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K_1 OF SEPARATIVE EXCHANGE RINGS AND C*-ALGEBRAS WITH REAL RANK ZERO

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Abstract. For any (unital) exchange ring R whose finitely generated projective modules satisfy the separative cancellation property ($A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B$), it is shown that all invertible square matrices over R can be diagonalized by elementary row and column operations. Consequently, the natural homomorphism $GL_1(R) \to K_1(R)$ is surjective. In combination with a result of Huaxin Lin, it follows that for any separative, unital C*-algebra A with real rank zero, the topological $K_1(A)$ is naturally isomorphic to the unitary group $U(A)$ modulo the connected component of the identity. This verifies, in the separative case, a conjecture of Shuang Zhang.

Introduction

The extent to which matrices over a ring R can be diagonalized is a measure of the complexity of R, as well as a source of computational information about R and its free modules. Two natural properties offer themselves as "best possible": (1) that an arbitrary matrix can be reduced to a diagonal matrix on left and right multiplication by suitable invertible matrices, or (2) that an arbitrary invertible matrix can be reduced to a diagonal one by suitable elementary row and column operations. The second property has an immediate K-theoretic benefit, in that it implies that the Whitehead group $K_1(R)$ is a natural quotient of the group of units of R. Our main goal here is to prove property (2) for exchange rings (definition below) satisfying a cancellation condition which holds very widely (and conceivably for all exchange rings). This theorem, when applied to C*-algebras with real rank zero (also defined below), verifies a conjecture of Shuang Zhang in an extensive class of C*-algebras.

The class of exchange rings has recently taken on a unifying role for certain direct sum cancellation problems in ring theory and operator algebra. In particular, exchange rings encompass both (von Neumann) regular rings (this is an old and easy observation) on the one hand, and C*-algebras with real rank zero [**3**, Theorem 7.2] on the other. Within this class, a unifying theme for a number of open problems is the property of *separative cancellation* for finitely generated projective modules, namely the condition

$$A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B$$

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(see [**2**, **3**]). For example, if R is a separative exchange ring, then the (K-theoretic) stable rank of R can only be 1, 2, or ∞ [**3**, Theorem 3.3], and every regular square matrix over R is equivalent (via multiplication by invertible matrices) to a diagonal matrix [**2**, Theorem 2.4]. We prove below that invertible matrices over separative exchange rings can be diagonalized via elementary row and column operations. Recently, Perera [**25**] has applied our methods to the problem of lifting units modulo an ideal I in a ring R, assuming that I satisfies non-unital versions of separativity and the exchange property. In this case, a unit u of R/I lifts to a unit of R if and only if the class of u in $K_1(R/I)$ is in the kernel of the connecting homomorphism $K_1(R/I) \to K_0(I)$ [**25**, Theorem 3.1].

We defer discussion of the C*-algebraic aspects of our results to Section 3, except for the following remark. While earlier uses of the exchange property and separativity for C*-algebras can easily be written out in standard C*-theoretic terms – e.g., with direct sums and isomorphisms of finitely generated projective modules replaced by orthogonal sums and Murray-von Neumann equivalences of projections – our present methods do not lend themselves to such a translation. In particular, although our main C*-algebraic application may be stated as a diagonalization result for unitary matrices, all of the steps in our proofs involve manipulations with non-unitary matrices.

Throughout the paper, we consider only unital rings and C*-algebras. We reserve the term *elementary operation* for the row (respectively, column) operation in which a left (respectively, right) multiple of one row (respectively, column) of a matrix is added to a different row (respectively, column). Similarly, we reserve the name *elementary matrix* for a transvection $I + re_{ij}$ where I is an identity matrix, e_{ij} is one of the usual matrix units for some $i \neq j$, and r is an element of the base ring. Thus, as usual, an elementary row (respectively, column) operation on a matrix A corresponds to multiplying A on the left (respectively, right) by an elementary matrix.

Note that while odd permutation matrices usually cannot be expressed as products of elementary matrices, certain signed permutation matrices can be. For example,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

In particular, the operation of replacing rows R_i and R_j (respectively, columns C_i and C_j) with the rows R_j and $-R_i$ (respectively, the columns C_j and $-C_i$) can be achieved as a sequence of three elementary operations. Therefore any entry of a matrix can be moved to any other position by a sequence of elementary row and column operations, at the possible expense of moving other entries and multiplying some by -1.

For any ring R, let $E_n(R)$ denote the subgroup of $GL_n(R)$ generated by the elementary matrices. If $GL_n(R)$ is generated by $E_n(R)$ together with the subgroup $D_n(R)$ of invertible diagonal matrices, then R is said to be a *GE_n-ring* [**10**, p. 5]. Further, R is a *GE-ring* provided it is a GE_n-ring for all n. If R is a GE_n-ring then $E_n(R)$ is a normal subgroup of $GL_n(R)$, and so $GL_n(R) = D_n(R)E_n(R) = E_n(R)D_n(R)$. Of course, this means that every invertible $n \times n$ matrix over a GE_n-ring can be diagonalized using only elementary row (respectively, column) operations.

It is easy to check that all rings with stable rank 1 are GE-rings. Note that if R is a GE-ring, then the natural homomorphism from $GL_1(R)$, the group of units of R, to $K_1(R)$

is surjective. For comparison, we recall the well-known fact that if R has stable rank d, then the natural map $GL_d(R) \to K_1(R)$ is surjective (e.g., [**12**, Theorem 40.42]).

1. Exchange rings and separativity

Although our notions and results will be right-left symmetric, all modules considered in this paper will be right modules. A module M over a ring R has the *finite exchange property* [**11**] if for every R-module A and any decompositions

$$A = M' \oplus N = A_1 \oplus \cdots \oplus A_n$$

with $M' \cong M$, there exist submodules $A_i' \subseteq A_i$ such that

$$A = M' \oplus A_1' \oplus \cdots \oplus A_n'.$$

(It follows from the modular law that A_i' must be a direct summand of A_i for all i.) It should be emphasized that the direct sums in this definition are internal direct sums of submodules of A. One advantage of the resulting internal direct sum decompositions (as opposed to isomorphisms with external direct sums) rests on the fact that direct summands with common complements are isomorphic – e.g., $N \cong \bigoplus_{i=1}^n A_i'$ above since each of these summands of A has M' as a complementary summand.

Following Warfield [**29**], we say that R is an *exchange ring* if R_R satisfies the finite exchange property. By [**29**, Corollary 2], this definition is left-right symmetric. If R is an exchange ring, then every finitely generated projective R-module has the finite exchange property (by [**11**, Lemma 3.10], the finite exchange property passes to finite direct sums and to direct summands), and so the endomorphism ring of any such module is an exchange ring. Further, idempotents lift modulo all ideals of an exchange ring [**24**, Theorem 2.1, Corollary 1.3].

The class of exchange rings is quite large. It includes all semiregular rings (i.e., rings which modulo the Jacobson radical are (von Neumann) regular and have idempotent-lifting), all π-regular rings, and more; see [**1**, **28**, **29**]. Further, all unital C*-algebras with real rank zero are exchange rings [**3**, Theorem 7.2].

The following criterion for exchange rings was obtained independently by Nicholson and the second author.

Lemma 1.1. [**17**, p. 167; **24**, Theorem 2.1] *A ring R is an exchange ring if and only if for every element $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1-a)R$.* $\square$

In the above lemma, it is equivalent to ask that for any $a_1, a_2 \in R$ with $a_1R + a_2R = R$, there exists an idempotent $e \in a_1R$ such that $1 - e \in a_2R$. We shall also need the analogous property corresponding to sums of more than two right ideals:

Lemma 1.2. [**24**, Theorem 2.1, Proposition 1.11] *Let R be an exchange ring. If $I_1, \dots, I_n$ are right ideals of R such that $I_1 + \cdots + I_n = R$, then there exist orthogonal idempotents $e_1, \dots, e_n \in R$ such that $e_1 + \cdots + e_n = 1$ and $e_j \in I_j$ for all j.* $\square$

We reiterate that a ring R is *separative* provided the following cancellation property holds for finitely generated projective right (equivalently, left) R-modules A and B:

$$A \oplus A \cong A \oplus B \cong B \oplus B \quad \Longrightarrow \quad A \cong B.$$

See [**3**] for the origin of this terminology and for a number of equivalent conditions. We shall need the following one:

Lemma 1.3. [**2**, Proposition 1.2; **3**, Lemma 2.1] *A ring R is separative if and only if whenever A, B, C are finitely generated projective right R-modules such that $A \oplus C \cong B \oplus C$ and C is isomorphic to direct summands of both A^n and B^n for some n, then $A \cong B$.* $\square$

Note, in particular, that if R is separative and A, B, C are finitely generated projective right R-modules, then we can certainly cancel C from $A \oplus C \cong B \oplus C$ whenever A and B are generators in Mod-R.

Separativity seems to hold quite widely within the class of exchange rings; for instance, it holds for all known classes of regular rings (cf. [**3**]). In fact, the existence of non-separative exchange rings is an open problem.

It is clear from either form of the condition that a ring R is separative in case the finitely generated projective R-modules enjoy cancellation with respect to direct sums, which in turn holds in case R has stable rank 1. In fact, for exchange rings, cancellation of finitely generated projective modules is equivalent to stable rank 1 [**31**, Theorem 9]. Separativity, however, is much weaker than stable rank 1. For example, any regular right self-injective ring is separative (e.g., [**15**, Theorem 10.34(b)]), but such rings can have infinite stable rank – e.g., the ring of all linear transformations on an infinite dimensional vector space.

2. K_1 of separative exchange rings

We use the notation $A \lesssim^{\oplus} B$ to denote that a module A is isomorphic to a direct summand of a module B.

Lemma 2.1. *Let R be an exchange ring and $e_1, \dots, e_n \in R$ idempotents. Then there exists an idempotent $e \in e_1R + \cdots + e_nR$ such that $e_1R \leq eR$ and $e_iR \lesssim^{\oplus} eR$ for all i. In particular, $ReR = Re_1R + \cdots + Re_nR$.*

Proof. By induction, it suffices to do the case $n = 2$. Now

$$R = e_1R \oplus (1 - e_1)R = e_2R \oplus (1 - e_2)R$$

and e_1R has the finite exchange property, so there exist decompositions $e_2R = A \oplus B$ and $(1 - e_2)R = A' \oplus B'$ such that $R = e_1R \oplus A \oplus A'$. Then we can choose an idempotent $e \in R$ such that $eR = e_1R \oplus A$. Obviously $e_1R \leq eR$, and since

$$e_1R \cong R/(A \oplus A') \cong B \oplus B',$$

we have $e_2R \lesssim^{\oplus} A \oplus e_1R = eR$. $\square$

Corollary 2.2. *Let R be an exchange ring and $a \in R$ such that $RaR = R$. Then there exist idempotents $e \in aR$ and $f \in Ra$ such that $ReR = RfR = R$.*

Proof. Write $R = \sum_{i=1}^n x_iaR$ for some x_i. By Lemma 1.2, there exist orthogonal idempotents $g_1, \dots, g_n \in R$ such that $g_1 + \cdots + g_n = 1$ and $g_i \in x_iaR$ for all i. Set $g_i = x_iay_i$ with $y_i = y_ig_i$. Then $e_i := ay_ix_i$ is an idempotent in aR and $e_iR \cong g_iR$. By Lemma 2.1, there exists an idempotent $e \in \sum_{i=1}^n e_iR$ such that $e_iR \lesssim^{\oplus} eR$ for all i. Then $e \in aR$ and $g_iR \lesssim^{\oplus} eR$ for all i, so all $g_i \in ReR$, and thus $ReR = R$.

The existence of f follows by symmetry. $\square$

Lemma 2.3. *Let R be any ring and $A \in GL_n(R)$. If A has an idempotent entry, then A can be reduced by elementary row and column operations to the form*

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{bmatrix}.$$

Proof. By elementary operations, we can move the idempotent entry, call it e, into the $1,1$ position. If $n = 1$, then e is invertible, so $e = 1$ and we are done. Now assume that $n > 1$, and let

$$\begin{bmatrix} e & b_2 & b_3 & \cdots & b_n \end{bmatrix}$$

be the first row of A. By elementary column operations, we can subtract eb_i from the i-th entry for each $i \geq 2$. Thus, we can assume that $b_2, \dots, b_n \in (1-e)R$. Since A is invertible, $eR + b_2R + \cdots + b_nR = R$, and so it follows that $b_2R + \cdots + b_nR = (1-e)R$. Hence, by elementary column operations we can add $1-e$ to the first entry. Now we have a 1 in the $1,1$ position, and the rest is routine. $\square$

Since we shall need to perform a number of operations on the top rows of invertible matrices, it is convenient to work with the rows alone. Recall that any row $[\, a_1 \quad a_2 \quad \cdots \quad a_n \,]$ of an invertible matrix over a ring R is *right unimodular*, that is, $\sum_{i=1}^n a_iR = R$. Elementary column operations apply to such a row just by viewing it as a $1 \times n$ matrix. Such operations amount to multiplying the row on the right by an elementary matrix. Since our rings need not be commutative, elementary column operations can only introduce right-hand coefficients.

Lemma 2.4. *Let R be an exchange ring and $\alpha = [\, a_1 \quad a_2 \quad \cdots \quad a_n \,]$ a right unimodular row over R. Then α can be transformed by elementary column operations to a row $[\, b_1 \quad b_2 \quad \cdots \quad b_n \,]$ such that $R = b_1R \oplus \cdots \oplus b_nR$ and each $b_i \in a_iRa_i$.*

Proof. Since $\sum_{i=1}^n a_iR = R$, Lemma 1.2 gives us orthogonal idempotents $e_1, \dots, e_n \in R$ such that $e_1 + \cdots + e_n = 1$ and $e_i \in a_iR$ for all i, say $e_i = a_ir_i$. By elementary column operations, we can subtract $e_ia_1 = a_ir_ia_1$ from the first entry of α for each $i \geq 2$. This transforms α to $\alpha' = [\, e_1a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_n \,]$. Note that $e_1 \in e_1a_1R$. Thus, we can repeat the above process for each entry, and transform α' to the row $[\, e_1a_1 \quad e_2a_2 \quad \cdots \quad e_na_n \,]$, with entries $e_ia_i \in a_iRa_i$. Moreover, $e_ia_iR = e_iR$, and therefore $R = \bigoplus_{i=1}^n e_ia_iR$. $\square$

Corollary 2.5. *Let R be an exchange ring and $\alpha = [\, a_1 \quad a_2 \quad \cdots \quad a_n \,]$ a right unimodular row over R, with $n \geq 2$. Then α can be transformed by elementary column operations to a row $[\, b_1 \quad b_2 \quad \cdots \quad b_n \,]$ such that $Rb_1R = R$ and $b_i \in a_iRa_i$ for all $i \geq 2$.*

Proof. By Lemma 2.4, we may assume that $R = \bigoplus_{i=1}^n a_iR$. It follows that all $a_i \in Rb_1$ where $b_1 = a_1 + \cdots + a_n$ (multiply b_1 on the left by the orthogonal idempotents arising from the given decomposition of R_R). Thus $Rb_1R = R$. By elementary column operations, we can add $a_2, \dots, a_n$ to the first entry of α, and thus transform it to $[\, b_1 \quad a_2 \quad \cdots \quad a_n \,]$. $\square$

Recall that an element x in a ring R is *(von Neumann) regular* provided there exists an element $y \in R$ such that $xyx = x$, equivalently, provided xR is a direct summand of R_R.

If y can be chosen to be a unit in R, then x is said to be *unit-regular*. A regular element $x \in R$ is unit-regular if and only if $R/xR \cong \mathrm{r.ann}(x)$, where $\mathrm{r.ann}(x)$ denotes the right annihilator of x in R (cf. [**15**, proof of Theorem 4.1]).

Corollary 2.6. *Let R be an exchange ring and $\alpha = [\,a_1 \quad a_2 \quad \cdots \quad a_n\,]$ a right unimodular row over R, with $n \geq 2$. Then α can be transformed by elementary column operations to a row $[\,c_1 \quad c_2 \quad \cdots \quad c_n\,]$ such that c_2 is a regular element, $c_2 \in Ra_2$, and $c_2R = (1-g)R$ for an idempotent g with $RgR = R$.*

Proof. By Corollary 2.5, we may assume that $Ra_1R = R$. By Corollary 2.2, there exists an idempotent $e \in a_1R$ such that $ReR = R$. By elementary column operations, we can subtract ea_2 from the second entry of α, so there is no loss of generality in assuming that $a_2 \in (1-e)R$. (At this stage, our current a_2 is only a <u>left</u> multiple of the original a_2. This is why the conclusions of the lemma state $c_2 \in Ra_2$ rather than $c_2 \in a_2Ra_2$.) Now using Lemma 2.4, we can transform α to a row $[\,c_1 \quad c_2 \quad \cdots \quad c_n\,]$ such that $R = \bigoplus_{i=1}^n c_iR$ and $c_2 \in a_2Ra_2$. Then $c_2R = (1-g)R$ for some idempotent g, and c_2 is regular. Moreover, $(1-g)R = c_2R \subseteq a_2R \subseteq (1-e)R$ and so $Re \subseteq Rg$. Therefore $RgR = R$. □

Lemma 2.7. *Let R be an exchange ring and $A \in GL_n(R)$, with $n \geq 2$. Then A can be transformed by elementary row and column operations to a matrix whose $1,1$ entry d is regular, with $dR = (1-p)R$ and $Rd = R(1-q)$ for some idempotents p, q such that $RpR = RqR = R$.*

Proof. By Corollary 2.6, we can assume that the $1,2$ entry of A is a regular element c such that $cR = (1-g)R$ for some idempotent g with $RgR = R$. With elementary operations, we can move c to the $2,1$ position.

Now apply the transpose of Corollary 2.6 to the first column of A. Thus, A can be transformed by elementary row operations to a matrix whose $2,1$ entry is a regular element d such that $d \in cR$ and $Rd = R(1-q)$ for some idempotent q with $RqR = R$. Since d is regular, $dR = (1-p)R$ for some idempotent p. Then $(1-p)R \subseteq (1-g)R$, whence $Rg \subseteq Rp$ and so $RpR = R$.

Finally, use elementary operations to move d to the $1,1$ position. □

Theorem 2.8. *If R is a separative exchange ring, then R is a GE-ring, and so the natural homomorphism $GL_1(R) \to K_1(R)$ is surjective.*

Proof. We need to show that R is a GE_n-ring for all n. This is trivial for $n = 1$, hence we assume, by induction, that $n \geq 2$ and that R is a GE_{n-1}-ring. Let A be an arbitrary invertible $n \times n$ matrix over R.

By Lemma 2.7, we may assume that the $1,1$ entry d of A is regular, with $dR = (1-p)R$ and $Rd = R(1-q)$ for some idempotents p, q such that $RpR = RqR = R$. We claim that d is unit-regular. Note that because $RpR = RqR = R$, the projective modules pR and qR are generators.

Now $R = \mathrm{r.ann}(d) \oplus B = dR \oplus C$ for some B, C, and we have to prove that $\mathrm{r.ann}(d) \cong C$. Since $B \cong dB = dR$, we have $\mathrm{r.ann}(d) \oplus B \cong C \oplus B$. From $Rd = R(1-q)$, we get $\mathrm{r.ann}(d) = qR$ and so $\mathrm{r.ann}(d)$ is a generator. Since $C \cong R/dR \cong pR$, we see that C is a generator too. By Lemma 1.3, $\mathrm{r.ann}(d) \cong C$ as desired.

The unit-regularity of d gives $d = ue$ for some unit u and idempotent e. Set

$$U = \begin{bmatrix} u & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix};$$

then the matrix $U^{-1}A$ has an idempotent entry. By Lemma 2.3, there exist $E, F \in E_n(R)$ such that

$$EU^{-1}AF = \begin{bmatrix} 1 & 0 \\ 0 & A' \end{bmatrix}$$

where $A' \in GL_{n-1}(R)$. By our induction hypothesis, $A' \in E_{n-1}(R)D_{n-1}(R)$. It follows that

$$A \in D_n(R)E_n(R)D_n(R)E_n(R),$$

and therefore we have shown that R is a GE_n-ring. This establishes the induction step and completes the proof. $\square$

Remarks 2.9. **(a)** Observe that the proof of Theorem 2.8 did not use the full force of separativity, only the cancellation property ($A \oplus C \cong B \oplus C \implies A \cong B$) for finitely generated projective R-modules A, B, C with A and B generators.

(b) Theorem 2.8 includes, in particular, the result of Menal and Moncasi that every factor ring of a right self-injective ring is a GE-ring [**22**, Theorem 2.2]. To make the connection explicit, recall that right self-injective rings are semiregular (e.g., [**13**, Theorem 2.16, Lemma 2.18]) and hence exchange; thus, all their factor rings are exchange rings. Further, any right self-injective ring is separative (e.g., [**14**, Theorem 3]). It follows that factor rings of right self-injective rings are separative [**3**, Theorem 4.2].

(c) As a special case of Theorem 2.8, we obtain that any separative regular ring is a GE-ring, which gives a partial affirmative answer to a question of Moncasi [**23**, Questió 5].

(d) In the situation of Theorem 2.8, one naturally asks for a description of the kernel of the epimorphism $GL_1(R) \to K_1(R)$. This has been answered for unit-regular rings and regular right-self-injective rings by Menal and Moncasi [**22**, Theorems 1.6, 2.6], and for exchange rings with primitive factors artinian by Chen and Li [**9**, Theorem 3]. In all the above cases, $K_1(R) \cong GL_1(R)^{\mathrm{ab}}$ provided $\frac{1}{2} \in R$ [**22**, Theorems 1.7, 2.6; **9**, Corollary 7]. Further, $K_1(R) \cong GL_1(R)^{\mathrm{ab}}$ when R is either a C*-algebra with unitary 1-stable range or an AW*-algebra [**22**, Theorem 1.3, Corollary 2.11] (here the algebraic K_1 is meant). The unit-regular and AW* results correct and extend earlier work of Handelman [**18**, Theorem 2.4; **19**, Theorem 7].

Theorem 2.10. *If R is a separative exchange ring and A is a (von Neumann) regular $n \times n$ matrix over R, then A can be diagonalized using elementary row and column operations.*

Proof. By [**2**, Theorem 2.4], there exist $P, Q \in GL_n(R)$ such that PAQ is diagonal. By Theorem 2.8, $P = U_1V_1$ and $Q = V_2U_2$, where $U_1, U_2 \in D_n(R)$ and $V_1, V_2 \in E_n(R)$. So V_1AV_2 is a diagonal matrix obtained from A by elementary row and column operations. $\square$

Remark 2.11. When applying Theorem 2.10, note the distinction between invertible matrices and general matrices. An invertible matrix over a separative exchange ring can be diagonalized from either side (by Theorem 2.8), whereas the diagonalization of a general regular matrix sometimes requires elementary operations on both the rows and the columns. For example, the 2×2 matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ over a field cannot be diagonalized using only elementary row operations.

Example 2.12. Non-regular matrices over separative exchange rings need not be diagonalizable by elementary operations, even over finite dimensional algebras. For example, choose a field F and let

$$R = F[x_1, x_2, x_3, x_4]/\langle x_1, x_2, x_3, x_4 \rangle^2.$$

Then R has a basis $1, a_1, a_2, a_3, a_4$ such that $a_i a_j = 0$ for all i, j. Since R is clearly semiregular, it is an exchange ring; separativity is an easy exercise. In fact, since R is artinian, it has stable rank 1. Recall that this also implies that R is a GE-ring. Observe that every element of R is a sum of a scalar plus a nilpotent element, and that the product of any two nilpotent elements of R is zero.

Now consider the matrix $A = \left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right]$, whose entries are linearly independent nilpotent elements of R. We claim that any sequence of elementary row or column operations on A can only produce a matrix whose entries are linearly independent nilpotent elements. For instance, consider a product

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} c_{11} + bc_{21} & c_{12} + bc_{22} \\ c_{21} & c_{22} \end{bmatrix}$$

where $c_{11}, c_{12}, c_{21}, c_{22}$ are linearly independent nilpotent elements. Then $b = \beta + n$ for some $\beta \in F$ and some nilpotent element n, whence $bc_{21} = \beta c_{21}$ and $bc_{22} = \beta c_{22}$, and so the entries in the matrix product above are linearly independent (they are clearly nilpotent). The same thing happens with other elementary operations, establishing the claim.

Therefore no sequence of elementary operations on A can produce a matrix with a zero entry. In particular, A cannot be diagonalized by elementary operations. Since R is a GE-ring, it follows that A cannot be diagonalized by invertible matrices either, i.e., there do not exist $X, Y \in GL_2(R)$ such that XAY is diagonal. Thus the first of the natural properties discussed in the introduction is not implied by the second. □

3. K_1 of separative C*-algebras with real rank zero

In connection with his work on the structure of multiplier algebras (e.g., [**32**, **33**, **34**]), Shuang Zhang has conjectured [unpublished] that if A is any unital C*-algebra with real rank zero, the topological $K_1(A)$ is isomorphic to the unitary group $U(A)$ modulo the connected component of the identity, $U(A)^\circ$. We confirm this conjecture in case A is separative, which at the same time provides a unified approach to all known cases of the conjecture. The main interest of Zhang's conjecture is in the case when the stable rank of A is greater than 1, since it has long been known that $K_1^{\mathrm{top}}(A) \cong U(A)/U(A)^\circ$ for all unital C*-algebras A with stable rank 1 (e.g., this is equivalent to [**26**, Theorem 2.10]).

We consider only unital, complex C*-algebras in this section, and we refer the reader to [**4**, **30**] for background and notation for C*-algebras. In particular, we use $\sim$ and $\lesssim$ to denote Murray-von Neumann equivalence and subequivalence of projections, and we write $M_\infty(A)$ for the (non-unital) algebra of $\omega \times \omega$ matrices with only finitely many nonzero entries from an algebra A. We write $U(A)$ for the unitary group of a unital C*-algebra A, and $U(A)^\circ$ for the connected component of the identity in $U(A)$.

In the theory of operator algebras, it is customary to write $K_1(A)$ for the topological K_1-group of A (e.g., [**4**, Definition 8.1.1; **30**, Definition 7.1.1]), and we shall follow that practice here. Thus, $K_1(A) = GL_\infty(A)/GL_\infty(A)^\circ$. We then use the notation $K_1^{\mathrm{alg}}(A)$ to denote the algebraic K_1-group of A. Since $K_1^{\mathrm{alg}}(A)$ is the abelianization of $GL_\infty(A)$ (e.g., [**27**, Proposition 2.1.4, Definition 2.1.5]) and $K_1(A)$ is abelian (e.g., [**4**, Proposition 8.1.3; **30**, Proposition 7.1.2]), there is a natural surjective homomorphism $K_1^{\mathrm{alg}}(A) \to K_1(A)$. Finally, following Brown [**5**, p. 116], we say that A has K_1*-surjectivity* (respectively, K_1*-injectivity*) provided the natural homomorphism $U(A)/U(A)^\circ \to K_1(A)$ is surjective (respectively, injective).

The concept of *real rank zero* for a C*-algebra A has a number of equivalent characterizations (see [**6**]). One is the requirement that each self-adjoint element of A can be approximated arbitrarily closely by real linear combinations of orthogonal projections. (This is usually phrased as saying that the set of self-adjoint elements of A with finite spectrum is dense in the set of all self-adjoint elements.) It was proved in [**3**, Theorem 7.2] that A has real rank zero if and only if it is an exchange ring. Hence, the C*-algebras with real rank zero are exactly the C*-algebras to which our results above can be applied.

Given a C*-algebra A, all idempotents in matrix algebras $M_n(A)$ are equivalent to projections (e.g., [**4**, Proposition 4.6.2; **27**, Proposition 6.3.12]). Hence, A is separative if and only if

$$p \oplus p \sim p \oplus q \sim q \oplus q \qquad \Longrightarrow \qquad p \sim q$$

for projections $p, q \in M_\infty(A)$. An equivalent condition (analogous to Lemma 1.3) is that $p \oplus r \sim q \oplus r \implies p \sim q$ whenever $r \lesssim n.p$ and $r \lesssim n.q$ for some n. Separativity in A is thus equivalent to the requirement that all matrix algebras $M_n(A)$ satisfy the *weak cancellation* introduced by Brown and Pedersen [**5**, p. 116; **7**, p. 114]. They have shown that every extremally rich C*-algebra (see [**7**, p. 125]) with real rank zero is separative ([**8**], announced in [**5**, p. 116]). We would like to emphasize the question of whether non-separative exchange rings exist by focusing on the C* case:

Problem. Is every C*-algebra with real rank zero separative?

By combining Theorem 2.8 with a result of Lin, we obtain the following theorem.

Theorem 3.1. *If A is a separative, unital C*-algebra with real rank zero, then the natural map $U(A)/U(A)^\circ \to K_1(A)$ is an isomorphism.*

Proof. Lin proved K_1-injectivity for C*-algebras with real rank zero in [**20**, Lemma 2.2]. Hence, it only remains to show K_1-surjectivity. It is a standard fact that $U(A)$ and $GL_1(A)$ have the same image in $K_1(A)$ (e.g., [**4**, pp. 66, 67] or [**30**, proof of Proposition 4.2.6]). Now the natural map $GL_1(A) \to K_1(A)$ factors as the composition of natural maps $GL_1(A) \to K_1^{\mathrm{alg}}(A) \to K_1(A)$, the second of which is surjective. Since A has real rank zero, it is an exchange ring, and so the map $GL_1(A) \to K_1^{\mathrm{alg}}(A)$ is surjective by Theorem 2.8. Therefore the image of $U(A)$ in $K_1(A)$ is all of $K_1(A)$, as desired. □

Brown and Pedersen have proved that every separative, extremally rich C*-algebra has K_1-surjectivity ([**8**], announced in [**5**, p. 116; **7**, p. 114]). Since there are C*-algebras with real rank zero that are not extremally rich [**5**, Example, p. 117], Theorem 3.1 can be viewed as a partial extension of the Brown-Pedersen result within the class of C*-algebras with real rank zero.

We thank the referee for the following remark.

Remark 3.2. While Theorem 3.1 is neither unexpected nor new in the case of stable rank 1 (cf. the result of Rieffel cited above), it is perhaps surprising that there are many C*-algebras of real rank zero and stable rank 2 to which the theorem applies. To see this, consider C*-algebra extensions

$$0 \to I \to A \to B \to 0$$

in which I and B have real rank zero and A is unital. By theorems of Zhang ([**35**, Lemma 2.4]; cf. [**6**, Theorem 3.14]) and Lin and Rørdam [**21**, Proposition 4], A has real rank zero if and only if projections lift from B to A, if and only if the connecting map $K_0(B) \to K_1(I)$ in topological K-theory vanishes. In this case, by [**3**, Theorem 7.5], A will be separative provided I and B are both separative, and in particular if I and B have stable rank 1. However, by [**21**, Proposition 4], if I and B have stable rank 1, then A will have stable rank 2 provided the connecting map $K_1(B) \to K_0(I)$ does not vanish. It is easy to find specific extensions satisfying the above conditions, such as the examples analyzed in [**21**, End of Section 1] or [**16**].

We conclude with an application of Theorem 3.1 that extends an argument of Brown [**5**, Theorem 1], relating homotopy and unitary equivalence of projections, to a wider context within real rank zero. Projections p and q in a C*-algebra A are *unitarily equivalent* provided there exists a unitary element $u \in A$ such that $upu^* = q$; they are *homotopic* provided there is a continuous path $f : [0,1] \to \{\text{projections in } A\}$ such that $f(0) = p$ and $f(1) = q$. It is a standard fact that homotopic projections are unitarily equivalent (e.g., [**4**, Propositions 4.3.3, 4.6.5; **30**, Proposition 5.2.10]).

Theorem 3.3. *Let A be a separative, unital C*-algebra with real rank zero, let $p, q \in A$ be projections, and let $B = \overline{ApA} + \mathbb{C} \cdot 1$. Then p and q are homotopic in A if and only if $q \in B$ and p, q are unitarily equivalent in B.*

Proof. If p and q are homotopic in A, they are connected by a path of projections within A. Each projection along this path is homotopic to p and hence is unitarily equivalent to p. Thus, these projections all lie in ApA. In particular, $q \in B$, and p and q are homotopic in B. Consequently, p and q must be unitarily equivalent in B.

Conversely, assume that $q \in B$ and p, q are unitarily equivalent in B. By [**6**, Corollary 2.8, Theorem 2.5], the closed ideal $I = \overline{ApA}$ has real rank zero (as a non-unital C*-algebra), and so the unital C*-algebras B and pIp have real rank zero. We do not need separativity for B, just K_1-injectivity (by Lin's result). Since I is an ideal of A, any projections in $M_\infty(I)$ which are (Murray-von Neumann) equivalent in $M_\infty(A)$ are also equivalent in $M_\infty(I)$ (any implementing partial isometry necessarily lies in $M_\infty(I)$). Hence, the separativity of A implies that I is separative, and so pIp is separative. Therefore, by Theorem 3.1, pIp has K_1-surjectivity.

With the above information in hand, Brown's proof [**5**, Theorem 1] carries through in the present setting. We sketch the details for the reader's convenience. By hypothesis, $q = upu^*$ for some unitary $u \in U(B)$; let α denote the image of u in $K_1(B)$. Now $K_1(B) = K_1(I^\sim) = K_1(I)$, and because pIp is a full hereditary sub-C*-algebra of I, the natural map $K_1(pIp) \to K_1(I)$ is an isomorphism [**5**, Remark, p. 117]. Thus α is the image of some $\beta \in K_1(pIp)$. Since pIp has K_1-surjectivity, β is the image of some unitary $v_1 \in U(pIp)$. Let $v = v_1 + 1 - p$ and $w = uv^*$. Then w is a unitary in B such that $q = wpw^*$, and the image of w in $K_1(B)$ is zero. Since B has K_1-injectivity, $w \in U(B)^\circ$, from which it follows that p and q are homotopic. $\square$

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