0.5 percent interest compounded annually. You put in 1000 dollars at the beginning of 2011.

(a) How much money will be in the account a the beginning of 2015?

$\boxed{1000(1.005)^4 \text{ dollars}}$

(b) When will I have 1050 dollars?

$\boxed{\frac{\log(1.05)}{\log(1.005)} \text{ years after the beginning of 2011}}$.

(7) Suppose you get a credit card that does not require any paymments for the first six months. Then you purchase a stereo for 300 dollars with your credit card. The card compounds monthly and interest on the debt will continue to accrue. The credit card has an annal percentage rate (APR) of 17.9 percent. How much money do you owe at the end of the six months?

$P = 300$ dollars, $m = 12$ since the card compounds monthly (12 times a year), and the annual interest rate is $r = 17.9\%$ or 0.179.

Then $f(n) = 300(1+\frac{0.179}{12})^n$ dollars. So after six months, you owe $\boxed{f(6) = 300(1 + \frac{0.179}{12})^6 \text{ dollars}}$.

4. Linear Interpolation / Extrapolation, Graphing with Logarithmic Coordinates

(1) If $f(1) = 3$ and $f(5) = 5$, use linear interpolation to approximate $f(4)$.

$\boxed{f(4) \approx 9/2}$.

(2) The world population in 1990 was 5.4 billion and in 1995 was 5.8 billion.

(a) Use linear extrapolation to find the population in 2010.

$\boxed{\text{7 billion}}$.

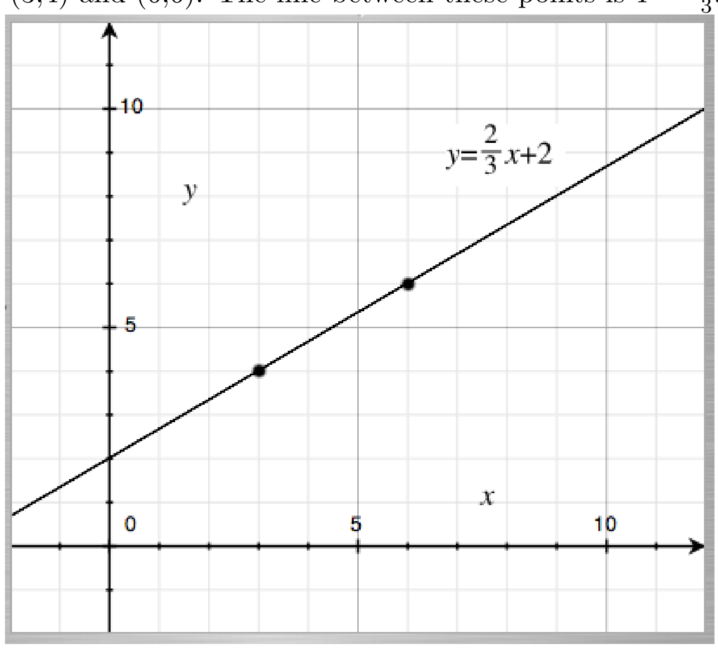
(b) If the population was really 7.2 billion, what was the percent error?

$(0.2/7.2) \cdot 100\% = \boxed{25/9 \text{ percent}}$.

(3) Suppose $f(x)$ is an exponentially increasing function and that $f(3) = 10^4$ and $f(6) = 10^6$.

(a) Graph x verses $Y = \log(f(x))$. Find the equation of the line through the two points on your graph of x verses $Y = \log(f(x))$.

The points are (3,4) and (6,6). The line between these points is $Y = \frac{2}{3}x + 2$.



(b) Use this line to approximate $f(5)$.

Plug 5 into the line to estimate $\log(f(5))$: $\log(f(5)) \approx Y = \frac{2}{3}(5) + 2 = \frac{16}{3}$. Then convert back to $f(x)$ coordinates: $\boxed{f(5) \approx 10^{16/3}}$.

5. Proportionality

(1) The amount of taxes a city collects is proportional to the population of the city. In 1980 the population was 2 million and it had increased to 3 million by 1992. If 4 billion dollars in taxes were collected in 1980 how much was collected in 1992?

$\boxed{\text{6 billion dollars}}$.

(2) The time it takes to buy a candy bar at the Arbor is proportional to the number of people in line and inversely proportional to the number of cash registers open. If it takes 5/7 of a minute when there are 5 people in line and 3 registers open, then how many minutes would it take if there are 56 people in line and 4 registers open?

$\boxed{\text{6 minutes}}$.

(3) Suppose x is proportional to y and y is inversely proportional to z.

(a) Is x proportional to z?

No.

(b) Is x inversely proportional to z?

Yes.

6. The Change in $f(x)$, Average Rate of Change, Instantaneous Rate of Change and Derivatives

(1) If x is increased from 6 to 7 how much does $\frac{1+x}{2+x}$ change by? Does the function increase or decrease when x goes from 6 to 7?

The function changes by $\boxed{1/72}$.

The function increases.

(2) (a) Find

$$\frac{\text{the change in } (x-2)(x+2) \text{ as } x \text{ increases from 1 to } 1+h}{h}$$

where h is a positive real number. Simplify your answer.

$\boxed{h+2}$

(b) Find

$$\lim_{h\to 0} \frac{\text{the change in } (x-2)(x+2) \text{ as } x \text{ increases from 1 to } 1+h}{h}$$

$\boxed{2}$

(3) Let $f(x) = x^2 + 4$.

(a) Find the average rate of change between $x = 2$ and $x = 3$.

$\boxed{5}$

(b) Find the average rate of change between $x = 2$ and $x = 2 + h$.

$\boxed{4 + h}$

(c) Using only part (b) and the limit definition of the derivative, what is the instantaneous rate of change at $x = 2$?

$\boxed{4}$

(d) If $f(x)$ represented the grams of bacteria in a petri dish x hours after 12pm, write out what the answer to part (c) means.

Part (c) tells us that at 4pm the grams of bacteria in the petri dish are increasing at a rate of 4 grams per hour.

(4) The total profit that a company has made measured in millions of dollars is $p(t)$ where t is the time measured in years with year zero corresponding to 1990.

(a) What is the meaning of $p'(5) = 0$?

The company was neither making nor losing money at the start of 1995.

(b) What are the units of $p'(t)$?

Millions of dollars per year

(c) What does it mean if $p'(t)$ is positive?

The company is making money at time t.

(5) If a commodity is priced at p dollars the number of items that sell is $Q(p)$.

(a) What does $Q(50) = 20,000$ mean?

At a price of 50 dollars 20,000 items of the commmodity are sold.

(b) What are the units of $Q'(p)$?

items per dollar

(c) What is the meaning of the statement that $Q'(50) = -200$?

When the price is 50 dollars, the rate at which sales are decreasing is 200 items per dollar.

7. Computing Derivatives

(1) Let $f(x) = 4 - x^2$. Use the limit definition of a derivative to find $f'(3)$

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{[4 - (3+h)^2] - [4 - 3^2]}{h} = \lim_{h\to 0} \frac{4 - 9 - 6h - h^2 + 5}{h} = \lim_{h\to 0} \frac{-6h - h^2}{h} = \lim_{h\to 0} (-6 - h) = \boxed{-6}.$$

(2) Let $f(x) = 2/x$. Use the limit definition of a derivative to find $f'(1)$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\frac{2}{1+h} - \frac{2}{1}}{h} = \lim_{h\to 0} \frac{2 - 2(1+h)}{h(1+h)} = \lim_{h\to 0} \frac{-2h}{h(1+h)} = \lim_{h\to 0} \frac{-2}{1+h} = \boxed{-2}.$$

(3) Find the derivative of the following functions:

(a) $f(x) = x^2 + 3x$

$\boxed{f'(x) = 2x + 3}$

(b) $f(x) = \frac{1}{x^2} + \sqrt{x}$

Rewrite $f(x) = x^{-2} + x^{\frac{1}{2}}$ and then

$f'(x) = -2x^{(-2-1)} + \frac{1}{2}x^{(\frac{1}{2}-1)} = -2x^{-3} + \frac{1}{2}x^{-\frac{1}{2}} = \frac{-2}{x^3} + \frac{1}{2\sqrt{x}}$. So $\boxed{f'(x) = \frac{-2}{x^3} + \frac{1}{2\sqrt{x}}}$.

(c) $f(x) = e + x$

$\boxed{f'(x) = 1}$

(d) $f(x) = 4e^{3x} + 2x - \pi$

$\boxed{f'(x) = 12e^{3x} + 2}$

(e) $f(x) = (x-1)(2x+1)$

First multiply out: $f(x) = 2x^2 - x - 1$ and then

$f'(x) = 4x - 1$. So $\boxed{f'(x) = 4x - 1}$

(f) $f(x) = \frac{x^3-2x^2+x}{x^2-x}$

First simplify: $f(x) = \frac{x(x-1)^2}{x(x-1)} = x - 1$ and then

$\boxed{f'(x) = 1}$.

(4) If you know that $f'(1) = 4$, $g'(1) = 2$, and $h(x) = 2f(x) + 3g(x)$, then what is $h'(1)$?

$h(x) = 2f(x)+3g(x)$, so $h'(x) = 2f'(x)+3g'(x)$. Then $h'(1) = 2f'(1)+3g'(1) = 2\cdot4+3\cdot2 = 8+6 = 14$. So $\boxed{h'(1) = 14}$.

8. Tangent Lines and Tangent Line Approximation

(1) Suppose $g(2) = 4$ and $g'(2) = -1$. Find the tangent line to $g(x)$ at $x = 2$.

We are looking for the equation of the line that goes through the point $(2,4)$ with slope -1. So using point slope form we find:

$y - 4 = -(x-2)$

$\boxed{y = -x + 6}$

(2) Let $f(x) = x^2 - 5x + 9$.

(a) Find the equation of the tangent line to $f(x)$ at $x = 3$.

$f(3) = 9 - 15 + 9 = 3$, so our line will go through the point $(3,3)$.

$f'(x) = 2x - 5$. Now plug in 3 to get the slope of the line: $f'(3) = 6 - 5 = 1$.

So we are looking for the equation of the line with slope 1 that goes through the point $(3,3)$. The answer is $\boxed{y = x}$.

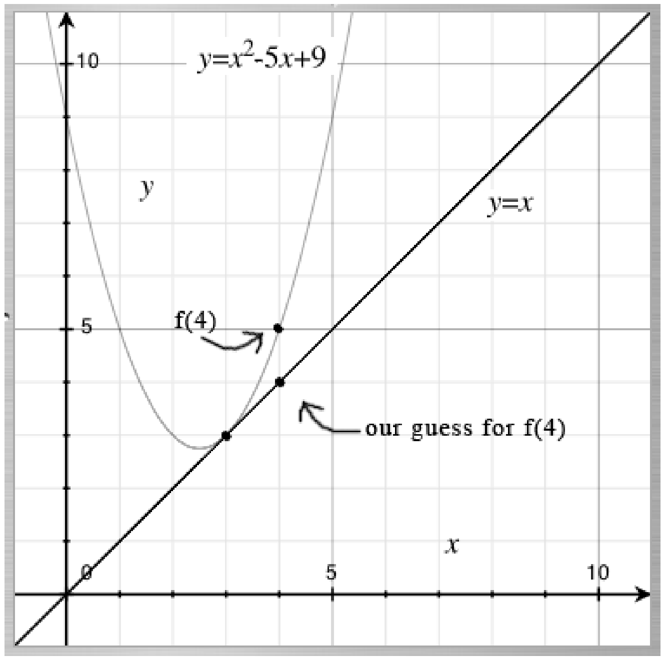
(b) Use your tangent line to estimate $f(4)$.

Plug 4 into our equation: $\boxed{f(4) \approx 4}$.

(c) What is the percentage error?

$f(4)$ is actually $4^2 - 5(4) + 9 = 16 - 20 + 9 = 5$.

So the percentage error is $\frac{5-4}{5} \cdot 100\% = \frac{1}{5} \cdot 100\% = \boxed{20\%}$.



(3) Let $f(x) = x^2 + e$.

(a) Find the equation of the tangent line to $f(x)$ at $x = 1$.

$f(1) = 1 + e$, so our line will go through the point $(1, 1 + e)$.

$f'(x) = 2x$. Now plug in 1 to get the slope of the line: $f'(1) = 2$.

So we are looking for the equation of the line with slope 2 that goes through the point $(1, 1 + e)$.

$y = 2x + b$. Find b by plugging in the point.

$1 + e = 2(1) + b \implies b = e - 1$.

So the line is given by $\boxed{\text{y=2x+e-1}}$.

(b) Use your tangent line to estimate $f(2)$.

Plug 2 into our equation: $y = 2 \cdot 2 + e - 1 = 3 + e$. So $\boxed{f(2) \approx 3 + e}$.

(4) Let $f(x) = 2e^{3x} + 1$.

(a) Find the equation of the tangent line to $f(x)$ at $x = 0$.

$\boxed{\text{y=6x+3}}$.

(b) Use your tangent line to estimate $f(1)$.

$\boxed{f(1) \approx 9}$.

(5) Let $f(x) = \sqrt{x}$.

(a) Find the equation of the tangent line to $f(x)$ at $x = 1$.

The point is $(1, 1)$ and the slope is $f'(1) = \frac{1}{2}(1)^{-\frac{1}{2}} = \frac{1}{2}$.

This gives the equation $\boxed{y = \frac{1}{2}x + \frac{1}{2}}$.

(b) Use your tangent line to estimate $\sqrt{2}$.

This question is asking in disguise to estimate $f(2) = \sqrt{2}$. Plug in 2 to the line to get $\boxed{\sqrt{2} = f(2) \approx \frac{3}{2}}$.

9. Second Derivative / Max-min problems

(1) The height above the ground of an object t seconds after it is launched vertically upwards is $h(t) = 100t - 5t^2$ meters.

(a) What is the velocity of the object after 3 seconds?

$\boxed{f'(3) = 70 \text{ m/s}}$.

(b) What is the acceleration after 3 seconds

$\boxed{f''(3) = -10 \text{ m/s}^2}$.

(c) How many seconds after launch did the object hit the ground?

$\boxed{20 \text{ seconds}}$.

(2) Let $f(x) = x^3 - 6x^2 + 9x + 2$.

(a) Find all the critical points and determine if they correspond to a local maximum or a local minimum.

$f'(x) = 3x^2 - 12x + 9$. Set this equal to zero to find the critical points:

$0 = 3x^2 - 12x + 9 = 3(x-3)(x-1)$.

So x=3 and x=1 are both critical points.

Use the second derivative to determine if they are a local minimum or a local maximum:

$f''(x) = 6x - 12$.

$f''(1) = 6(1) - 12 = -6 < 0$ So $f(x)$ obtains a local maximum at $x = 1$.

Next compute $f(1) = 1 - 6 + 9 + 2 = 6$.

So $\boxed{f(1) = 6 \text{ is a local maximum}}$.

$f''(3) = 6(3) - 12 = 6 > 0$ So $f(x)$ obtains a local minimum at $x = 3$.

Finally, compute $f(3) = (3)^3 - 6(3^2) + 9(3) + 2 = 3^2(3 - 6 + 3) + 2 = 2$.

So $\boxed{f(3) = 2 \text{ is a local minimum}}$.

(3) A commuter railway has 800 passengers per day and charges each one two dollars per day. For each 4 cents that the fare is increased, 5 fewer people will go by train. What is the greatest profit that can be earned.

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(4) On a certain island there are two populations of deer. After t years the numbers of deer in the two populations are $p(t) = 200e^t$ and $q(t) = 1000e^{-t}$. When is the total population smallest?

The total population is $f(t) = p(t) + q(t) = 200e^t + 1000e^{-t}$.

$f'(t) = 200e^t - 1000e^{-t}$.

Set $f'(t) = 0$ and solve for t:

$0 = 200e^t - 1000e^{-t}$.

$200e^t = 1000e^{-t}$.

$e^t = 5e^{-t}$.

$e^{2t} = 5$.

2t=ln(5)

$t = \frac{\ln(5)}{2}$

The total population is the smallest when $\boxed{t = \frac{\ln(5)}{2}}$.

Another acceptable answer: $\boxed{t = \frac{\log(5)}{2\log(e)}}$ (they are equivalent).

(5) A cylindrical metal can is to have no lid. It is to have a volume of 64π inches cubed. What height minimizes the amount of metal used?

$\boxed{4}$

(6) A rectangular field will have one side made of a brick wall and the other three sides made of wooden fence. Brick wall costs 10 dollars per meter and wooden fence costs 20 dollars for 4 meters. the area of the field is to be 600 square meters. What length should the brick wall be to give the lowest total cost of wall plus fence?

$\boxed{20}$