The complete dimension theory of partially ordered systems with equivalence and orthogonality

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Contents

Abstract

We develop dimension theory for a large class of structures of the form $(L, \leq, \perp, \sim)$, where $(L \leq)$ is a partially ordered set, $\perp$ is a binary relation on L, and $\sim$ is an equivalence relation on L, subject to certain axioms. We call these structures *espaliers*. For x, y, $z \in L$, we say that $z = x \oplus y$ holds, if $x \perp y$ and z is the supremum of $\{x, y\}$. The *dimension theory* of L is the universal $\sim$-invariant homomorphism from $(L, \oplus, 0)$ to a partial commutative monoid S. We say that S is the *dimension range* of L. Particular examples of espaliers are the following:

(i) Let B be a complete Boolean algebra. For x, $y \in B$, we say that $x \perp y$ if $x \wedge y = 0$, and we take $\sim$ to be any zero-separating, unrestrictedly additive and refining equivalence relation on B (for instance, equality).

(ii) Let R be a right self-injective von Neumann regular ring. We denote by L the lattice of all direct summands of a given nonsingular injective right R-module, for instance, the lattice of finitely generated right ideals of R. For A, $B \in L$, we say that $A \perp B$ if $A \cap B = \{0\}$, and $A \sim B$ if $A \cong B$.

(iii) More generally, let L be a complete, meet-continuous, complemented, modular lattice. For x, $y \in L$, we say that $x \perp y$ if $x \wedge y = 0$, and $x \sim y$ if x and y are projective by (finite) decomposition.

(iv) Let A be an AW*-algebra. We denote by L the lattice of projections of A, and take the standard orthogonality and equivalence relations on L. For p, $q \in L$, then, $p \perp q$ if $pq = 0$, and $p \sim q$ if p and q are *Murray-von Neumann equivalent*, that is, there exists $x \in A$ such that $p = x^*x$ and $q = xx^*$.

We prove that the dimension range of any espalier $(L, \leq, \perp, \sim)$ is a *lower interval* of a commutative monoid of the form

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma), \tag{*}$$

where Ω_{I}, Ω_{II}, and Ω_{III} are complete Boolean spaces, and where we put, for every ordinal γ,

$$\mathbb{Z}_\gamma = \mathbb{Z}^+ \cup \{\aleph_\xi \mid 0 \leq \xi \leq \gamma\},$$
$$\mathbb{R}_\gamma = \mathbb{R}^+ \cup \{\aleph_\xi \mid 0 \leq \xi \leq \gamma\},$$
$$\mathbf{2}_\gamma = \{0\} \cup \{\aleph_\xi \mid 0 \leq \xi \leq \gamma\},$$

endowed with their interval topology and natural addition operations. Conversely, we prove that every lower interval of a monoid of the form (*) can be represented as the dimension range of an espalier arising from each of the contexts (i)–(iv) above. The context of W*-algebras requires the spaces Ω_{I}, Ω_{II}, and Ω_{III} to be *hyperstonian*, and no further restriction is needed.

This subsumes many earlier dimension-theoretic results, and, in applications, completes theories developed for examples such as (i)–(iv) above.

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CHAPTER 1

Introduction

1-1. Background

The central theme of this paper is, as indicated by the title, *dimension theory*. Basically, an equivalence relation $\sim$ is given on a structure L, and our goal is to elucidate the quotient structure $L/{\sim}$. We shall be interested in cases where L is a *complete lattice* endowed with a notion of orthogonality subject to a number of axioms. To make it clear that the motivations for this are widespread and by no means confined to lattice theory, we start by discussing what is known about some fundamental examples. In Chapter 5, we will apply our general theory to these examples, thus showing the improvements that it brings to them.

1-1.1. Abstract measure theory. One of the most basic examples of what could be called a "dimension theory" arises from measure theory. To pick a favorite, we first consider the Lebesgue measure m on the real line $\mathbb{R}$. It is defined on the Boolean algebra $\mathcal{B}$ of all Lebesgue-measurable subsets of $\mathbb{R}$. However, it fails total additivity of measure, for every subset of $\mathbb{R}$ is the union of singletons, which have Lebesgue measure zero. To bring back total additivity, the standard way is to say that m is defined not on $\mathcal{B}$, but on the quotient algebra $B = \mathcal{B}/\mathcal{N}$, where $\mathcal{N}$ is the ideal of null sets. The Boolean algebra B and the resulting map from B to $[0, +\infty]$, which we still denote by m, have the following properties:

(a) B is a *complete* Boolean algebra.

(b) The map m is *unrestrictedly additive*, that is, the following equality holds:

$$\mathsf{m}\left(\bigvee_{i\in I} x_i\right) = \sum_{i\in I} \mathsf{m}(x_i),$$

for any *disjoint* family $(x_i)_{i\in I}$ of elements of B. The notation $\bigvee_{i\in I} x_i$ stands for the *join* (i.e., supremum) of the set $\{x_i \mid i \in I\}$ in B.

(c) For all x, $y \in B$, if y is a translate of x (that is, $y = \alpha + x$ for some real number α), then $\mathsf{m}(x) = \mathsf{m}(y)$.

Rule (b) above seems somehow puzzling at first glance, because of the apparent possibility of an uncountable index set I. However, since the Boolean algebra B is countably saturated, all infinite joins in B are, really, *countable* joins, so that, in (b), all the x_i-s are majorized by the join of countably many of them.

For x, $y \in B$, we define the relation $x \sim y$ to hold, if there are disjoint families $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ of elements of B such that $x = \bigvee_{i\in I} x_i$ and $y = \bigvee_{i\in I} y_i$, and y_i is a translate of x_i, for all $i \in I$. It is not difficult to verify that $\sim$ is an *equivalence relation* on B. Furthermore, by (b) and (c) above, $x \sim y$ implies that $\mathsf{m}(x) = \mathsf{m}(y)$, for all x, $y \in B$.

It is harder to verify that the converse of the above fact also holds, namely: $\mathsf{m}(x) = \mathsf{m}(y)$ implies that $x \sim y$, for all $x, y \in B$. This fact is due to S. Banach and A. Tarski, see [**2**], or [**52**, Theorem 9.17]. Hence the quotient set $B/{\sim}$ is isomorphic, *via* the measure m, to the interval $[0, +\infty]$. A moment's reflection shows that $B/{\sim}$ can be endowed with a *partial addition*, defined by the rule

$$[x] + [y] = [x \vee y], \quad \text{for all disjoint } x, y \in B,$$

that endows it with a structure of *partial commutative monoid* (see Definition 2-1.2), and that the measure m factors through an isomorphism of partial monoids between $B/{\sim}$ and $[0, +\infty]$. We see in this particular case that $B/{\sim}$ is a *total* monoid, that is, the addition of $B/{\sim}$ is defined everywhere.

Now let us consider the *converse* of the above paragraph. That is, we are given a Boolean algebra B, endowed with an equivalence relation $\sim$, and we wish to find the structure of $B/{\sim}$. While this problem in full generality can lead to almost any structure, we focus the study by making the following assumptions on B and $\sim$, that are satisfied for the example above:

(1) B is a complete Boolean algebra.
(2) $x \sim 0$ implies that $x = 0$, for all $x \in B$.
(3) (see Axiom (L6) of Definition 4-1.1) The relation $\sim$ is *unrestrictedly refining*, that is, for every $a \in L$ and every disjoint family $(b_i)_{i \in I}$ of elements of L, if $a \sim \bigvee_{i \in I} b_i$, then there exists a decomposition $a = \bigvee_{i \in I} a_i$, with $(a_i)_{i \in I}$ disjoint, such that $a_i \sim b_i$ for all $i \in I$.
(4) (see Axiom (L7) of Definition 4-1.1) The relation $\sim$ is *unrestrictedly additive*, that is, for all disjoint families $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ of elements of L, if $a_i \sim b_i$ for all $i \in I$, then $\bigvee_{i \in I} a_i \sim \bigvee_{i \in I} b_i$.

The most basic example of this situation is for $B = \mathfrak{P}(\Omega)$, the powerset algebra of an infinite set Ω, where $\sim$ is the relation of *equipotency* on subsets of Ω, that is, $X \sim Y$ if and only if there exists a bijection from X onto Y. If γ is the unique ordinal such that $|\Omega| = \aleph_\gamma$, then $L/{\sim}$ is isomorphic to the monoid

$$\mathbb{Z}_\gamma = \mathbb{Z}^+ \cup \{\aleph_\xi \mid 0 \leq \xi \leq \gamma\},$$

endowed with the addition that extends the natural addition of the set $\mathbb{Z}^+$ of nonnegative integers and such that $n + \aleph_\beta = \aleph_\alpha + \aleph_\beta = \aleph_\beta$, for all $n \in \mathbb{Z}^+$ and all ordinals α, β such that $\alpha \leq \beta \leq \gamma$, see page 38. So, if $\mu \colon B \to \mathbb{Z}_\gamma$ is the map defined by the rule $\mu(X) = |X|$, for all $X \in B$, then μ factors through $\sim$, thus defining an isomorphism from $B/{\sim}$ onto $\mathbb{Z}_\gamma$.

As we shall see in this paper, it is still possible, in the general case, to obtain a "measure" μ on B such that $B/{\sim}$ is isomorphic to the range of μ. The range of the measure μ is not necessarily $[0, +\infty]$ and not even some $\mathbb{Z}_\gamma$ (as in the example above), but rather a certain set of continuous functions from a complete Boolean space (i.e., extremally disconnected compact Hausdorff topological space) Ω to a monoid of the form $\mathbb{R}^+ \cup \{\aleph_\xi \mid 0 \leq \xi \leq \gamma\}$ (or a submonoid of this monoid). A similar result is achieved by D. Maharam in [**40**], in a slightly different context—for instance, all sums and joins are countable joins, while B satisfies the countable chain condition. This is not the only restriction imposed in Maharam's work, as, for example, Axiom III, page 281 in [**40**], that rules out what we will call later the "Type III" case.

1-1.2. Nonsingular injective modules over self-injective regular rings. Let R be a (von Neumann) regular, right self-injective ring, let M be a nonsingular injective right R-module. We order the set L of all direct summands of M by inclusion and we endow it with the relation of isomorphism, $\cong$. The *dimension theory* of M is the study of the structure of $L/{\cong}$. We say that a family $(X_i)_{i\in I}$ of elements of L is *orthogonal*, if the sum of the submodules X_i is a direct sum. We recall some fundamental properties of L and $\cong$ (references will be given in Section 5-3):

(1) L is a complete lattice, that is, every subset of L has a supremum.
It is known that the infimum of a family $(X_i)_{i\in I}$ of elements of L is their *intersection*, $\bigcap_{i\in I} X_i$.
(2) L is *complemented*, that is, every element X of L has a complement (that is, an element Y of L such that $X \oplus Y = M$).
(3) L is *meet-continuous*, that is, for every $X \in L$ and every upward directed family $(Y_i)_{i\in I}$ of elements of L, the following equality holds:

$$X \cap \bigvee_{i\in I} Y_i = \bigvee_{i\in I} (X \cap Y_i).$$

(4) L is *modular*, that is, the equality

$$X \cap (Y \vee Z) = (X \cap Y) \vee Z$$

holds, for all X, Y, $Z \in L$ such that $X \supseteq Z$.
(5) (see Axiom (L6) of Definition 4-1.1) The relation $\cong$ is *unrestrictedly refining*, that is, for every $X \in L$ and every orthogonal family $(Y_i)_{i\in I}$ of elements of L, if $X \cong \bigvee_{i\in I} Y_i$, then there exists an orthogonal decomposition $X = \bigvee_{i\in I} X_i$ such that $X_i \cong Y_i$ for all $i \in I$.
(6) (see Axiom (L7) of Definition 4-1.1) The relation $\cong$ is *unrestrictedly additive*, that is, for all orthogonal families $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ of elements of L, if $X_i \cong Y_i$ for all $i \in I$, then $\bigvee_{i\in I} X_i \cong \bigvee_{i\in I} Y_i$.

We observe that the supremum in L of a family $(X_i)_{i\in I}$ of elements of L is not given by the sum of submodules $\sum_{i\in I} X_i$, but by its *injective hull*, $\mathrm{E}\left(\sum_{i\in I} X_i\right)$ (which can be identified with a unique submodule of M because M is injective and nonsingular).

As in Subsection 1-1.1, the quotient set $L/{\cong}$ can be endowed with a structure of *partial commutative monoid*, under the addition given by the rule

$$[X] + [Y] = [X \oplus Y] \quad \text{if } X \cap Y = \{0\},$$

for all X, $Y \in L$.

Essentially by using Axioms (1)–(6) above, the structure of $L/{\cong}$ has been completely elucidated in several particular cases. For example, in case M is *directly finite* (i.e., M is not isomorphic to any proper direct summand of itself), $L/{\cong}$ is isomorphic to a lower subinterval (with respect to the componentwise ordering) of a monoid of the form

$$M = \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}^+ \cup \{\infty\}) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}^+ \cup \{\infty\}), \tag{1-1.1}$$

where Ω_{I} and Ω_{II} are complete Boolean spaces; see Chapter 11 in K. R. Goodearl and A. K. Boyle [**18**]. In the general case, there are a monoid M of the form given by (1-1.1) and a direct power N of a monoid of the form $\{0\}\cup\{\aleph_\xi \mid \xi < \gamma\}$ (for a certain ordinal γ) such that $L/{\cong}$ embeds into $M \times N$, see Chapters 12 and 13 in [**18**], and the variations in [**17**, Chapter 12]. Further results along these lines were obtained by C. Busqué [**7**], who showed, in particular, that the second factor of the embedding above, namely the map $L/{\cong} \to N$, actually sends $L/{\cong}$ to $\mathbf{C}(\Omega, \{0\} \cup \{\aleph_\xi \mid \xi < \gamma\})$ for a suitable complete Boolean space Ω (containing $\Omega_{\mathrm{I}} \sqcup \Omega_{\mathrm{II}}$) [**7**, Proposition 4.7]. However, these embeddings do not provide an isomorphism of $L/{\cong}$ onto a *lower subset* of a monoid of continuous functions. The difficulties are already visible in case R is a complete Boolean algebra (viewed as a ring), due to an example of K. Eda [**11**]: There exists a complete Boolean algebra R such that the injective hull of the free R-module of rank $\aleph_0$ contains a direct sum of $\aleph_1$ copies of itself (see the discussion of Problem 18 in [**17**, p. 374]). Here the image of the embedding obtained from [**18**], [**17**], and [**7**] contains a function with all values at least $\aleph_2$, but not the constant function with value $\aleph_1$.

1-1.3. Conditionally complete, meet-continuous, sectionally complemented, modular lattices. For elements a, b, and c in a lattice L with zero, we say that $c = a \oplus b$, if $c = a \vee b$ and $a \wedge b = 0$. We say that L is *sectionally complemented*, if for all a, $b \in L$ such that $a \leq b$, there exists $x \in L$ such that $a \oplus x = b$. If L is *modular*, that is, the implication

$$x \geq z \Longrightarrow x \wedge (y \vee z) = (x \wedge y) \vee z$$

holds, for all x, y, $z \in L$, then the partial operation $\oplus$ gives L a structure of *partial commutative monoid*. Completeness and meet-continuity of L are defined as in (1) and (3) of Subsection 1-1.2. So, in particular, if M is a nonsingular injective right module over a right self-injective regular ring R, then the lattice of all direct summands of M is complete, meet-continuous, sectionally complemented, and modular. The classical von Neumann *continuous geometries*, see J. von Neumann [**51**] or F. Maeda [**38**], are obtained by adding the conditions that L has a unit (that is, a largest element) and is join-continuous.

At this point, we seem to be stymied because of the following problem. We cannot claim outright that our lattice-theoretical context could lead to generalizations of Subsection 1-1.2, for there is no such thing *a priori* as "isomorphism of submodules" between the elements of L. In the case of continuous geometries, it is easy to remedy this by replacing isomorphism by *perspectivity*, which turns out to be *transitive* (this is a difficult result, due to J. von Neumann [**51**]). Elements a and b of a lattice L are *perspective*, in notation $a \sim b$, if there exists $x \in L$ such that $a \wedge x = b \wedge x$ and $a \vee x = b \vee x$. For continuous geometries, the structure of $L/{\sim}$ is completely understood, see [**51**] and, for the general, reducible case, T. Iwamura [**26**]—namely, $L/{\sim}$ is isomorphic to a lower segment of the positive cone of a Dedekind complete lattice-ordered group. The paper J. Harding and M. F. Janowitz [**25**] shows how a reducible continuous geometry can be represented as the space of continuous sections of a bundle of irreducible continuous geometries, thus shedding more light on the transition from irreducible continuous geometries to reducible ones.

However, for a general complete, meet-continuous, sectionally complemented, modular lattice L, the relation of perspectivity $\sim$ on L is not transitive as a rule—see, for example, the obvious case where L is the subspace lattice of an infinite-dimensional vector space over a field. Hence, we have to find a better candidate than $\sim$ to replace isomorphism of submodules. A natural guess is of course the *transitive closure* $\approx$ of $\sim$ (usually called *projectivity*), but this relation fails to be additive, as defined in Axiom (L7) of Definition 4-1.1, and as isomorphism of submodules should be. The final answer is, in fact, nontrivial, and it follows from the theory of *normal equivalences* introduced by the second author in Chapters 10–13 of [**56**]. Namely, there is (fortunately!) *exactly one* "reasonable" candidate for isomorphism, and it is the binary relation $\equiv$ on L defined by the rule

$$a \equiv b \text{ if there are decompositions } a = x_0 \oplus x_1 \text{ and } b = y_0 \oplus y_1, \quad \text{with } x_0 \sim y_0 \text{ and } x_1 \sim y_1,$$

for all a, $b \in L$. The transitivity of $\equiv$ is proved in Theorem 13.2 of [**56**], while the complete additivity of $\equiv$ (Axiom (L7) of Definition 4-1.1, see also item (6) of Subsection 1-1.2) is proved as in Proposition 13.9 of [**56**] by replacing countable families by arbitrary families. The quotient $L/{\equiv}$ is then a lower subset of the so-called *dimension monoid* $\operatorname{Dim} L$ of L, which, as its name indicates, is a (commutative) monoid. The dimension theory of L is elucidated here in Theorem 5-2.6. In the context of Subsection 1-1.2, that is, L is the lattice of all direct summands of a given nonsingular injective right module over a right self-injective regular ring, it is then the case that $\equiv$ is identical to submodule isomorphism on L, see Lemma 10.2 and Theorem 13.2 of [**56**].

Of crucial importance for all the proofs of these results is a result of I. Halperin and J. von Neumann [**22**] that states that $x \approx y$ and $x \wedge y = 0$ implies that $x \sim y$, for all x, $y \in L$. This result is extended in [**56**] to countably meet-continuous lattices, where it is used to prove that the quotient $L/{\equiv}$ is then a so-called *generalized cardinal algebra*, see Proposition 13.10 of [**56**]. However, even in case the lattice-theoretical version of direct finiteness (see Subsection 1-1.2) holds in L, no analogue of an embedding into monoids of the form (1-1.1) had been found before the present work.

1-1.4. Lattices of projections of W*- and AW*-algebras. We recall that an *AW*-algebra* is a C*-algebra A such that the right annihilator of any subset X of A has the form pA, for a projection p of A (a *projection* of A is an element p of A such that $p = p^2 = p^*$). We denote by L the set of projections of A. Let $a \leq b$ hold, if $ab = a$ (equivalently, $ba = a$), for all a, $b \in L$. Thus $\leq$ is a partial ordering on L. Orthogonality of any projections a and b, in notation $a \perp b$, is defined by $ab = 0$, and (Murray-von Neumann) equivalence is defined by the rule

$$a \sim b \text{ if there exists } x \in A \text{ such that } a = x^*x \text{ and } b = xx^*.$$

Much of the structure of L was developed axiomatically by I. Kaplansky in his monograph [**30**]. Some of the axioms and methods we use were inspired by Kaplansky's work, as was the structure theory for nonsingular injective modules constructed by Goodearl and Boyle [**18**]. Readers familiar with either of those works will recognize the parallels below.

Again, the quotient $L/{\sim}$ can be endowed with a structure of partial commutative monoid, where addition is given by the rule

$$[a] + [b] = [a + b], \quad \text{if } ab = 0, \text{ for all } a, b \in L,$$

where $[p]$ denotes the $\sim$-equivalence class of a projection p of A. The amount of known general information on the structure of $L/{\sim}$ is more fragmentary than for the examples considered in previous sections, due to fewer axioms satisfied. For example, the analogues of properties (3) (meet-continuity) and (4) (modularity) considered in Subsection 1-1.2 fail for projections of AW*-algebras as a rule. In F. J. Murray and J. von Neumann [**42**], a $[0, +\infty]$-valued "dimension function" is constructed on the projections of any *W*-factor* (i.e., indecomposable von Neumann algebra); Kaplansky showed that the same construction could be carried out for AW*-factors. Still in the indecomposable case, it is known that the closed two-sided ideals are well-ordered, see F. B. Wright [**58**]. Most of what was known about $L/{\sim}$ in the general case could be obtained from more general, often lattice-theoretical works that we shall discuss now.

1-1.5. Lattice-theoretical generalizations. A common feature of the structures considered in Subsections 1-1.1–1-1.4 is that they all involve a complete, sectionally complemented lattice L, a binary relation $\perp$ on L, and an equivalence relation $\sim$ on L. It has been observed early that even apart from the classical study of continuous geometries, the dimension theory of a given structure could be done by just studying the associated structure $(L, \perp, \sim)$. Furthermore, these structures will be *ordered* structures, so that we shall write $(L, \leq, \perp, \sim)$ instead of $(L, \perp, \sim)$:

— It is in S. Maeda [**39**] that the most general axiomatization of the structures $(L, \leq, \perp, \sim)$ is given. It holds for all the examples considered in Subsections 1-1.1–1-1.4, and this allows to construct "dimension functions"—corresponding to the measures of Subsection 1-1.1—on L that, in the "finite" case, separate the elements of L.
— In L. H. Loomis [**35**], another axiomatization is used, that involves an *orthocomplementation* on L, thus it does not apply to the examples considered in Subsections 1-1.2 and 1-1.3.
— In P. A. Fillmore [**12**], a further axiomatization of the structures $(L, \perp, \sim)$ is introduced, that does not assume completeness of L but rather *countable* completeness, and that assumes an orthocomplementation (thus, again, it does not encompass Subsections 1-1.2 and 1-1.3). One of the main results is that the structure $L/{\sim}$ is a generalized cardinal algebra (as in Subsection 1-1.3). Furthermore, under some countability assumptions, L is complete and $L/{\sim}$ is isomorphic to a lower subset of a monoid of the form (1-1.1), see [**12**, Theorem 3.12].

Nevertheless, in each class of examples considered in Subsections 1-1.1–1-1.4, some of the dimension-theoretical properties that one could have expected to hold were still missing from the known results. For example, there has been no general treatment of the reducible Type III case; it was seemingly not even clear whether or not it had to be treated as a pathology.

1-2. Results and methods

In view of the various examples presented in Section 1-1 and of what is known about them, the main goals of this paper are the following:

(1) To capture in a convenient set of axioms the various properties of the structures $(L, \leq, \perp, \sim)$ encountered in these examples. This set of axioms should be sufficient to develop a *complete* dimension theory of these structures, that is, a complete description of the structures $L/{\sim}$, without additional assumptions such as finiteness or chain conditions.

(2) To develop a set of monoid-theoretical axioms that should be satisfied by the structures $L/{\sim}$.
(3) Although the set of axioms obtained in (2) is quite complicated, our third goal will be to give a simple description of the structures satisfying the axioms of (2) in terms of continuous functions on complete Boolean spaces.

We shall now give some details about our road to these goals.

Espaliers. The relevant structures $(L, \leq, \perp, \sim)$ will be called *espaliers*, see Definition 4-1.1. The axiom system defining espaliers is stronger than the axiom system $(1, \alpha)$, $(1, \beta)$, ..., $(1, \zeta)$, $(2, \alpha)$, ..., $(2, \zeta)$ considered by S. Maeda in [**39**]. Nevertheless, these axioms are sufficient for our purposes—for instance, all the examples considered in Section 1-1 are espaliers. The only drastic generalization that we will introduce is to state that the underlying partial ordering of an espalier $(L, \leq, \perp, \sim)$ defines a *partial*, as opposed to total, lattice, so that for elements a and b of L, the meet (i.e., infimum) $a \wedge b$ of $\{a, b\}$ always exists, but the join (i.e., supremum) of $\{a, b\}$ exists only in case $\{a, b\}$ is majorized. This small generalization affects neither the proofs nor even the results—the structures $L/{\sim}$ are partial structures anyway—and it paves the way for further algebraic constructions on espaliers, such as *amalgamation*.

Continuous dimension scales. In parallel to this, we shall develop a system of monoid-theoretical axioms, (M1)–(M6) (see Definition 3-1.1), that captures the structures (partial commutative monoids) $L/{\sim}$, for an espalier L. This axiom system is rather complicated, but it completely isolates what monoid theory we need to understand the structures $L/{\sim}$. Among these axioms is a variant of *conditional completeness* (see Axiom (M2)), that is, every nonempty subset admits an infimum for the algebraic (pre)ordering (see Definition 2-1.3), but there are other, less natural-looking axioms, such as (M6).

The partial commutative monoids satisfying Axioms (M1)–(M6) will be called *continuous dimension scales*. They are unrelated to H. Lin's "continuous scales" introduced in [**33, 34**].

The relation between espaliers and continuous dimension scales is then given by the following (see Theorem 4-3.9).

Theorem A. *Let $(L, \leq, \perp, \sim)$ be an espalier. Then the partial commutative monoid $L/{\sim}$ of all $\sim$-equivalence classes of elements of L is a continuous dimension scale.*

Descriptions of continuous dimension scales. At first glance, Theorem A may appear as the ultimate goal of this paper. However, it provides only a list of properties of the partial monoids $L/{\sim}$, without giving any representation in terms of known structures. Moreover, although the axioms describing the structure of espalier seem to be almost the weakest possible to obtain a complete dimension theory, and thus, in some sense, unavoidable, this might not seem to be the case *a priori* for the axioms describing continuous dimension scales. We counter this by proving that there are no "missing" axioms for continuous dimension scales relative to espaliers.

Theorem B. *A partial commutative monoid S is a continuous dimension scale if and only if $S \cong L/{\sim}$ for some espalier $(L, \leq, \perp, \sim)$.*

Theorem B follows from the fact that most of our classes of examples of espaliers are *universal* in the sense that arbitrary continuous dimension scales can be represented (isomorphically) as lower subsets of the continuous dimension scales $L/{\sim}$ arising from these examples—see, for example, Theorems 5-1.13, 5-2.8, 5-3.14, 5-4.10.

As for a concrete representation of continuous dimension scales, we exhibit them as lower subsets (for the algebraic preordering) of product spaces of the form

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma),$$

where Ω_{I}, Ω_{II}, and Ω_{III} are complete Boolean spaces and, for any ordinal γ, the monoids $\mathbb{Z}_\gamma$, $\mathbb{R}_\gamma$, and $\mathbf{2}_\gamma$ are defined as

$$\begin{aligned}
\mathbb{Z}_\gamma &= \mathbb{Z}^+ \cup \{\aleph_\xi \mid 0 \leq \xi \leq \gamma\},\\
\mathbb{R}_\gamma &= \mathbb{R}^+ \cup \{\aleph_\xi \mid 0 \leq \xi \leq \gamma\},\\
\mathbf{2}_\gamma &= \{0\} \cup \{\aleph_\xi \mid 0 \leq \xi \leq \gamma\},
\end{aligned}$$

endowed with the natural addition and ordering, together with the interval topology (see Section 1-3). See page 38 for more details.

Theorem C. *Let S be a partial commutative monoid. Then S is a continuous dimension scale if and only if it can be embedded as a lower subset into a product monoid of the form*

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma),$$

where Ω_{I}, Ω_{II}, and Ω_{III} are complete Boolean spaces.

A more precise version of Theorem C is formulated in Theorem 3-8.9. The concrete version of Theorem B is that any lower subset of a monoid of the form

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma),$$

can be represented as $L/{\sim}$, for a suitable espalier L. More precisely, we show that L may arise from each of the above contexts—abstract measure theory, nonsingular injective modules over self-injective regular rings, meet-continuous complemented modular lattices, and projection lattices of AW*-algebras, see Sections 5-1–5-4. For projection lattices of W*-algebras, there is an additional restriction on the spaces Ω_{I}, Ω_{II}, Ω_{III}—namely, they are *hyperstonian*, see Corollary 5-4.8. In addition, it is worth noticing that although the embedding in Theorem C is not unique as a rule, it is determined by the condition that it "commutes with projections" and its value at the elements of a *finitary unit* of S (Definition 3-7.6), see Theorem 3-9.10. Finally, all this extends to "continuous dimension scales" that are no longer *sets*, but rather *proper classes*. The corresponding common extensions of the abovementioned "existence" and "uniqueness" statements hold, and they are presented in Theorem 3-10.5.

In order to make the results and methods of this paper accessible to the widest audience, we have avoided the use of forcing and Boolean-valued models for most proofs. Exceptions to this rule are the proofs of D-universality for the classes of Boolean espaliers (Theorem 5-1.13) and espaliers of projections of AW*-algebras (Theorem 5-4.9), as reasonable "forcing-free" proofs do not seem to be available.

1-3. Notation and terminology

Disjoint unions of sets will be denoted by $\sqcup$, $\bigsqcup$, so that, for example, $X = \bigsqcup_{i\in I} X_i$ means that $X = \bigcup_{i\in I} X_i$ and that $X_i \cap X_j = \varnothing$, for all distinct i, $j \in I$.

Following the usual set-theoretical terminology, we denote by ω the set of all natural numbers. We identify any natural number n with $\{0, 1, \ldots, n-1\}$. Any ordinal α is identified with the set of all ordinals less than α. A cardinal is an initial ordinal. Following well-established set-theoretical practice, for an ordinal α, the notations ω_α and $\aleph_\alpha$ both denote the α-th infinite cardinal, except that the first one is viewed as an ordinal while the second one is viewed as a cardinal.

If P is a partially preordered set, a subset X of P is a *lower subset* (resp., *upper subset*) of P if $x \le y$ and $y \in X$ (resp., $x \in X$) implies that $x \in X$ (resp., $y \in X$), for all x, $y \in P$. For an element a of P, we denote by $(a]$ (resp., $[a)$) the lower subset (resp., upper subset) of P generated by a. A subset X of P is *coinitial*, if $[X) = \bigcup_{a\in X}[a)$ is equal to P. If P has a least element 0, a subset X of P is *dense* in P, if $X \setminus \{0\}$ is coinitial in $P \setminus \{0\}$. We say that X is an *antichain* of P, if $0 \notin X$ and $(a] \cap (b] = \{0\}$ for any distinct a, $b \in X$. If X and Y are subsets of P, we abbreviate the statement

$$\forall (x, y) \in X \times Y,\ x \le y$$

by $X \le Y$. If $X = \{a\}$ (resp., $Y = \{a\}$), we write $a \le Y$ (resp., $X \le a$).

The *interval topology* on P is the least topology of P for which all the intervals of the form $(a]$ or $[a)$, for $a \in P$, are closed.

We shall consider the interval topology only in the totally ordered, complete case. The relevant result is then the following, see, for example, [**6**, §X.12].

Proposition 1-3.1 (O. Frink). *Let $(E, \le)$ be a totally ordered set. We suppose that E is* complete*, that is, every subset of E has an infimum in E. Then the interval topology of E is compact Hausdorff.*

If G is a partially ordered group, G^+ denotes the positive cone of G. We put $\mathbb{N} = \mathbb{Z}^+ \setminus \{0\}$. We say that G is *directed*, if is upward directed as a partially ordered set; we say that G satisfies the *interpolation property*, if for all a_0, a_1, b_0, $b_1 \in G$ such that $a_0, a_1 \le b_0, b_1$, there exists $x \in G$ such that $a_0, a_1 \le x \le b_0, b_1$. We say that G is *Dedekind complete*, if it is directed and every nonempty majorized subset of G has a supremum. It is well-known that every Dedekind complete partially ordered group is abelian, see, for example, [**6**, Theorem 28]. We shall write such groups using additive notation.

For any point x in a topological space Ω, we denote by $\mathcal{N}^\Omega(x)$ (or $\mathcal{N}(x)$ if Ω is understood) the set of all open neighborhoods of x in Ω. For a subset X of Ω, we denote by $\mathring{X}$ the *interior* of X and by $\overline{X}$ the *closure* of X in Ω. If K is a totally ordered set, endowed with its interval topology, a map $f\colon \Omega \to K$ is *lower semicontinuous* (resp., *upper semicontinuous*), if the set $\{x \in \Omega \mid f(x) \le \alpha\}$ (resp., $\{x \in \Omega \mid \alpha \le f(x)\}$) is closed, for every $\alpha \in K$.

A topological space Ω is *extremally disconnected*, if the closure of every open subset of Ω is open. We use the terminology *complete Boolean space* as a synonym for *extremally disconnected compact Hausdorff topological space*. See Section 3-3 for more detail on these concepts. Complete Boolean spaces are also called *Stone spaces* (or *stonian spaces*) in the literature.

CHAPTER 2

Partial commutative monoids

2-1. Basic results about partial commutative monoids

2-1.1. Partial commutative monoids. Many monoid-theoretical objects we shall deal with through this paper are not monoids, but just *partial* monoids. The following fundamental example provides us with a large supply of partial monoids.

EXAMPLE 2-1.1. Let $(M, +, 0)$ be a commutative monoid. For a subset S of M satisfying the two following properties

(i) $0 \in S$;
(ii) $x + y \in S$ implies that $x,\ y \in S$, for all $x,\ y \in M$,

we endow S with the partial addition $+_S$ defined by

$$a +_S b = c, \text{ only in case } c \in S,$$

for any $a,\ b \in S$. We call S a *partial submonoid* of M.

Observe that we do not merely consider *all* subsets of M, but only those that satisfy the conditions (i) and (ii) above—they are exactly the nonempty lower subsets of M for the *algebraic preordering* of M, see Definition 2-1.3.

It turns out that the properties of partial submonoids of commutative monoids are captured by the following definition.

DEFINITION 2-1.2. A *partial commutative monoid* is a structure $(S, +, 0)$, where $+$ is a partial binary operation on S which satisfies the following properties:

(a) $+$ is *associative*, that is, for all $a,\ b,\ c \in S$, $(a + b) + c$ is defined if and only if $a + (b + c)$ is defined, and then, both have the same value.
(b) $+$ is *commutative*, that is, for all $a,\ b \in S$, $a + b$ is defined if and only if $b + a$ is defined, and then, both have the same value.
(c) There exists an element, denoted by 0 (*necessarily unique*), of S such that $a + 0 = a$, for all $a \in S$.

We generalize to this context the classical definition of the algebraic preordering on a commutative monoid.

DEFINITION 2-1.3. Let $(S, +, 0)$ be a partial commutative monoid. The *algebraic preordering* on S is the (reflexive, transitive) binary relation $\leq$ defined on S by the rule

$$a \leq b \text{ if and only if } a + x = b, \text{ for some } x \in S.$$

An element $u \in S$ is an *order-unit*, if every element of S lies below nu (defined), for some $n \in \mathbb{Z}^+$.

The following definition is of course a direct generalization of Example 2-1.1.

Definition 2-1.4. A *partial submonoid* of a partial commutative monoid S is a lower subset T of S (for the algebraic preordering of S) containing 0 as an element, endowed with the partial addition defined by

$$a + b = c \text{ if and only if } a + b = c \text{ in } S \text{ and } c \in T, \text{ for all } a,\, b \in T.$$

We omit the trivial proof of the following result.

Proposition 2-1.5. *Every partial submonoid* (*as in Example* 2-1.1) *of a partial commutative monoid is a partial commutative monoid.*

The following class of embeddings will be of special interest.

Definition 2-1.6. Let A and B be partial commutative monoids, and let $\varphi\colon A \to B$. We say that φ is a *lower embedding*, if the following conditions hold:

(i) φ is a homomorphism of partial monoids.
(ii) φ is one-to-one, and $\varphi(x) \leq \varphi(y)$ implies that $x \leq y$, for all $x,\, y \in A$.
(iii) The range of φ is a lower subset of B, with respect to the algebraic preordering of B.

Hence, a lower embedding from A into B identifies A with a lower subset (with respect to the algebraic preordering) of B, endowed with the structure of partial submonoid as in Definition 2-1.4.

The following result shows that all partial commutative monoids can be obtained from Example 2-1.1.

Proposition 2-1.7. *Every partial commutative monoid admits a lower embedding into a commutative monoid.*

Proof. Let $(S, +, 0)$ be a partial commutative monoid. Let ∞ be any object such that $\infty \notin S$, and put $S^\bullet = S \cup \{\infty\}$. We define on $S^\bullet$ the binary operation $+^\bullet$ defined by the rule

$$a +^\bullet b = \begin{cases} a + b, & \text{if } a,\, b \in S \text{ and } a + b \text{ is defined in } S, \\ \infty, & \text{otherwise,} \end{cases} \qquad \text{for all } a,\, b \in S^\bullet.$$

It is easy to verify that $(S^\bullet, +^\bullet, 0)$ is a commutative monoid and that the inclusion map from S into $S^\bullet$ is a lower embedding. $\square$

Remark 2-1.8. A noticeable effect of Proposition 2-1.7 is to make computations in partial commutative monoids much more convenient. For example, suppose that we have to prove that an equality of the form $A = B$ holds in a given partial commutative monoid S, *via* a sequence of equalities $A = C_0 = C_1 = \cdots = C_n = B$, where A, B, and the C_i are finite sums of elements of S. We assume in addition that the sum defining A is defined in S. Instead of having to verify that all the terms C_i are defined in S and pairwise equal, it is sufficient to argue in $S^\bullet$ that $A = C_0 = C_1 = \cdots = C_n = B$, without having to worry about undefined terms.

This applies, in particular, to the following Lemmas 2-1.9, 2-1.10, and 2-1.11.

Lemma 2-1.9. *Let $(S, +, 0)$ be a partial monoid, with algebraic preordering $\leq$. Let a, b, a', $b' \in S$. If $a + b$ is defined and $a' \leq a$ and $b' \leq b$, then $a' + b'$ is defined, and $a' + b' \leq a + b$.*

In any given partial commutative monoid S, we define inductively the statement $a = \sum_{i<n} a_i$ to hold, for $n < \omega$, a, a_0, $\dots$, $a_{n-1} \in S$, as follows:

(i) $a = \sum_{i<0} a_i$ if and only if $a = 0$.
(ii) $a = \sum_{i<n+1} a_i$ if and only if $a = \left(\sum_{i<n} a_i\right) + a_n$.

If the operation of S is denoted by $\oplus$, then we shall write $\oplus_{i<n} a_i$ instead of $\sum_{i<n} a_i$.

LEMMA 2-1.10. *Let $(S, +, 0)$ be a partial commutative monoid. For all $n < \omega$, all a, a_0, ..., $a_{n-1} \in S$, and every permutation σ of n,*

$$a = \sum_{i<n} a_i \qquad \textit{if and only if} \qquad a = \sum_{i<n} a_{\sigma(i)}.$$

By Lemma 2-1.10, for a finite set I and elements a, a_i (for $i \in I$) of S, we can define unambiguously the statement $a = \sum_{i\in I} a_i$ to hold, if $a = \sum_{j<n} a_{\sigma(j)}$, where n is the cardinality of I and σ is any bijection from n onto I.

LEMMA 2-1.11. *Let $(S, +, 0)$ be a partial commutative monoid. Let I and J be finite sets, let $\pi\colon I \twoheadrightarrow J$ be a surjective map, let $(a_i)_{i\in I}$ be a family of elements of S, and let $a \in S$. Then the following are equivalent:*

(i) $a = \sum_{i\in I} a_i$.
(ii) *For all $j \in J$, the term $\sum_{i\in\pi^{-1}\{j\}} a_i$ is defined, and, if we denote its value by b_j, then $a = \sum_{j\in J} b_j$.*

2-1.2. Partial refinement monoids.

DEFINITION 2-1.12. We say that a partial commutative monoid $(S, +, 0)$ has the *refinement property*, or is a *partial refinement monoid*, if for all a_0, a_1, b_0, $b_1 \in S$ such that $a_0 + a_1 = b_0 + b_1$, there are elements $c_{i,j}$ in S, for i, $j < 2$, such that the equalities $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$ hold, for all $i < 2$.

The information contained in the equalities $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$ for all $i < 2$ will often be condensed in the format of a *refinement matrix* as follows:

	b_0	b_1
a_0	$c_{0,0}$	$c_{0,1}$
a_1	$c_{1,0}$	$c_{1,1}$

These notations can be easily generalized to refinement matrices of arbitrary, finite or even infinite, dimensions. These notations are also very widely used in [**56**].

Define a *refinement monoid* as a commutative monoid satisfying the refinement property. In a spirit similar to Proposition 2-1.7, we shall now prove (see Proposition 2-1.13) that every partial refinement monoid can be obtained as a lower subset of a refinement monoid. The proof of Proposition 2-1.7 does not apply for this result, because $S^\bullet$ fails in general to satisfy refinement even if S has refinement. We shall use instead a procedure adapted to refinement monoids.

PROPOSITION 2-1.13. *Every partial refinement monoid S admits a lower embedding into a refinement monoid $\widetilde{S}$. In addition, one can take $\widetilde{S}$ to be generated by S as a monoid, and such that the canonical embedding from S into $\widetilde{S}$ is universal among the homomorphisms of partial monoids from S to commutative monoids.*

PROOF. The following construction is a particular case of the construction presented in Chapter 4 of [**56**]—with the notations used there, $\widetilde{S} = \mathrm{Dim}(S, +, =)$. However, in this context, a direct verification is easy, so we give an outline here.

Let S be a partial refinement monoid. We endow the set $\mathbb{S}$ of all finite, nonempty sequences of elements of S with the binary relation $\equiv$ defined by the rule

$(a_i)_{i<m} \equiv (b_j)_{j<n}$ if there are $c_{i,j} \in S,$ for $i < m$ and $j < n,$ such that

$$a_i = \sum_{j<n} c_{i,j}, \text{ for all } i < m, \text{ and } b_j = \sum_{i<n} c_{i,j}, \text{ for all } j < n.$$

By using the refinement property in S, it is not difficult to verify that $\equiv$ is an equivalence relation on $\mathbb{S}$. For any $s \in \mathbb{S}$, we denote by $[s]$ the equivalence class of s modulo $\equiv$. We endow the quotient $\widetilde{S} = \mathbb{S}/{\equiv}$ with the binary addition $+$ defined by

$$[s] + [t] = [s \frown t], \text{ for all } s,\, t \in \mathbb{S},$$

where $s \frown t$ denotes the *concatenation* of s and t. It is straightforward to verify that $\widetilde{S}$, endowed with $+$, is a refinement monoid. For any $a \in S$, we denote by $j(a)$ the equivalence class modulo $\equiv$ of the finite sequence (a) of length one. Then $j(0)$ is the zero element of $\widetilde{S}$, and j is a lower embedding from S into $\widetilde{S}$.

For the remainder of the proof, we shall identify S with its image in $\widetilde{S}$ under the embedding j. Thus the elements of $\widetilde{S}$ are exactly the finite sums $\sum_{i<m} a_i$, where $m \in \mathbb{N}$ and $a_0, \dots, a_{m-1} \in S$, and the equality $\sum_{i<m} a_i = \sum_{j<n} b_j$ holds if and only if there exists a refinement matrix of the form

	$b_j\ (j < n)$
$a_i\ (i < n)$	$c_{i,j}$

for some elements $c_{i,j}$ (for $i < m$ and $j < n$) of S. Obviously $\widetilde{S}$ is generated by S as a monoid.

Now we verify the second assertion of Proposition 2-1.13. Let M be a commutative monoid and let $f\colon S \to M$ be a homomorphism of partial monoids. Let $\overline{g}\colon \mathbb{S} \to M$ be the map defined by the rule

$$\overline{g}((a_i)_{i<n}) = \sum_{i<n} f(a_i), \text{ for all } (a_i)_{i<n} \in \mathbb{S}.$$

Then $\overline{g}(j(0)) = 0_M$, $\overline{g}(s \frown t) = \overline{g}(s) + \overline{g}(t)$, and $s \equiv t$ implies that $\overline{g}(s) = \overline{g}(t)$, for all $s,\, t \in \mathbb{S}$. Hence $\overline{g}$ can be factored through $\equiv$, thus yielding a homomorphism $g\colon \widetilde{S} \to M$ that extends f. Since S generates $\widetilde{S}$ as a monoid, g is the only homomorphism with this property. □

For any partial refinement monoid S, we take $\widetilde{S}$ to be the refinement monoid having all the properties described in Proposition 2-1.13. By the given universal property, $\widetilde{S}$ is unique up to isomorphism. We will also identify S with its canonical image in $\widetilde{S}$.

Our next lemma collects some basic information about $\widetilde{S}$. For $n \in \mathbb{N}$, we put

$$nS = \left\{ \sum_{i<n} x_i \mid x_0, \dots, x_{n-1} \in S \right\} \subseteq \widetilde{S}.$$

Definition 2-1.14. A partial refinement monoid S is *conical*, if $x + y = 0$ implies that $x = y = 0$, for all $x,\, y \in S$. In other words, $x \leq 0$ implies that $x = 0$, for all $x \in S$.

LEMMA 2-1.15. *Let S be a partial refinement monoid. Then the following assertions hold:*

(i) *nS is a lower subset of $\widetilde{S}$, for all $n \in \mathbb{N}$.*
(ii) *If S is cancellative (i.e., $a+c=b+c$ in S implies that $a=b$), then $\widetilde{S}$ is cancellative.*
(iii) *If S is conical, then $\widetilde{S}$ is conical.*

PROOF. (i) is an easy consequence of refinement in $\widetilde{S}$.

(ii) Folklore. A proof can be found in Lemma 3.6 in [**56**].

(iii) Let x, $y \in \widetilde{S}$ such that $x+y=0$. Write $x=\sum_{i<m} x_i$ and $y=\sum_{j<n} y_j$ for some m, $n \in \mathbb{N}$ and $x_0, \dots, x_{m-1}, y_0, \dots, y_{n-1} \in S$. Then, for all $i<m$, $x_i \leq x \leq x+y=0$ in $\widetilde{S}$, thus, since S is a lower subset of $\widetilde{S}$, $x_i \leq 0$ in S. Hence, as S is conical, $x_i=0$, so $x=0$. Hence $y=0$. □

2-2. Direct decompositions of partial refinement monoids

In this section, we shall fix a conical partial refinement monoid S. We shall denote by $\leq$ the algebraic preordering of S.

DEFINITION 2-2.1. An *ideal* of S is a nonempty subset I of S such that $a+b \in I$ if and only if $a \in I$ and $b \in I$, for all a, $b \in S$ such that $a+b$ is defined.

We define elements a and b of S to be *orthogonal*, in notation, $a \perp b$, if $x \leq a, b$ implies that $x=0$, for all $x \in S$. If X and Y are subsets of S, then we define $X \perp Y$ to hold if $x \perp y$ for all $(x,y) \in X \times Y$. We shall put

$$X^{\perp} = \{s \in S \mid s \perp x, \text{ for all } x \in X\}, \qquad \text{for all } X \subseteq S. \tag{2-2.1}$$

If $X=\{x\}$, a singleton, then we write $x^{\perp}$ instead of $\{x\}^{\perp}$. In particular, $X \perp Y$ if and only if $Y \subseteq X^{\perp}$, if and only if $X \subseteq Y^{\perp}$.

LEMMA 2-2.2.

(i) *$a \perp c$ and $b \perp c$ implies that $a+b \perp c$, for all a, b, $c \in S$ such that $a+b$ is defined.*
(ii) *The set $X^{\perp}$ is an ideal of S, for all $X \subseteq S$.*
(iii) *$a \perp b$ and a, $b \leq c$ implies that $a+b$ is defined and $a+b \leq c$.*

PROOF. (i) Let $x \leq c, a+b$. By refinement, there are a', $b' \in S$ such that $a' \leq a$, $b' \leq b$, and $x=a'+b'$. So $a' \leq a, c$, whence $a'=0$. Similarly, $b'=0$, so $x=0$, thus proving $a+b \perp c$.

(ii) is an obvious consequence of (i).

(iii) By the definition of $\leq$, there are a', $b' \in S$ such that $a+a'=b+b'=c$. By applying refinement to the equality $a+a'=b+b'$ and by using the assumption that $a \perp b$, we obtain $t \in S$ such that $a'=b+t$ and $b'=a+t$. Since $a+a'$ is defined, $a+b$ is defined, and $c=a+b+t \geq a+b$. □

NOTATION 2-2.3. For $n \in \mathbb{N}$ and $X_0, \dots, X_{n-1} \subseteq S$, we put

$$X_0+\cdots+X_{n-1} = \left\{ x \in S \mid \exists (x_0, \dots, x_{n-1}) \in X_0 \times \cdots \times X_{n-1}, \quad x=\sum_{i<n} x_i \right\}.$$

We shall also write $\sum_{i<n} X_i$ instead of $X_0 + \cdots + X_{n-1}$. If $X_i = X$ for all i, then we shall abbreviate this further by nX. If $X_i \perp X_j$ for all $i \neq j$, then we shall write $X_0 \oplus \cdots \oplus X_{n-1}$, or $\bigoplus_{i<n} X_i$, instead of $\sum_{i<n} X_i$, and we shall say that the sum of the X_i is *orthogonal*.

Lemma 2-2.4. *Let $n \in \mathbb{N}$ and let S_i, for $i < n$, be nonempty subsets of S such that $S = \bigoplus_{i<n} S_i$. Then the following hold:*

(i) $S_i = \left(\bigoplus_{j \neq i} S_j\right)^{\perp}$, *for all $i < n$. In particular, S_i is an ideal of S.*
(ii) *For all $x \in S$, there exists a unique decomposition $x = \sum_{i<n} x_i$ such that $x_i \in S_i$ for all $i < n$.*

Proof. (i) The sum of all the S_j is orthogonal, thus so is the sum of all S_j, for $j \neq i$. Furthermore, $S_i \perp S_j$, for all $j \neq i$, so $S_j \subseteq S_i^{\perp}$. Hence, by using Lemma 2-2.2(ii), $\bigoplus_{j \neq i} S_j \subseteq S_i^{\perp}$. Conversely, let $x \in S_i^{\perp}$. By assumption, there exists a decomposition $x = \sum_{j<n} x_j$, where $x_j \in S_j$, for all $j < n$. But $x_i \in S_i$, thus $x \perp x_i$; whence $x_i = 0$, so $x \in \bigoplus_{j \neq i} S_j$. Hence $S_i = \left(\bigoplus_{j \neq i} S_j\right)^{\perp}$. By Lemma 2-2.2(ii), it follows that S_i is an ideal of S.

(ii) Suppose $x = \sum_{i<n} x_i = \sum_{i<n} y_i$, with elements x_i, $y_i \in S_i$, for all $i < n$. Since S satisfies refinement, there exists a refinement matrix of the form

	$y_j \ (j < n)$
$x_i \ (i < n)$	$z_{i,j}$

with elements $z_{i,j} \in S$, for all i, $j < n$. But if $i \neq j$, then $S_i \perp S_j$, whence $z_{i,j} = 0$. Hence, $x_i = z_{i,i} = y_i$, for all $i < n$. $\square$

Remark 2-2.5. The direct product $\prod_{i<n} S_i$ can be naturally endowed with a structure of partial monoid, by defining the addition componentwise. In the context of Lemma 2-2.4, we obtain a map

$$\varphi\colon S \to \prod_{i<n} S_i, \quad x \mapsto (x_i)_{i<n} \text{ with } x = \sum_{i<n} x_i, \quad x_i \in S_i \text{ for all } i < n.$$

This map is a one-to-one homomorphism of partial monoids. However, it is *not*, in general, surjective: for arbitrary $x_i \in S_i$, for $i < n$, the sum $\sum_{i<n} x_i$ may not be defined. But of course, if S is a (total) monoid, then φ is an isomorphism.

Proposition 2-2.6. *In the context of Remark 2-2.5, φ is a lower embedding from S into $\prod_{i<n} S_i$.*

Proof. Only part (iii) of the definition of a lower embedding is not completely trivial.

Let $x \in S$ and $(y_i)_{i<n} \in \prod_{i<n} S_i$ such that $(y_i)_{i<n} \leq \varphi(x)$. Put $\varphi(x) = (x_i)_{i<n}$, so $y_i \leq x_i$, for all $i < n$. By the definition of φ, $\sum_{i<n} x_i$ is defined, and equal to x. By Lemma 2-1.9, $\sum_{i<n} y_i$ is also defined. Denote its value by y. By the definition of φ, $(y_i)_{i<n} = \varphi(y)$. $\square$

2-3. Projections of partial refinement monoids

Standing hypothesis: S is a conical partial refinement monoid.
We denote again by $\leq$ the algebraic preordering of S.

DEFINITION 2-3.1. A *projection* of S is an endomorphism p of $(S, +, 0)$ such that

$$x \in p(x) + (pS)^{\perp}, \qquad \text{for all } x \in S.$$

In particular, if p is a projection of S, then $p(x) \leq x$, for all $x \in S$. Thus $p(0) = 0$. Furthermore, p preserves the algebraic preordering of S, see Definition 2-1.3.

PROPOSITION 2-3.2. *Let p be an endomorphism of S. Then the following are equivalent:*

(i) *p is a projection of S.*
(ii) *There are ideals S_0 and S_1 of S such that*
  (a) *$S = S_0 \oplus S_1$.*
  (b) *$p(x_0 + x_1) = x_0$, for all $(x_0, x_1) \in S_0 \times S_1$ such that $x_0 + x_1$ is defined.*

PROOF. (ii)$\Rightarrow$(i) is easy.

(i)$\Rightarrow$(ii) Assume (i). We put $S_0 = pS$ and $S_1 = (pS)^{\perp}$. By the definition of a projection, $S = S_0 + S_1$. Since $S_0 \perp S_1$, it follows that $S = S_0 \oplus S_1$. In particular, S_0 and S_1 are ideals of S (see Lemma 2-2.4(i)). For $x \in S$, let $y \in S_1$ such that $x = p(x) + y$. If $x = x_0 + x_1$ in S such that $x_i \in S_i$ for all $i < 2$, then, by Lemma 2-2.4(ii), $p(x) = x_0$ and $y = x_1$. $\square$

COROLLARY 2-3.3. *Every projection of S is idempotent.*

In the context of Proposition 2-3.2(ii), we observe that $S_0 = pS$ while $S_1 = p^{-1}\{0\} = S_0^{\perp} = (pS)^{\perp}$. In particular, p is determined by S_0 alone, so we shall call p the *projection of S onto S_0*.

DEFINITION 2-3.4. A *direct summand* of S is a subset X of S such that $S = X \oplus Y$, for some $Y \subseteq S$.

Of course, by Lemma 2-2.4, X is then an ideal of S, and $Y = X^{\perp}$, so $S = X \oplus X^{\perp}$.

It follows that the direct summands of S are exactly the ranges of the projections of S.

Furthermore, by exchanging the roles of S_0 and S_1, we obtain another projection, which we shall denote by $p^{\perp}$. Formally, $p^{\perp}$ is the unique projection of S such that $p^{\perp}S = (pS)^{\perp}$ and $(p^{\perp})^{-1}\{0\} = pS$. We observe that $p^{\perp\perp} = p$.

NOTATION 2-3.5. Let $\operatorname{Proj} S$ denote the set of projections of S. We shall also often use the notation $\operatorname{Proj}^* S = \operatorname{Proj} S \setminus \{0\}$.

We shall now study the structure of $\operatorname{Proj} S$, towards Proposition 2-3.11.

LEMMA 2-3.6. *Let p, $q \in \operatorname{Proj} S$. The the following holds:*

(i) *$S = (pS \cap qS) \oplus (pS \cap q^{\perp}S) \oplus (p^{\perp}S \cap qS) \oplus (p^{\perp}S \cap q^{\perp}S)$.*
(ii) *Let r denote the projection from S onto $pS \cap qS$. Then $r = pq = qp$.*

PROOF. (i) It is obvious that all ideals $pS \cap qS$, $pS \cap q^{\perp}S$, $p^{\perp}S \cap qS$, and $p^{\perp}S \cap q^{\perp}S$ are pairwise orthogonal. Let $x \in S$. Since $S = pS + (pS)^{\perp}$, there exists a decomposition $x = x_0 + x_1$, where $x_0 \in pS$ and $x_1 \in (pS)^{\perp}$. For $i < 2$, $x_i \in qS + (qS)^{\perp}$, thus $x_i = x_{i,0} + x_{i,1}$, for some $x_{i,0} \in qS$ and $x_{i,1} \in (qS)^{\perp}$. Since

pS and $(pS)^\perp$ are ideals of S, $x_{0,0} \in pS \cap qS$, $x_{0,1} \in pS \cap q^\perp S$, $x_{1,0} \in p^\perp S \cap qS$, and $x_{1,1} \in p^\perp S \cap q^\perp S$. Observe that $x = x_{0,0} + x_{0,1} + x_{1,0} + x_{1,1}$.

(ii) Since both p and q act as the identity on $pS \cap qS$, so does pq. Furthermore, $q(x) = 0$ for all $x \in q^\perp S$ and $pq(x) = p(x) = 0$ for all $x \in p^\perp S \cap qS$, so, $pq = r$. By symmetry, $r = qp$. □

We shall put $p \wedge q = pq = qp$, for all p, $q \in \operatorname{Proj} S$.

Corollary 2-3.7. *The structure* $(\operatorname{Proj} S, \wedge)$ *is a semilattice* (*i.e., an idempotent commutative monoid*).

We endow $\operatorname{Proj} S$ with the partial ordering $\leq$ defined by

$$p \leq q \text{ if and only if } p \wedge q = p, \qquad \text{for all } p,\ q \in \operatorname{Proj} S.$$

For this partial ordering, $p \wedge q$ is, of course, the infimum of $\{p, q\}$. The least element of $\operatorname{Proj} S$ is 0 (the zero map), while the greatest element of $\operatorname{Proj} S$ is id_S (the identity on S).

Lemma 2-3.8. *Let p, $q \in \operatorname{Proj} S$. Then the following holds:*

(i) $p \leq q$ *if and only if* $pS \subseteq qS$ *if and only if* $p(x) \leq q(x)$ *holds, for all* $x \in S$.

(ii) $p \wedge q = 0$ *if and only if* $q \leq p^\perp$.

Proof. (i) If $p \leq q$, then $pS = qpS \subseteq qS$.

Suppose now that $pS \subseteq qS$. Let $x \in S$. The inequality $p(x) \leq q(x)$ holds for all $x \in pS$ (because then $p(x) = x = q(x)$) and for all $x \in p^\perp S$ (because then $p(x) = 0 \leq q(x)$), so it holds for all $x \in S$ since $S = pS + p^\perp S$.

If $p(x) \leq q(x)$ for all $x \in S$, then $p(x) \in qS$ since qS is an ideal of S, so $qp(x) = p(x)$. Hence $p \leq q$.

(ii) By Lemma 2-3.6, $p \wedge q = 0$ if and only if $pS \cap qS = \{0\}$. Hence, $p \wedge q = 0$ if and only if $qS \subseteq (pS)^\perp = p^\perp S$, if and only if $q \leq p^\perp$ by (i) above. □

Corollary 2-3.9. *The map* $p \mapsto p^\perp$ *is an involutive anti-automorphism of* $(\operatorname{Proj} S, \leq)$.

Proof. We already know that $p^{\perp\perp} = p$, for all $p \in \operatorname{Proj} S$. Furthermore, by Lemma 2-3.8, $p \leq q$ implies that $q^\perp \leq p^\perp$, for all p, $q \in \operatorname{Proj} S$. □

Since $(\operatorname{Proj} S, \leq)$ is a meet-semilattice, we thus obtain the following.

Corollary 2-3.10. $(\operatorname{Proj} S, \leq)$ *is a lattice.*

So we denote by $p \vee q$ the supremum of $\{p, q\}$, for all p, $q \in \operatorname{Proj} S$. We can strengthen Corollary 2-3.10 right away.

Proposition 2-3.11. $(\operatorname{Proj} S, \leq)$ *is a Boolean algebra.*

Proof. By Corollary 2-3.10, $(\operatorname{Proj} S, \leq)$ is a lattice. Furthermore, $p^\perp$ is a complement of p, for all $p \in \operatorname{Proj} S$. Hence, to conclude the proof, it suffices to prove distributivity. The argument below is classical, and it can be traced back to Glivenko's work, see, for example, [**6**, §V.11].

So, let p, q, $r \in \operatorname{Proj} S$. We put

$$s = p \wedge (q \vee r) \qquad \text{and} \qquad t = (p \wedge q) \vee (p \wedge r).$$

Then $t^\perp \wedge p \wedge q = t^\perp \wedge p \wedge r = 0$, which implies, by Lemma 2-3.8(ii), that $t^\perp \wedge p \leq q^\perp$ and $t^\perp \wedge p \leq r^\perp$, thus, meeting both inequalities, $t^\perp \wedge p \leq q^\perp \wedge r^\perp$. By Corollary 2-3.9, $q^\perp \wedge r^\perp = (q \vee r)^\perp$, so it follows that $t^\perp \wedge p \wedge (q \vee r) = 0$, that is, by Lemma 2-3.8(ii), $t^\perp \leq s^\perp$; whence $s \leq t$. But the converse inequality $s \geq t$ is obvious, thus $s = t$. □

Notation 2-3.12. For p, q, $r \in \operatorname{Proj} S$, let $r = p \oplus q$ hold just in case $r = p \vee q$ and $p \wedge q = 0$.

Lemma 2-3.13. *Let p, $q \in \operatorname{Proj} S$ such that $p \wedge q = 0$. Then*

$$(p \vee q)(x) = p(x) + q(x), \qquad \textit{for all } x \in S.$$

Proof. Let $x \in S$, and put $r = p \vee q$. We apply the definition of a projection to p and to q. So there are $u \in (pS)^\perp$ and $v \in (qS)^\perp$ such that $r(x) = pr(x) + u = qr(x) + v$. We observe that $pr(x) = p(x)$ and $qr(x) = q(x)$. By applying the refinement property to the equality $p(x) + u = q(x) + v$ and by observing that $p(x) \perp q(x)$, we obtain $t \in S$ such that $u = q(x) + t$ and $v = p(x) + t$. On the one hand, $t \leq u, v$, thus $t \in p^\perp S \cap q^\perp S = r^\perp S$, see Lemma 2-3.6. On the other hand, $t \leq r(x)$. Hence, $t = 0$, so $r(x) = p(x) + u = p(x) + q(x)$. □

Notation 2-3.14. For x, y, $z \in S$, $z = x \wedge y$ is the statement

$$z \leq x, y \qquad \text{and} \qquad \forall t, \quad t \leq x, y \Rightarrow t \leq z.$$

We define, dually, the statement $z = x \vee y$. Note that z is uniquely defined by either statement only in case $\leq$ is antisymmetric.

Similarly, one can define the notations $a = \bigwedge_{i \in I} a_i$ and $a = \bigvee_{i \in I} a_i$.

Proposition 2-3.15. *Let p, $q \in \operatorname{Proj} S$, let $x \in S$. Then the following statements are satisfied:*

$$(p \wedge q)(x) = p(x) \wedge q(x), \qquad (p \vee q)(x) = p(x) \vee q(x).$$

Proof. We put $r = p \wedge q$ and $s = p \vee q$.

By Lemma 2-3.8, $r(x) \leq p(x), q(x)$. Let $y \in S$ such that $y \leq p(x), q(x)$. Since pS and qS are ideals of S (see Lemma 2-2.4(i)), $y \in pS \cap qS$, so $p(y) = q(y) = y$. Thus $y = pq(y) = r(y) \leq r(x)$. Hence $r(x) = p(x) \wedge q(x)$.

By Lemma 2-3.8, $p(x), q(x) \leq s(x)$. Let $y \in S$ such that $p(x), q(x) \leq y$. Thus, *a fortiori*, $p(x), (p^\perp \wedge q)(x) \leq y$. Since $\operatorname{Proj} S$ is a Boolean algebra, $s = p \oplus (p^\perp \wedge q)$. It follows, by Lemma 2-3.13, that $s(x) = p(x) + (p^\perp \wedge q)(x)$, thus $s(x) \leq y$ by Lemma 2-2.2(iii). Hence $s(x) = p(x) \vee q(x)$. □

If S is a *total* (as opposed to partial) monoid, then the projections of S correspond to direct decompositions of S, thus, they preserve arbitrary suprema and infima. For our partial structures, the corresponding result still holds.

Lemma 2-3.16. *Let p be a projection of S. For every family $(a_i)_{i \in I}$ of elements of S and every $a \in S$,*

(i) *If $I \neq \varnothing$, then $a = \bigwedge_{i \in I} a_i$ implies that $p(a) = \bigwedge_{i \in I} p(a_i)$.*
(ii) *Suppose that any two elements of S have a meet. Then $a = \bigvee_{i \in I} a_i$ implies that $p(a) = \bigvee_{i \in I} p(a_i)$.*

Note. The natural settings of Lemma 2-3.16 are in situations where S is antisymmetric as well. However, that condition is not, strictly speaking, necessary, if we use the interpretation of the symbols $\bigwedge$ and $\bigvee$ given in Notation 2-3.14.

PROOF. (i) Of course, $p(a) \leq p(a_i)$, for all i. Let b be a minorant of $\{p(a_i) \mid i \in I\}$. Since I is nonempty, b belongs to pS, so $p(b) = b$. Since b is also a minorant of $\{a_i \mid i \in I\}$, $b \leq a$. Hence, $b = p(b) \leq p(a)$.

(ii) Of course, $p(a_i) \leq p(a)$, for all i. Let b be a majorant of $\{p(a_i) \mid i \in I\}$. Since $p(a)$ is also a majorant of $\{p(a_i) \mid i \in I\}$ and by assumption, $c = p(a) \wedge b$ exists and it is a majorant of $\{p(a_i) \mid i \in I\}$. From $c \leq p(a)$ it follows that $c + p^{\perp}(a)$ is defined. Furthermore, $a_i = p(a_i) + p^{\perp}(a_i) \leq c + p^{\perp}(a)$, for all $i \in I$, thus $a \leq c + p^{\perp}(a)$. Therefore, $p(a) \leq c \leq b$. □

DEFINITION 2-3.17. Suppose that S is antisymmetric. For a, b, $c \in S$, let $c = b \smallsetminus a$ mean that c is the least $x \in S$ such that $b \leq a + x$. We say that c is the *least difference* of b and a.

LEMMA 2-3.18. *Suppose that S is antisymmetric, let $a \leq b$ in S. If $b \smallsetminus a$ exists, then $b = a + (b \smallsetminus a)$.*

PROOF. Put $c = b \smallsetminus a$. Since $a \leq b$, there exists $d \in S$ such that $b = a + d$, thus, by the definition of the least difference, $c \leq d$, whence $b \leq a + c \leq a + d = b$. Therefore, since S is antisymmetric, $b = a + c$. □

LEMMA 2-3.19. *Suppose that S is antisymmetric and that $b \smallsetminus a$ exists for all a, $b \in S$ such that $a \leq b$. Then $p(b \smallsetminus a) = p(b) \smallsetminus p(a)$, for all a, $b \in S$ such that $a \leq b$ and all $p \in \operatorname{Proj} S$.*

PROOF. From $b \leq a + (b \smallsetminus a)$ it follows that $p(b) \leq p(a) + p(b \smallsetminus a)$, thus $p(b) \smallsetminus p(a) \leq p(b \smallsetminus a)$. Conversely, put $d = p(b) \smallsetminus p(a)$. Then $p(b) \leq p(a) + d$ by definition, while we also have $p^{\perp}(b) \leq p^{\perp}(a) + p^{\perp}(b \smallsetminus a)$, thus, adding the two inequalities together (and observing that, since $d \leq p(b \smallsetminus a)$, $d + p^{\perp}(b \smallsetminus a)$ is defined), we obtain the inequality $b \leq a + d + p^{\perp}(b \smallsetminus a)$. It follows that $b \smallsetminus a \leq d + p^{\perp}(b \smallsetminus a)$, whence, by applying p, we obtain that $p(b \smallsetminus a) \leq d$. Therefore, $d = p(b \smallsetminus a)$. □

2-4. General comparability

Standing hypothesis: S is a conical partial refinement monoid.
We denote again by $\leq$ the algebraic preordering of S.

DEFINITION 2-4.1. We say that S has *general comparability*, if for all x, $y \in S$, there exists $p \in \operatorname{Proj} S$ such that $p(x) \leq p(y)$ and $p^{\perp}(x) \geq p^{\perp}(y)$.

We give a sufficient condition that implies general comparability.

LEMMA 2-4.2. *Suppose that S satisfies the following axioms:*

(i) *$\forall a, b$, $\exists c, x, y$ such that $a = c + x$, $b = c + y$, and $x \perp y$.*
(ii) *$S = a^{\perp} + a^{\perp\perp}$, for all $a \in S$.*

Then S satisfies general comparability.

PROOF. Let a, $b \in S$. Consider c, x, y as in (i). By (ii), there exists $p \in \operatorname{Proj} S$ such that $pS = x^{\perp}$ and $p^{\perp}S = x^{\perp\perp}$. So $p(x) = 0$ and $p^{\perp}(y) = 0$, whence $p(a) \leq p(b)$ and $p^{\perp}(b) \leq p^{\perp}(a)$. □

LEMMA 2-4.3. *Suppose that S has general comparability and that the algebraic preordering of S is antisymmetric. Let a, $b \in S$. The following assertions hold:*

(i) *The pair $\{a, b\}$ has an infimum.*
(ii) *If the pair $\{a, b\}$ is majorized, then it has a supremum.*

A partially ordered set satisfying (i) and (ii) above is sometimes called a *chopped lattice*.

PROOF. By general comparability, there exists $p \in \operatorname{Proj} S$ such that $p(a) \leq p(b)$ and $p^\perp(a) \geq p^\perp(b)$. So $c = p(a) + p^\perp(b)$ is defined, and $c \leq a$, b. Furthermore, it is easy to verify that $c = a \wedge b$.

Similarly, if the pair $\{a, b\}$ is majorized by an element e, then $d = p(b) + p^\perp(a)$ is defined, and $d \leq e$. It is easy to verify that $d = a \vee b$. $\square$

NOTATION 2-4.4. For a, $b \in S$, let $a \ll b$ hold, if $a + b = b$. We also say that b *absorbs* a.

We recall the following axiom, see [**53, 54**]:

The pseudo-cancellation property:

$$\forall a, b, c, \quad a + c = b + c \Rightarrow \exists d,\ \exists u, v \ll c \text{ such that } a = d + u \text{ and } b = d + v.$$

If $a + c = b + c$, $a = d + u$, $b = d + v$, and u, $v \ll c$, then $b + u$ is defined (because $u \leq c$ and $b + c$ is defined) and $b + u = d + u + v \geq a$. Hence we obtain the following weaker version of the pseudo-cancellation property:

$$\forall a, b, c, \quad a + c \leq b + c \Rightarrow \exists x \ll c \text{ such that } \ a \leq b + x.$$

LEMMA 2-4.5. *Suppose that S has general comparability. Then S satisfies the pseudo-cancellation property.*

PROOF. Suppose $a + c = b + c$ in S. By using refinement, we find a refinement matrix as follows:

	b	c
a	t	a'
c	b'	c'

By general comparability, there exists $p \in \operatorname{Proj} S$ such that $p(a') \leq p(b')$ and $p^\perp(b') \leq p^\perp(a')$. Let u, $v \in S$ such that $p(a') + v = p(b')$ and $p^\perp(b') + u = p^\perp(a')$. Then

$$\begin{aligned} p(c) &= p(b') + p(c') &&= p(a') + p(c') + v &&= p(c) + v, \\ p^\perp(c) &= p^\perp(a') + p^\perp(c') &&= p^\perp(b') + p^\perp(c') + u &&= p^\perp(c) + u, \end{aligned}$$

It follows that $u + c = v + c = c$. Put $d = t + p(a') + p^\perp(b')$. By using Lemma 2-3.13, we obtain the equalities

$$\begin{aligned} a &= t + a' = t + p(a') + p^\perp(a') = d + u \\ b &= t + b' = t + p(b') + p^\perp(b') = d + v. \end{aligned}$$

$\square$

COROLLARY 2-4.6. *Suppose that S has general comparability. Then S is* separative*, that is, it satisfies the statement*

$$\forall a, b, c, \quad (a + c = b + c \text{ and } c \leq a, b) \Rightarrow a = b.$$

PROOF. Let a, b, $c \in S$ such that $a + c = b + c$ and $c \leq a$, b. By Lemma 2-4.5, there are d, a', $b' \in S$ such that $a = d + a'$, $b = d + b'$, and a', $b' \ll c$. In particular, a', $b' \leq c$, thus, since $a + c$ and $b + c$ are defined, $a + b'$ and $b + a'$ are defined, see Lemma 2-1.9. Note that $a + b' = b + a'$. However, $c \leq a$, b, thus, since $a' \ll c$, we obtain that $a' \ll b$, so $b + a' = b$. Similarly, $a + b' = a$. Therefore, $a = b$. $\square$

Definition 2-4.7. An element c of S is

— *directly finite—ii*, if $x + c = c$ implies that $x = 0$, for all $x \in S$,

— *cancellable*, if $x + c = y + c \in S$ implies that $x = y$, for all $x, y \in S$.

We say that S is *stably finite*, if every element of S is directly finite. We denote by S_{fin} the subset of S consisting of all directly finite elements.

It is obvious that every cancellable element is directly finite. By Lemma 2-4.5, we obtain immediately the following converse.

Lemma 2-4.8. *Suppose that S has general comparability. Then every directly finite element of S is cancellable.*

2-5. Boolean-valued partial refinement monoids

Standing hypothesis: S is a conical partial refinement monoid.
We denote again by $\leq$ the algebraic preordering of S.

For elements a and b of S, it follows from Proposition 2-3.15 that the set of all projections p of S such that $p(a) \leq p(b)$ is closed under finite join. We shall now consider a stronger statement.

Definition 2-5.1. For $a, b \in S$, we shall denote by $\|a \leq b\|$ the largest projection p of S such that $p(a) \leq p(b)$ if it exists. Hence, $\|a \leq b\| \in \operatorname{Proj} S$.

We say that S is *Boolean-valued*, if the Boolean value $\|a \leq b\|$ is defined, for all $a, b \in S$.

Notation 2-5.2. For $a, b \in S$, if both $\|a \leq b\|$ and $\|b \leq a\|$ are defined, we put $\|a = b\| = \|a \leq b\| \wedge \|b \leq a\|$.

Lemma 2-5.3. *Assume that S has general comparability. Let $a \in S$, and suppose that $\|a = 0\|$ is defined. Then the following assertions hold:*

(i) $a^{\perp} = \|a = 0\| \, S$.

(ii) $a^{\perp\perp} = \|a = 0\|^{\perp} \, S$.

(iii) $S = a^{\perp} \oplus a^{\perp\perp}$.

Proof. (i) Put $p = \|a = 0\|$. For $x \in pS$ such that $x \leq a$, we have $x = p(x) \leq p(a) = 0$. Hence $pS \subseteq a^{\perp}$.

Conversely, let $x \in a^{\perp}$. By general comparability, there exists $q \in \operatorname{Proj} S$ such that $q(a) \leq q(x)$ and $q^{\perp}(x) \leq q^{\perp}(a)$. Since $a \perp x$, the equalities $q(a) = q^{\perp}(x) = 0$ hold. It follows from the definition of p that $q \leq p$. Therefore, $p^{\perp}(x) \leq q^{\perp}(x) = 0$, so $x \in pS$.

(ii) follows immediately from (i), while (iii) follows immediately from (i), (ii), and the fact that $\|a = 0\|$ is a projection of S. $\square$

Lemma 2-5.4. *Assume that S has general comparability. Let $a \in S$. Then $\|a = 0\|$ exists if and only if $S = a^{\perp} \oplus a^{\perp\perp}$.*

Proof. If $\|a = 0\|$ exists, then $S = a^{\perp} \oplus a^{\perp\perp}$ by Lemma 2-5.3(iii). Conversely, suppose that $S = a^{\perp} \oplus a^{\perp\perp}$. So there exists a unique projection p of S such that $pS = a^{\perp}$. From $p(a) \leq a$ and $p(a) \in a^{\perp}$ it follows that $p(a) = 0$. Let $q \in \operatorname{Proj} S$ such that $q(a) = 0$. We claim that $q(x) \perp a$, for any $x \in S$. Indeed, let $y \in S$ such that $y \leq q(x), a$. From $y \leq q(x)$ it follows that $q(y) = y$, thus $y = q(y) \leq q(a) = 0$, so $y = 0$, thus establishing our claim. So $qS \subseteq a^{\perp} = pS$, whence $q \leq p$ by Lemma 2-3.8(i). Therefore, $p = \|a = 0\|$. $\square$

DEFINITION 2-5.5. Let a, $b \in S$. We say that a is *removable* from b, and we write $a \ll_{\mathrm{rem}} b$, if the following conditions hold:

(i) $a \leq b$.
(ii) $b \leq a + x$ implies that $b \leq x$, for all $x \in S$.

In particular, we observe that $a \ll_{\mathrm{rem}} b$ implies that $a + b \leq b$ (the converse does not hold as a rule). In particular, in case S is antisymmetric, $a \ll_{\mathrm{rem}} b$ implies that $a \ll b$.

LEMMA 2-5.6. *Let a, b, $c \in S$ such that either $a \ll_{\mathrm{rem}} b \leq c$ or $a \leq b \ll_{\mathrm{rem}} c$. Then $a \ll_{\mathrm{rem}} c$.*

PROOF. In both cases, it is trivial that $a \leq c$.

Suppose that $a \ll_{\mathrm{rem}} b \leq c$. Let $x \in S$ such that $c \leq a + x$. So $b \leq a + x$, thus, since $a \ll_{\mathrm{rem}} b$, $b \leq x$, that is, $x = b + y$ for some y. Hence $c \leq a + x = a + b + y$. But $a \ll_{\mathrm{rem}} b$, thus $a + b \leq b$, so $c \leq b + y = x$. So $a \ll_{\mathrm{rem}} c$.

Suppose now that $a \leq b \ll_{\mathrm{rem}} c$. Let $x \in S$ such that $c \leq a + x$. So $c \leq b + x$, thus (since $b \ll_{\mathrm{rem}} c$) $c \leq x$. So, again, $a \ll_{\mathrm{rem}} c$. $\square$

LEMMA 2-5.7. *Suppose that S is antisymmetric* (*that is, the algebraic preordering of S is antisymmetric*) *and that S has pseudo-cancellation. For all a, b, $c \in S$, the following assertions hold:*

(i) *$a \leq b \leq a + c$ implies that there exists $x \leq c$ such that $b = a + x$.*
(ii) *If $a \leq b$ in S, then $a \ll_{\mathrm{rem}} b$ if and only if $b = a + x$ implies that $b = x$, for all $x \in S$.*

PROOF. (i) Since $a \leq b$, there exists $y \in S$ such that $b = a + y$. Hence $a + y \leq a + c$, thus, by pseudo-cancellation, there exists $u \ll a$ such that $y \leq u + c$. By refinement, there are $v \leq u$ and $x \leq c$ such that $y = v + x$. Since S is antisymmetric, $v \ll a$. Hence, $b = a + y = a + x$, with $x \leq c$.

(ii) We prove the nontrivial direction. So, suppose that $b = a + x$ implies $b = x$, for all $x \in S$. Now let $x \in S$ such that $b \leq a + x$. By (i) above, there exists $y \leq x$ such that $b = a + y$. By assumption, $b = y$; whence $b \leq x$. $\square$

LEMMA 2-5.8. *Suppose that S is antisymmetric and satisfies general comparability. Let a, $b \in S$.*

(i) *If $a \ll_{\mathrm{rem}} b$, then $p(a) \ll_{\mathrm{rem}} p(b)$, for all $p \in \operatorname{Proj} S$.*
(ii) *Let $(p_i)_{i \in I}$ be a family of projections of S. We assume that both $\overline{a} = \bigvee_{i \in I} p_i(a)$ and $\overline{b} = \bigvee_{i \in I} p_i(b)$ are defined. If $p_i(a) \ll_{\mathrm{rem}} p_i(b)$ for all $i \in I$, then $\overline{a} \ll_{\mathrm{rem}} \overline{b}$.*
(iii) *Let $n < \omega$, let $(p_i)_{i<n}$ be a finite sequence of projections of S, and let $p = \bigvee_{i<n} p_i$. If $p_i(a) \ll_{\mathrm{rem}} p_i(b)$ for all $i < n$, then $p(a) \ll_{\mathrm{rem}} p(b)$.*

PROOF. (i) It is clear that $p(a) \leq p(b)$. Now let $x \in S$ such that $p(a)+x = p(b)$. So,

$$\begin{aligned} b &= p(b) + p^{\perp}(b) \\ &= p(a) + p^{\perp}(b) + x \\ &= p(a) + p^{\perp}(a) + p^{\perp}(b) + x && \text{(because } p^{\perp}(a) \ll p^{\perp}(b)) \\ &= a + p^{\perp}(b) + x. \end{aligned}$$

Since $a \ll_{\mathrm{rem}} b$, it follows that $p^{\perp}(b) + x = b$; whence $p(b) = p(x) = x$. By Lemma 2-5.7(ii), $p(a) \ll_{\mathrm{rem}} p(b)$.

(ii) Observe first that $\overline{a} \leq \overline{b}$. Let $x \in S$ such that $\overline{a} + x = \overline{b}$. Observe that $p_i(a) \leq \overline{a} \leq a$ for all $i \in I$; hence $p_i(\overline{a}) = p_i(a)$. Similarly, $p_i(\overline{b}) = p_i(b)$. Therefore, $p_i(a)+p_i(x) = p_i(b)$, for all i, hence, since $p_i(a) \ll_{\mathrm{rem}} p_i(b)$, $p_i(b) = p_i(x) \leq x$. This holds for all i, whence $\overline{b} \leq x$, so $x = \overline{b}$. The conclusion follows from Lemma 2-5.7(ii).

(iii) By Proposition 2-3.15, $p(a) = \bigvee_{i<n} p_i(a)$ and $p(b) = \bigvee_{i<n} p_i(b)$. The conclusion follows then from (ii). □

Corollary 2-5.9. *Suppose that S is antisymmetric and satisfies general comparability. For all a, b, $c \in S$, if $a \ll_{\mathrm{rem}} b, c$, then $a \ll_{\mathrm{rem}} b \wedge c$.*

Proof. By general comparability, there exists $p \in \operatorname{Proj} S$ such that $p(b) \leq p(c)$ and $p^{\perp}(c) \leq p^{\perp}(b)$. Then, as in the proof of Lemma 2-4.3, $b \wedge c = p(b) + p^{\perp}(c)$. By Lemma 2-5.8(i), $p(a) \ll_{\mathrm{rem}} p(b) = p(b \wedge c)$ and $p^{\perp}(a) \ll_{\mathrm{rem}} p^{\perp}(c) = p^{\perp}(b \wedge c)$. Hence, Lemma 2-5.8(iii) implies that $a \ll_{\mathrm{rem}} b \wedge c$. □

Definition 2-5.10. An element a of S is *purely infinite*, if $2a = a$.

We denote by $S|_{\infty}$ the set of all purely infinite elements of S.

We observe that the only element of S which is both directly finite and purely infinite is 0.

Lemma 2-5.11. *Suppose that S is antisymmetric and satisfies general comparability. Then $S|_{\infty}$ is closed under finite infima and suprema.*

Proof. This is clear from the descriptions of pairwise infima and suprema given in the proof of Lemma 2-4.3. □

Lemma 2-5.12. *Suppose that S is antisymmetric. Let a, $b \in S$ such that $a \leq b$. If either a or b is purely infinite, then $a \ll b$.*

Lemma 2-5.13. *Suppose that S is antisymmetric, Boolean-valued, and that it has general comparability. Let $a \in S|_{\infty}$ and $b \in S$ such that $a \leq b$. Put $p = \|b \leq a\|$. Then $p^{\perp}(a) \ll_{\mathrm{rem}} p^{\perp}(b)$.*

Proof. By the definition of p, $p(b) \leq p(a)$. Since S is antisymmetric, $p(a) = p(b)$. Furthermore, $p^{\perp}(a) \leq p^{\perp}(b)$ (because $a \leq b$).

Let $x \in S$ such that

$$p^{\perp}(b) = p^{\perp}(a) + x. \tag{2-5.1}$$

By general comparability, there exists $q \in \operatorname{Proj} S$ such that

$$q(x) \leq q(a) \tag{2-5.2}$$

$$q^{\perp}(a) \leq q^{\perp}(x) \tag{2-5.3}$$

By applying q to (2-5.1), we obtain that

$$qp^{\perp}(b) = qp^{\perp}(a) + q(x). \tag{2-5.4}$$

However, $x \leq p^{\perp}(b)$, thus $p^{\perp}(x) = x$, so $q(x) = qp^{\perp}(x) = p^{\perp}q(x) \leq p^{\perp}q(a) = qp^{\perp}(a)$. Hence, by Lemma 2-5.12, $qp^{\perp}(a) + q(x) = qp^{\perp}(a)$, so, by (2-5.4), $qp^{\perp}(b) = qp^{\perp}(a)$. By the definition of p, $qp^{\perp} \leq p$, thus, since $qp^{\perp} \leq p^{\perp}$, $qp^{\perp} = 0$, that is, $q \leq p$. Hence $p^{\perp} \leq q^{\perp}$, thus, by (2-5.3), $p^{\perp}(a) \leq p^{\perp}(x) = x$. Hence, by (2-5.1) and by Lemma 2-5.12, $x = p^{\perp}(b)$. We conclude the proof by Lemma 2-5.7(ii). □

We now introduce a useful definition.

DEFINITION 2-5.14. For $a \in S$, the *central cover* of a, denoted by $\mathrm{cc}(a)$, is defined as $\|a = 0\|^{\perp}$.

Note that $a^{\perp} = \mathrm{cc}(a)^{\perp} S$, by Lemma 2-5.3(i).

COROLLARY 2-5.15. *Suppose that S is antisymmetric, Boolean-valued, and that it has general comparability. Let $a \in S|_{\infty}$ and $b \in S$ such that $a \leq b$. Then $a \ll_{\mathrm{rem}} b$ if and only if $q(b) \not\leq q(a)$ for all nonzero projections $q \leq \mathrm{cc}(b)$.*

PROOF. Assume first that $a \ll_{\mathrm{rem}} b$, and let $q \leq \mathrm{cc}(b)$ be a projection such that $q(b) \leq q(a)$. Then $b \leq q(a) + q^{\perp}(b) \leq a + q^{\perp}(b)$, and it follows from the assumption $a \ll_{\mathrm{rem}} b$ that $b \leq q^{\perp}(b)$. Thus $q(b) = 0$, so $q \wedge \mathrm{cc}(b) = 0$, and hence $q = 0$.

Conversely, assume that $q(b) \not\leq q(a)$ for all nonzero projections $q \leq \mathrm{cc}(b)$, set $p = \|b \leq a\|$, and observe that $p \perp \mathrm{cc}(b)$. By Lemma 2-5.13, $p^{\perp}(a) \ll_{\mathrm{rem}} p^{\perp}(b)$, and so $\mathrm{cc}(b)(a) \ll_{\mathrm{rem}} \mathrm{cc}(b)(b)$. Therefore $a \ll_{\mathrm{rem}} b$. $\square$

LEMMA 2-5.16. *Assume that S is antisymmetric, Boolean-valued, and satisfies general comparability. Let $a \in S$ and let $p \in \operatorname{Proj} S$.*

(i) $\mathrm{cc}(a) \leq p$ *if and only if* $a \in pS$.
(ii) $\mathrm{cc}(p(a)) = p \wedge \mathrm{cc}(a)$.
(iii) *Suppose that $a = \bigvee_{i \in I} a_i$, for a family $(a_i)_{i \in I}$ of elements of S. Then* $\mathrm{cc}(a) = \bigvee_{i \in I} \mathrm{cc}(a_i)$.
(iv) $\mathrm{cc}(a \wedge b) = \mathrm{cc}(a) \wedge \mathrm{cc}(b)$, *for all a, $b \in S$.*

PROOF. (i) $\mathrm{cc}(a) \leq p$ if and only if $p^{\perp} \leq \|a = 0\|$, if and only if $p^{\perp}(a) = 0$, if and only if $p(a) = a$, if and only if $a \in pS$.

(ii) For all $q \in \operatorname{Proj} S$, $q \leq \|p(a) = 0\|$ if and only if $qp(a) = 0$, if and only if $qp \leq \|a = 0\|$, if and only if $q \leq \|a = 0\| \vee p^{\perp}$. Hence

$$\|p(a) = 0\| = \|a = 0\| \vee p^{\perp}.$$

Therefore, $\mathrm{cc}(p(a)) = \|p(a) = 0\|^{\perp} = p \wedge \mathrm{cc}(a)$.

(iii) By (i), for any $p \in \operatorname{Proj} S$, $\mathrm{cc}(a) \leq p$ if and only if $a \in pS$, if and only if $a_i \in pS$ for all i (by Lemma 2-3.16(ii)), if and only if $\mathrm{cc}(a_i) \leq p$ for all i, if and only if $\bigvee_{i \in I} \mathrm{cc}(a_i) \leq p$. The conclusion of (iii) follows.

(iv) It suffices to prove the inequality

$$\|a \wedge b = 0\| = \|a = 0\| \vee \|b = 0\| . \tag{2-5.5}$$

Since $a \wedge b \leq a$, b, the inequality $\|a \wedge b = 0\| \geq \|a = 0\| \vee \|b = 0\|$ is obvious. Conversely, put $p = \|a \wedge b = 0\|$. By general comparability, there are q, $r \in \operatorname{Proj} S$ such that $q(a) \leq q(b)$, $r(b) \leq r(a)$, and $p = q \vee r$. It follows that

$$\begin{aligned}
0 &= q(a \wedge b) && \text{(because } q \leq p\text{)}\\
&= q(a) \wedge q(b) && \text{(by Lemma 2-3.16(i))}\\
&= q(a) && \text{(because } q(a) \leq q(b)\text{),}
\end{aligned}$$

hence $q \leq \|a = 0\|$. Similarly, $r \leq \|b = 0\|$, so $p \leq \|a = 0\| \vee \|b = 0\|$. This completes the proof of (2-5.5). $\square$

2-6. Least and largest difference functions

Standing hypothesis: S is a partial refinement monoid satisfying the following additional properties:
(1) *S is antisymmetric.*
(2) *S has general comparability.*
(3) *S is Boolean-valued.*
(4) *Every element of S is the sum of a directly finite element and a purely infinite element.*

LEMMA 2-6.1. *For all $a \in S$, there exists $p \in \operatorname{Proj} S$ such that $p(a)$ is directly finite and $p^{\perp}(a)$ is purely infinite.*

PROOF. By assumption on S, there are elements x and y in S such that $a = x + y$, x is purely infinite, and y is directly finite. By general comparability, there exists $p \in \operatorname{Proj} S$ such that $p(x) \leq p(y)$ and $p^{\perp}(y) \leq p^{\perp}(x)$. Since y is directly finite and $p(x) \leq p(y) \leq y$, $p(x)$ is directly finite. But $p(x)$ is purely infinite, thus $p(x) = 0$, and so $p(a) = p(y)$ is directly finite. Since $p^{\perp}(x) \geq p^{\perp}(y)$ with $p^{\perp}(x)$ purely infinite, $p^{\perp}(a) = p^{\perp}(x) + p^{\perp}(y) = p^{\perp}(x)$ by Lemma 2-5.12. Therefore, $p^{\perp}(a)$ is purely infinite. □

COROLLARY 2-6.2. *For any $a \in S$, the following assertions hold:*
(i) *There exists a largest purely infinite element u of S such that $u \leq a$.*
(ii) *The element u is also the largest $s \in S$ such that $s \ll a$.*
(iii) *There exists a unique $v \in S$ such that $a = u + v$ and $u \perp v$.*
(iv) *The element v is directly finite.*

PROOF. By Lemma 2-6.1, there exists $p \in \operatorname{Proj} S$ such that $p(a)$ is directly finite and $p^{\perp}(a)$ is purely infinite. Set $u = p^{\perp}(a)$ and $v = p(a)$. Observe that $u \leq a$ and u is purely infinite, so $u \ll a$.

(ii) For any $s \in S$, $s \ll a$ implies that $p(s) \ll p(a)$. Since $p(a)$ is directly finite, $p(s) = 0$, and thus $s = p^{\perp}(s) \leq p^{\perp}(a) = u$.

(i) For any purely infinite $t \in S$ such that $t \leq a$, it follows from Lemma 2-5.12 that $t \ll a$, whence $t \leq u$ by part (ii).

(iii), (iv) We already have $a = u + v$ with $u \perp v$ and v directly finite. For any $w \in S$, if $a = u + w$ with $u \perp w$, then $u + w = u + v$, thus, by refinement (and since $u \perp v, w$), $v = w$. □

NOTATION 2-6.3. For any $a \in S$, we shall denote by $\frac{a}{\infty}$ the largest purely infinite element u of S such that $u \leq a$.

LEMMA 2-6.4. *Let a, $b \in S$ such that $a + b$ is defined. Then*
$$\frac{a+b}{\infty} = \frac{a}{\infty} + \frac{b}{\infty}.$$

PROOF. First, $\frac{a}{\infty} + \frac{b}{\infty}$ is purely infinite and below $a + b$, thus $\frac{a}{\infty} + \frac{b}{\infty} \leq \frac{a+b}{\infty}$.

Conversely, put $c = \frac{a+b}{\infty}$. Then $c + a + b = a + b$, thus, by canceling the directly finite parts of a and b (use Lemma 2-4.8), $c + a + \frac{b}{\infty} = a + \frac{b}{\infty}$, thus, again, $c + \frac{a}{\infty} + \frac{b}{\infty} = \frac{a}{\infty} + \frac{b}{\infty}$. In particular, $c \leq \frac{a}{\infty} + \frac{b}{\infty}$. □

Our next result involves the *least difference* function introduced in Definition 2-3.17.

PROPOSITION 2-6.5. *Let $a \leq b$ in S; then $b \smallsetminus a$ exists.*

PROOF. By Lemma 2-6.1, there exists $q \in \operatorname{Proj} S$ such that $q(a)$ is directly finite and $q^{\perp}(a)$ is purely infinite. Let $c_0 \in S$ such that

$$q(a) + c_0 = q(b). \tag{2-6.1}$$

Put $p = \left\| q^{\perp}(b) \leq q^{\perp}(a) \right\|$. Since $qq^{\perp} = 0$, the inequality $q \leq p$ holds. Observe also the following equality:

$$pq^{\perp}(b) = pq^{\perp}(a). \tag{2-6.2}$$

Since $a \leq b$ and $q^{\perp}(a)$ is purely infinite, it follows from Lemma 2-5.13 that $p^{\perp}q^{\perp}(a) \ll_{\mathrm{rem}} p^{\perp}q^{\perp}(b)$, that is, since $p^{\perp} \leq q^{\perp}$,

$$p^{\perp}(a) \ll_{\mathrm{rem}} p^{\perp}(b). \tag{2-6.3}$$

In particular, we obtain the relation

$$p^{\perp}(a) \ll p^{\perp}(b). \tag{2-6.4}$$

Since $c_0 \leq q(b) \leq p(b)$, $c = c_0 + p^{\perp}(b)$ is defined. So we obtain that

$$\begin{aligned}
b &= p^{\perp}(b) + pq^{\perp}(b) + q(b) \\
&= p^{\perp}(a) + p^{\perp}(b) + pq^{\perp}(a) + q(a) + c_0 \qquad \text{(by (2-6.1), (2-6.2), and (2-6.4))} \\
&= a + c_0 + p^{\perp}(b) \\
&= a + c.
\end{aligned}$$

Furthermore, let $x \in S$ such that $b \leq a + x$. So $q(b) \leq q(a) + q(x)$, that is, $q(a) + c_0 \leq q(a) + q(x)$. Thus, since $q(a)$ is directly finite and by Lemma 2-4.8, we obtain

$$c_0 \leq q(x). \tag{2-6.5}$$

Furthermore, $p^{\perp}(b) \leq p^{\perp}(a) + p^{\perp}(x)$, thus, by (2-6.3), we obtain that

$$p^{\perp}(b) \leq p^{\perp}(x). \tag{2-6.6}$$

By adding (2-6.5) and (2-6.6) together, we thus obtain that $c \leq q(x) + p^{\perp}(x) \leq p(x) + p^{\perp}(x) = x$. So we have verified that $c = b \smallsetminus a$. □

A similar result holds for the existence of the "largest difference".

PROPOSITION 2-6.6. *Let $a \leq b$ in S. Then there exists a largest element c of S such that $a + c \leq b$, and then $b = a + c$.*

The element c of the statement above will be denoted by $b - a$, the *largest difference* of b and a.

PROOF. By the definition of the algebraic preordering, there exists $d \in S$ such that $a + d = b$. So $c = \frac{a}{\infty} + d$ is defined (because $\frac{a}{\infty} \leq a$) and $c \leq b$. From $\frac{a}{\infty} \ll a$ it follows that $a + c = b$. If $x \in S$ is such that $a + x \leq b$, then, by pseudo-cancellation (see Lemma 2-4.5), $x \leq d + y$ for some $y \ll a$, so $x \leq d + \frac{a}{\infty} = c$. □

COROLLARY 2-6.7. *Let $a, b \in S$, let X be a nonempty subset of S. We assume that $a + x$ is defined for all $x \in X$.*

(i) *If $b = \bigwedge X$, then $a + b = \bigwedge(a + X)$.*
(ii) *If $b = \bigvee X$ and $a + X$ is majorized, then $a + b = \bigvee(a + X)$.*

Proof. (i) Pick $x \in X$. Since $a + x$ is defined and $b \leq x$, $a + b$ is defined. Furthermore, $a + b \leq a + X$. Conversely, let $c \leq a + X$. By Lemma 2-4.3, $c' = a \vee c$ is defined, and $a \leq c' \leq a + X$. By Proposition 2-6.5, $c' \smallsetminus a \leq X$, so $c' \smallsetminus a \leq b$. Therefore, by adding a on both sides of this inequality, we obtain that $c \leq c' = a + (c' \smallsetminus a) \leq a + b$.

(ii) Pick a majorant c of $a + X$. In particular, $c \geq a$. By Proposition 2-6.6, $c - a \geq X$, so $c - a \geq b$. Since $c = a + (c - a)$, $a + b$ is defined and $a + b \leq c$. This holds for any majorant c of $a + X$. Since $a + b$ is itself a majorant of $a + X$, it is the supremum of $a + X$. $\square$

CHAPTER 3

Continuous dimension scales

3-1. Basic properties; the monoids $\mathbb{Z}_\gamma$, $\mathbb{R}_\gamma$, and $\mathbf{2}_\gamma$

The fundamental definition underlying this chapter is the following.

Definition 3-1.1. A *continuous dimension scale* is a partial commutative monoid S which satisfies the following axioms.

(M1) S has refinement (see Definition 2-1.12), and the algebraic preordering on S is antisymmetric.
(M2) Every nonempty subset of S admits an infimum. Equivalently, every majorized subset of S admits a supremum.
(M3) S has general comparability (see Definition 2-4.1).
(M4) S is Boolean-valued (see Definition 2-5.1).
(M5) Every element a of S can be written $a = x + y$, where x is directly finite (Definition 2-4.7) and y is purely infinite (Definition 2-5.10).
(M6) Let a, b be purely infinite elements of S. If $a \ll_{\mathrm{rem}} b$ (see Definition 2-5.5), then the set of all purely infinite elements x of S such that $a \ll_{\mathrm{rem}} x$ and $x^\perp = b^\perp$ (see (2-2.1), page 21) has a least element.

A continuous dimension scale S is *bounded*, if it has a largest element.

All axioms (M1)–(M5) have been considered in Chapter 2. Axiom (M6) is a newcomer, whose importance will appear in Section 3-5.

We shall first give an alternative axiomatization of continuous dimension scales. In order to prepare for this, we first prove the following result, which extends the result of Lemma 2-4.2.

Proposition 3-1.2. *Let S be a partial refinement monoid satisfying the following properties:*

(1) *S is antisymmetric.*
(2) *Any two elements of S have a meet.*
(3) *S satisfies Axiom* (M5).

Then the following assertions are equivalent:

(i) *S satisfies the following axioms:*
  (N1) *$\forall a, b$, $\exists c, x, y$ such that $a = c + x$, $b = c + y$, and $x \perp y$.*
  (N2) *$S = a^\perp + a^{\perp\perp}$, for all $a \in S$.*
  (N3) *$b \smallsetminus a$ exists, for all a, $b \in S$ such that $a \leq b$.*
(ii) *S is Boolean-valued and it satisfies general comparability.*

Proof. (i)⇒(ii) Let S satisfy (N1), (N2), and (N3). The fact that S satisfies general comparability follows from Lemma 2-4.2. Now we prove that S is Boolean-valued. So let a, $b \in S$. By (N3), $c = a \smallsetminus (a \wedge b)$ exists. For any projection p

of S,

$$
\begin{aligned}
p(a) \leq p(b) \quad &\text{if and only if} \quad & p(a) &= p(a) \wedge p(b) & \\
&\text{if and only if} \quad & p(a) &= p(a \wedge b) & \text{(by Lemma 2-3.16(i))} \\
&\text{if and only if} \quad & p(a) \smallsetminus p(a \wedge b) &= 0 & \\
&\text{if and only if} \quad & p(c) &= 0 & \text{(by Lemma 2-3.19).}
\end{aligned} \tag{3-1.1}
$$

By (N2), $S = c^{\perp} + c^{\perp\perp}$. By Lemma 2-2.2(ii), $c^{\perp}$ and $c^{\perp\perp}$ are ideals of S, thus, since $c^{\perp} \cap c^{\perp\perp} = \{0\}$, $S = c^{\perp} \oplus c^{\perp\perp}$, hence, by Lemma 2-5.4, $p = \|c = 0\|$ exists. Therefore, by (3-1.1), $\|a \leq b\|$ exists, and $\|a \leq b\| = \|c = 0\|$.

(ii)$\Rightarrow$(i) Suppose that S is Boolean-valued and satisfies general comparability. We verify that S satisfies (N1)–(N3).

(N1) By general comparability, there exists $p \in \operatorname{Proj} S$ such that $p(a) \leq p(b)$ and $p^{\perp}(b) \leq p^{\perp}(a)$. Let $x, y \in S$ such that $p(a) + y = p(b)$ and $p^{\perp}(b) + x = p^{\perp}(a)$. Furthermore, since $p(a) \leq p(b)$ and $b = p(b) + p^{\perp}(b)$, the element $c = p(a) + p^{\perp}(b)$ is defined. From $x \leq p^{\perp}(a)$ and $y \leq p(b)$ it follows that $x \perp y$. Finally,

$$
\begin{aligned}
a &= p(a) + p^{\perp}(a) = c + x, \\
b &= p(b) + p^{\perp}(b) \ = c + y.
\end{aligned}
$$

Hence we have obtained (N1).

(N2) follows immediately from Lemma 2-5.3.

(N3) follows immediately from Proposition 2-6.5. □

Corollary 3-1.3. *Let S be a partial commutative monoid. Then S is a continuous dimension scale if and only if it satisfies the axioms* (M1), (M2), (M5), (M6), (N1), (N2), *and* (N3).

Remark 3-1.4. It follows from Corollary 3-1.3 that for a partial commutative monoid S, to be a continuous dimension scale is equivalent to the conjunction of the *second-order* axiom (M2) and a finite list of *first-order* axioms.

As a corollary of this alternative description of continuous dimension scales, we observe the following.

Definition 3-1.5. The *direct product* of a family $(S_i)_{i \in I}$ of partial commutative monoids is obtained by endowing the ordinary cartesian product $S = \prod_{i \in I} S_i$ with the partial addition defined by

$$(a_i)_{i \in I} + (b_i)_{i \in I} = (a_i + b_i)_{i \in I},$$

for all $(a_i)_{i \in I}, (b_i)_{i \in I} \in S$.

Of course, Definition 3-1.5 is an obvious generalization of the definition of the product of finitely many partial commutative monoids introduced in Remark 2-2.5.

It is trivial that the direct product of any family of partial commutative monoids is a partial commutative monoid. Far less trivial is the following preservation result.

Lemma 3-1.6. *Any direct product of a family of continuous dimension scales is a continuous dimension scale.*

Proof. We use the characterization of continuous dimension scales obtained in Corollary 3-1.3. The proof is relatively long but very easy, so we will not give the details of it but rather the basic idea. A key point is to verify that the operations $x \mapsto x^{\perp}$, $x \mapsto x^{\perp\perp}$, $(x, y) \mapsto x \wedge y$, $(x, y) \mapsto y \smallsetminus x$ (for $x \leq y$), and the relations $x \leq y$, $x \ll_{\mathrm{rem}} y$, $x \perp y$, and $y \in x^{\perp\perp}$ can be "read componentwise", that is, for example, if $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$, then $x \leq y$ if and only if $x_i \leq y_i$ for all i, $x^{\perp} = \prod_{i \in I} x_i^{\perp}$, and so on. Once these simple facts are established, the verification of the axioms (M1), (M2), (M5), (M6), (N1), (N2), and (N3) is routine. □

For a continuous dimension scale S and elements a and b in a lower subset T of S, the orthogonality of a and b means the same in S and in T. We capture this pattern in a definition.

Definition 3-1.7.

(i) A *statement* $\varphi(x_1, \dots, x_n)$ in the language of partial commutative monoids is *absolute*, if for any continuous dimension scale S, every lower subset T of S, and all elements $a_1, \dots, a_n \in T$, S satisfies $\varphi(a_1, \dots, a_n)$ if and only if T satisfies $\varphi(a_1, \dots, a_n)$.

(ii) A *definable function* $y = \varphi(x_1, \dots, x_n)$ in the language of partial commutative monoids is *absolute*, if for any continuous dimension scale S, every lower subset T of S, and all elements $a_1, \dots, a_n \in T$, $\varphi(a_1, \dots, a_n)$ is defined in S if and only if it is defined in T, and then both values are equal.

Lemma 3-1.8. *The following statements*

(i) $x \leq y$;
(ii) $x \perp y$;
(iii) $x^{\perp} \subseteq y^{\perp}$;
(iv) $x^{\perp} = y^{\perp}$;
(v) $x \ll_{\mathrm{rem}} y$

and the following function

(vi) $z = y \smallsetminus x$

are absolute.

Proof. Most items are trivial, except perhaps (v) and (vi). Let T be a lower subset of a continuous dimension scale S, let $a \leq b$ in T. Since S is a continuous dimension scale, $c = b \smallsetminus a$ is defined in S. By Lemma 2-3.18, $b = a + c$, thus, since T is a lower subset of S, $c \in T$. It readily follows that $c = b \smallsetminus a$ in T, which concludes the proof of (vi) (because c always exists). As $a \ll_{\mathrm{rem}} b$ if and only if $a \leq b$ and $b \smallsetminus a = b$, item (v) follows immediately. □

As an easy consequence of Corollary 3-1.3 and Lemma 3-1.8, we obtain the following.

Lemma 3-1.9. *Let S be a continuous dimension scale, let T be a lower subset of S, viewed as a partial submonoid of S (see Definition 2-1.4). Then T is a continuous dimension scale.*

We refer to Lemma 3-7.1 for more information on lower subsets of continuous dimension scales.

We observe that trying to use the original axioms (M1)–(M6) for the proof of Lemma 3-1.6 would have been much more difficult, since we would have needed to understand the projections of the product $\prod_{i\in I} S_i$. By using Corollary 3-1.3, the proof is still somewhat tedious, but essentially trivial.

We present another way to produce continuous dimension scales.

LEMMA 3-1.10. *Let I be an upwards directed partially ordered set, let $(S_i)_{i\in I}$ be a family of continuous dimension scales such that S_i is a lower subset of S_j, for all $i \leq j$ in I. Then $\bigcup_{\in I} S_i$ is a continuous dimension scale.*

PROOF. The set S is, of course, endowed with the union of all the partial commutative monoid operations on all the S_i-s. By using Lemma 3-1.8, it is easy to verify that S satisfies (M1), (M5), (N1), (N2), and (N3).

Let X be a nonempty subset of S; so there exists $i \in I$ such that $X \cap S_i \neq \varnothing$. Denote by a_j the meet of $X \cap S_j$, for all $j \geq i$ in I. Then $i \leq j \leq k$ implies that $a_j \geq a_k$, in particular, all the a_j-s belong to S_i, and the meet of all the a_j-s in S_i is also the meet of X in S. Hence S satisfies (M2).

Let $a \ll_{\mathrm{rem}} b$ in $S|_\infty$. There exists $i \in I$ such that a, $b \in S_i$. It follows from Lemma 3-1.8 that the statement $a \ll_{\mathrm{rem}} b$ holds in all S_j with $j \geq i$, thus, since S_j is a continuous dimension scale, the set of all elements $x \in S_j|_\infty$ such that $a \ll_{\mathrm{rem}} x$ and $x^\perp = b^\perp$ (we use Lemma 3-1.8) has a least element, say, c_j. It follows again from Lemma 3-1.8 that $c_j = c_i$, for all $j \geq i$; denote by c this element, then $c \in S|_\infty$ and c is minimum in the set of all elements $x \in S|_\infty$ such that $a \ll_{\mathrm{rem}} x$ and $x^\perp = b^\perp$. Hence S satisfies (M6). By Corollary 3-1.3, S is a continuous dimension scale. □

We now provide fundamental examples of continuous dimension scales.

For an ordinal γ, the monoids $\mathbb{Z}_\gamma$, $\mathbb{R}_\gamma$, and $\mathbf{2}_\gamma$ are defined in the Introduction:

$$\begin{aligned}
\mathbb{Z}_\gamma &= \mathbb{Z}^+ \cup \{\aleph_\xi \mid 0 \leq \xi \leq \gamma\},\\
\mathbb{R}_\gamma &= \mathbb{R}^+ \cup \{\aleph_\xi \mid 0 \leq \xi \leq \gamma\},\\
\mathbf{2}_\gamma &= \{0\} \cup \{\aleph_\xi \mid 0 \leq \xi \leq \gamma\}.
\end{aligned}$$

We call the elements of $\mathbb{Z}^+$, $\mathbb{R}^+$, and $\{0\}$ the *finite elements* of $\mathbb{Z}_\gamma$, $\mathbb{R}_\gamma$, and $\mathbf{2}_\gamma$, respectively. We endow each of the sets $\mathbb{Z}_\gamma$, $\mathbb{R}_\gamma$, and $\mathbf{2}_\gamma$ with the addition that extends the natural addition on the finite elements and on the alephs (so $\aleph_\alpha + \aleph_\beta = \aleph_\beta$ if $\alpha \leq \beta$), and such that every finite element is absorbed by every aleph. Hence $\mathbb{Z}_\gamma$, $\mathbb{R}_\gamma$, and $\mathbf{2}_\gamma$ are monoids (and not just partial ones). We also observe that the finite elements of $\mathbb{Z}_\gamma$, $\mathbb{R}_\gamma$, and $\mathbf{2}_\gamma$ are precisely the directly finite ones.

Therefore, the algebraic ordering on each of the structures $\mathbb{Z}_\gamma$, $\mathbb{R}_\gamma$, and $\mathbf{2}_\gamma$ is the natural *total* ordering.

PROPOSITION 3-1.11. *For every ordinal γ, the monoids $\mathbb{Z}_\gamma$, $\mathbb{R}_\gamma$, and $\mathbf{2}_\gamma$ are totally ordered continuous dimension scales.*

PROOF. All axioms (M1)–(M6) are trivially satisfied, except perhaps refinement. Let S be one of the structures $\mathbb{Z}_\gamma$, $\mathbb{R}_\gamma$, or $\mathbf{2}_\gamma$. Since every element of S is either cancellable or purely infinite, S satisfies the following weak form of the pseudo-cancellation property:

$$\forall a, b, c, \quad a + c = b + c \Rightarrow \exists x \ll c, a \leq b + x.$$

By Proposition 1.23 of [**53**], since S is totally ordered, it has refinement. □

Of course, one could also have verified refinement directly, but at the expense of a few more calculations. For example, start with the observation that $\mathbb{Z}^+$, $\mathbb{R}^+$, and $\{0\}$ have refinement; note the easy fact that adjoining a new infinity element to any refinement monoid produces another refinement monoid; use ordinal induction.

We shall also see in Proposition 3-2.1 that the positive cone of any Dedekind complete lattice ordered group is a continuous dimension scale. Furthermore, Proposition 3-1.11 will be considerably extended in Theorem 3-3.6.

3-2. Dedekind complete lattice-ordered groups

We recall that every Dedekind complete partially ordered group is abelian, see [**6**, Theorem 28]. Much more is true.

PROPOSITION 3-2.1. *Let G be a Dedekind complete lattice-ordered group. Then G^+ is a continuous dimension scale.*

PROOF. It is trivial that the algebraic preordering on G^+ is antisymmetric. It is well-known that G^+ (or, more generally, the positive cone of any lattice-ordered group) satisfies refinement, see, for example, [**5**, Théorème 1.2.16]. Hence G^+ satisfies (M1).

Axiom (M2) is just a reformulation of the fact that G is Dedekind complete.

Axioms (M5) and (M6) are trivially satisfied, because every element of G^+ is directly finite.

To verify that G^+ satisfies Axiom (M3), it suffices to verify that it satisfies the assumptions of Lemma 2-4.2. For a, $b \in G^+$, if we put $c = a \wedge b$, $x = a - c$, and $y = b - c$, then $a = c + x$, $b = c + y$, and $x \wedge y = 0$. This takes care of Assumption (i).

For x, $y \in G$, we define $x \perp y$ to hold, if $|x| \wedge |y| = 0$, and we put

$$X^\perp = \{g \in G \mid g \perp x \text{ for all } x \in X\},$$

the *polar* of X. Since G is Dedekind complete, the direct factors of G are exactly the polar subsets of G, see [**5**, Théorème 11.2.4]. In particular, $G = \{a\}^\perp + \{a\}^{\perp\perp}$, for all $a \in G$. Assumption (ii) follows.

Finally, we verify that G^+ satisfies Axiom (M4). Let a, $b \in G^+$. We put $c = a - a \wedge b$. For a polar subset H of G, G is the orthogonal sum of H and $H^\perp$ (see, for example, [**6**, Theorem 27]). In particular, the projection p_H of G onto H (relatively to $H^\perp$) is an idempotent homomorphism of lattice-ordered groups. By Proposition 2-3.2, the projections of G^+ are exactly the restrictions to G^+ of the maps of the form p_H, for a polar subset H of G. For any such H, $p_H(a) \leq p_H(b)$ if and only if $p_H(c) = 0$, that is, $H \subseteq \{c\}^\perp$. Hence, the restriction of $p_{\{c\}^\perp}$ to G^+ is the largest projection p of G^+ such that $p(a) \leq p(b)$. □

LEMMA 3-2.2. *Let S be a conical partial refinement monoid. If S is cancellative and satisfies Axiom* (M2)*, then $\widetilde{S}$* (*see Proposition* 2-1.13) *is the positive cone of a Dedekind complete lattice-ordered group.*

PROOF. By Proposition 2-1.13, $\widetilde{S}$ is a refinement monoid, and it is generated by S as a monoid. By Lemma 2-1.15(ii), $\widetilde{S}$ is a cancellative commutative monoid, thus it is the positive cone G^+ of a directed, partially preordered abelian group G. Since S is conical, so is $\widetilde{S}$ (see Lemma 2-1.15(iii)), and hence G is, in fact, partially ordered. Since $G^+ = \widetilde{S}$ has refinement, G is an interpolation group, see Proposition 2.1 in [**16**].

Claim 1. *Let a, b, $c \in S$ such that c is the infimum of $\{a, b\}$ in S. Then c is also the infimum of $\{a, b\}$ in G.*

Proof of Claim. Let $x \in G$ be a minorant of $\{a, b\}$. Since G has interpolation, there exists $y \in G$ such that 0, $x \leq y \leq a$, b. So $y \in S$, hence $x \leq y \leq c$ by the assumption on c. □ Claim 1.

Claim 2. *G is lattice-ordered.*

Proof. We put $X = \{x \in G^+ \mid s \wedge x \text{ exists in } G, \text{ for all } s \in S\}$. It follows from Claim 1 that X contains S. Let a, $b \in X$, we prove that $a + b \in X$. Let $s \in S$, put $s' = s - s \wedge a$. For any $x \in G$,

$$\begin{aligned} x \leq s,\ a+b \text{ if and only if } & x \leq s,\ s+b,\ a+b \\ \text{if and only if } & x \leq (s \wedge a) + s',\ (s \wedge a) + b \end{aligned}$$

(because $t \mapsto t + b$ is an order-automorphism of G)

$$\text{if and only if } x \leq (s \wedge a) + (s' \wedge b),$$

so $s \wedge (a + b) = (s \wedge a) + (s' \wedge b)$. So X is closed under addition, whence $X = G^+$. Then, replacing S by G^+ in the definition of X yields, by a similar argument, that $a \wedge b$ is defined, for all a, $b \in G$. □

Claim 3. *Let X be a nonempty subset of S and $a \in S$. If a is the supremum of X in S, then it is also the supremum of X in G.*

Proof of Claim. Let $b \in G$ be a majorant of X. Then, using Claim 2, $a \wedge b$ is also a majorant of X, but $0 \leq a \wedge b \leq a$, thus $a \wedge b \in S$. Since a is the supremum of X in S, $a = a \wedge b \leq b$. □ Claim 3.

Claim 4. *$2S$ satisfies Axiom* (M2).

Proof of Claim. Let $X = \{c_i \mid i \in I\}$ be a nonempty subset of $2S$, majorized by some element of $2S$, say, $a + b$, where a, $b \in S$. We prove that X admits a supremum in G.

By (i), $2S$ satisfies refinement, thus, for all $i \in I$, there are $a_i \leq a$ and $b_i \leq b$ in S such that $c_i = a_i + b_i$. Since $\{a_i \mid i \in I\}$ is a nonempty subset of S, majorized by a, it has, by assumption, a supremum in S, say, u. Observe that $a_i \leq u$ (in S) for all i, and put $a_i^* = u - a_i$. Then $\{b_i - a_i^* \wedge b_i \mid i \in I\}$ is a nonempty subset of S, majorized by b, thus it has a supremum, say, v, in S.

For all $i \in I$, $b_i \leq a_i^* \wedge b_i + v \leq a_i^* + v$, thus, by adding a_i to this inequality, we obtain that $c_i \leq u + v$.

So, to conclude the proof, it suffices to prove that $u + v$ is the least common majorant of X. So let x be a majorant of X. It follows from Claim 3 that u is the supremum of $\{a_i \mid i \in I\}$ in G. Then $a_i \leq c_i \leq x$, for all $i \in I$, thus $u \leq x$. Put $y = u - x$; observe that $y \in G^+$. For $i \in I$,

$$c_i = a_i + b_i \leq x = u + y = a_i + a_i^* + y,$$

so $b_i \leq a_i^* + y$. Since $b_i \leq b_i + y$ and G is lattice-ordered, $b_i \leq (a_i^* + y) \wedge (b_i + y) = a_i^* \wedge b_i + y$, so $b_i - a_i^* \wedge b_i \leq y$. This holds for all i, thus, by Claim 3, $v \leq y$, so $u + v \leq u + y = x$. □ Claim 4.

The conclusion of the proof is then easy: $2^n S$ is a lower subset of $2^{n+1}S$, for each $n < \omega$, and all the sets $2^n S$ satisfy (M2), thus their union, namely, G^+, also satisfies (M2). Therefore, G is Dedekind complete. □

3-3. Continuous functions on extremally disconnected topological spaces

We shall first present a very general result about continuous functions from extremally disconnected topological spaces to totally ordered sets with their interval topology. Some particular cases of this result are well-known. For example, if Ω is a complete Boolean space, then $\mathbf{C}(\Omega, \mathbb{R})$ is a Dedekind complete lattice ordered group, see Sätze 1 and 3 in [**43**] and Theorem 14 in [**47**]. We also refer to [**16**, Lemma 9.1] for a version of Proposition 3-3.2 for Ω basically disconnected and K an arbitrary closed interval of $\mathbb{R}$.

We recall a basic result of general topology, see [**20**, Theorem 10].

Proposition 3-3.1. *A topological space is the ultrafilter space of a complete Boolean algebra if and only if it is a complete Boolean space.*

Proposition 3-3.2. *Let Ω be an extremally disconnected topological space, let K be a complete totally ordered set. We endow K with its interval topology. Let $f\colon \Omega \to K$ be a lower semicontinuous map. Then the map $f^*\colon \Omega \to K$ defined by the rule*

$$f^*(x) = \bigwedge_{V \in \mathcal{N}(x)} \bigvee f[V], \quad \textit{for all } x \in \Omega,$$

is continuous, and it is the least continuous map $g\colon \Omega \to K$ such that $f \leq g$ (with respect to the componentwise ordering of $\mathbf{C}(\Omega, K)$).

Proof. Obviously, $f \leq f^*$. For any $\alpha \in K$, we define subsets of Ω by

$$F_\alpha = \{x \in \Omega \mid f(x) \leq \alpha\},\ G_\alpha = \{x \in \Omega \mid f^*(x) \leq \alpha\},\ G^*_\alpha = \{x \in \Omega \mid f^*(x) < \alpha\}.$$

We record a few basic facts about the sets F_α, G_α, G^*_α.

Claim 1. *For all $\alpha \in K$, the following assertions hold:*

(i) *F_α is closed.*
(ii) *$G_\alpha \subseteq F_\alpha$.*
(iii) *$\mathring{F}_\alpha = \mathring{G}_\alpha$.*

Proof of Claim. (i) follows from the lower semicontinuity of f.
(ii) follows from the fact that $f \leq f^*$.
(iii) By (ii), $\mathring{F}_\alpha$ contains $\mathring{G}_\alpha$. Put $V = \mathring{F}_\alpha$. For any $x \in V$, we have $V \in \mathcal{N}(x)$ and $\bigvee f[V] \leq \alpha$, hence $f^*(x) \leq \alpha$, that is, $x \in G_\alpha$. Hence V is contained in G_α, thus, since V is open, in $\mathring{G}_\alpha$. □ Claim 1.

To prove that f^* is continuous, it is sufficient to prove that G^*_α is open and that G_α is closed, for any $\alpha \in K$.

We start with G^*_α. For any $x \in G^*_\alpha$, there exists $V \in \mathcal{N}(x)$ such that $\bigvee f[V] < \alpha$, that is, there exists $\beta < \alpha$ such that $V \subseteq F_\beta$. Since V is open, it follows from Claim 1(iii) that $V \subseteq G_\beta$, whence $V \subseteq G^*_\alpha$.

Let now $x \in \overline{G_\alpha}$, we prove that $x \in G_\alpha$. Suppose first that α is right isolated, that is, $[\alpha, \gamma] = \{\alpha, \gamma\}$ for some $\gamma > \alpha$. Then $G_\alpha = G^*_\gamma$, and so G_α is open.

Since Ω is extremally disconnected, $\overline{G_\alpha}$ is clopen. On the other hand, it follows from Claim 1(i, ii) that $\overline{G_\alpha} \subseteq F_\alpha$. Therefore, by Claim 1(iii), $\overline{G_\alpha} \subseteq \mathring{F}_\alpha = \mathring{G}_\alpha$, so $x \in G_\alpha$.

Suppose now that α is not right isolated, and put $U_\beta = \mathring{F}_\beta$. Let $V \in \mathcal{N}(x)$. By assumption, $V \cap G_\alpha \neq \varnothing$, so there exists $y \in V$ such that $f^*(y) \leq \alpha$. By the definition of f^*, there exists $W \in \mathcal{N}(y)$ such that $\bigvee f[W] \leq \beta$, that is, $W \subseteq F_\beta$. Since W is open, $W \subseteq U_\beta$ as well, so $y \in U_\beta$ since $y \in W$. Therefore, every neighborhood of x meets U_β, that is, $x \in \overline{U_\beta}$. However, since Ω is extremally disconnected and U_β is open, $\overline{U_\beta}$ is open as well, thus $\overline{U_\beta} \in \mathcal{N}(x)$. Furthermore, $U_\beta \subseteq F_\beta$, thus $\overline{U_\beta} \subseteq F_\beta$, thus, by Claim 1(iii), $\overline{U_\beta} \subseteq G_\beta$. In particular, $f^*(x) \leq \beta$. This holds for all $\beta > \alpha$, whence $f^*(x) \leq \alpha$, that is, $x \in G_\alpha$.

Therefore, in both cases, G_α is closed. So we have proved the continuity of f^*.

Claim 2. *If f is continuous, then $f = f^*$.*

Proof of Claim. We prove in fact that for any $x \in \Omega$, if f is continuous at x, then $f(x) = f^*(x)$. We have already observed that $f(x) \leq f^*(x)$.

To prove the converse inequality, suppose first that $f(x)$ is right isolated, and let $\gamma > f(x)$ such that $[f(x), \gamma] = \{f(x), \gamma\}$. Since $f(x) < \gamma$ and since f is continuous, there exists $V \in \mathcal{N}(x)$ such that for all $y \in V$, $f(y) < \gamma$, that is, $f(y) \leq f(x)$. Thus $f^*(x) \leq f(x)$, hence $f^*(x) = f(x)$.

Suppose now that $f(x)$ is not right isolated. Since f is continuous at x, for any $\gamma > f(x)$, there exists $V \in \mathcal{N}(x)$ such that $f(y) < \gamma$ holds for all $y \in V$. Hence $f^*(x) \leq \bigvee f[V] \leq \gamma$. This holds for all $\gamma > f(x)$ and $f(x)$ is not right isolated, thus, again, $f^*(x) \leq f(x)$, hence $f^*(x) = f(x)$. $\square$ Claim 2.

If $g \geq f$ is a continuous map from Ω to K, then, by Claim 2, $g = g^* \geq f^*$; the minimality statement follows. $\square$

We introduce a few convenient notations.

Notation 3-3.3. We denote by $\mathbf{On}$ the proper class of all ordinals, and we extend the definitions of $\mathbb{Z}_\gamma$, $\mathbb{R}_\gamma$, and $\mathbf{2}_\gamma$ (see Section 3-1), defining further proper classes $\mathbb{Z}_\infty$, $\mathbb{R}_\infty$, and $\mathbf{2}_\infty$ as follows:

$$\mathbb{Z}_\infty = \mathbb{Z}^+ \cup \{\aleph_\alpha \mid \alpha \in \mathbf{On}\},$$
$$\mathbb{R}_\infty = \mathbb{R}^+ \cup \{\aleph_\alpha \mid \alpha \in \mathbf{On}\},$$
$$\mathbf{2}_\infty = \{0\} \cup \{\aleph_\alpha \mid \alpha \in \mathbf{On}\}.$$

For $\kappa \in \mathbf{2}_\infty$, we define κ^+ as the successor cardinal of κ if κ is an infinite cardinal, and we put $0^+ = \aleph_0$. That is, κ^+ is the immediate successor of κ in $\mathbf{2}_\infty$.

Notation 3-3.4. Let Ω be a set, let U be a subset of Ω, let K be a set with a distinguished zero element 0, let $f \colon \Omega \to K$. We denote by $f\rfloor_U$ the map from Ω to K defined by the rule

$$f\rfloor_U(x) = \begin{cases} f(x) & (\text{if } x \in U), \\ 0 & (\text{otherwise}), \end{cases} \qquad \text{for all } x \in \Omega.$$

Notation 3-3.5. Let Ω be a topological space, written as a disjoint union $\Omega = \bigsqcup_{i<n} \Omega_i$, where $n \in \mathbb{N}$ and $\Omega_0, \dots, \Omega_{n-1}$ are clopen subsets of Ω. Let K_0,

$\dots$, K_{n-1} be topological spaces. We define the set

$$\mathbf{C}(\Omega_0, K_0; \Omega_1, K_1; \dots; \Omega_{n-1}, K_{n-1})$$

as the set of all maps $f \colon \Omega \to \bigcup_{i<n} K_i$ such that $f|_{\Omega_i} \in \mathbf{C}(\Omega_i, K_i)$, for all $i < n$.

Of course, $\mathbf{C}(\Omega_0, K_0; \Omega_1, K_1; \dots; \Omega_{n-1}, K_{n-1})$ is naturally isomorphic to the direct product $\prod_{i<n} \mathbf{C}(\Omega_i, K_i)$. However, we find the present notation more convenient for such statements as Proposition 3-7.9.

THEOREM 3-3.6. *Let Ω be an extremally disconnected topological space, written as a disjoint union $\Omega = \Omega_{\mathrm{I}} \sqcup \Omega_{\mathrm{II}} \sqcup \Omega_{\mathrm{III}}$, for clopen subsets Ω_{I}, Ω_{II}, Ω_{III} of Ω. Let γ be an ordinal. Then the space*

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma; \Omega_{\mathrm{II}}, \mathbb{R}_\gamma; \Omega_{\mathrm{III}}, \mathbf{2}_\gamma)$$

is a continuous dimension scale.

PROOF. Let K be one of the totally ordered sets $\mathbb{Z}_\gamma$, $\mathbb{R}_\gamma$, or $\mathbf{2}_\gamma$, endowed with its interval topology and its natural monoid structure. We prove that $S = \mathbf{C}(\Omega, K)$ is a continuous dimension scale. The proof of the general case of Theorem 3-3.6 follows since the direct product of any family of continuous dimension scales is again a continuous dimension scale, see Lemma 3-1.6.

We first observe that S is a *total* (as opposed to partial) commutative monoid. It is obvious that the algebraic preordering of S is antisymmetric and that S has a largest element. Furthermore, let $(f_i)_{i\in I}$ be a nonempty family of elements of S. Define $f \colon \Omega \to K$ as the componentwise join of all the f_i, for $i \in I$. Then f is lower semicontinuous. By Proposition 3-3.2, there exists a least continuous map $f^* \colon \Omega \to K$ such that $f \leq f^*$. Then f^* is the supremum of $\{f_i \mid i \in I\}$ in S. Hence S satisfies Axiom (M2).

We now observe that $f\rfloor_U$ (see Notation 3-3.4) belongs to S, for any $f \in S$ and any clopen subset U of Ω. If f is the constant function with value α, for $\alpha \in K$, then we shall write $\alpha\rfloor_U$ instead of $f\rfloor_U$, and we shall of course write α instead of $\alpha\rfloor_\Omega$.

We shall use the following notation. For f, $g \in S$, we put

$$\begin{aligned}
[\![f \leq g]\!] &= \{x \in \Omega \mid f(x) \leq g(x)\},\\
[\![f = g]\!] &= \{x \in \Omega \mid f(x) = g(x)\},\\
[\![f < g]\!] &= \{x \in \Omega \mid f(x) < g(x)\}.
\end{aligned}$$

We observe that $[\![f \leq g]\!]$ and $[\![f = g]\!]$ are closed, while $[\![f < g]\!]$ is open.

For any $f \in S$, put $U_f = [\![f < \aleph_0]\!]$. Then U_f is open, thus, since Ω is extremally disconnected, $\overline{U_f}$ is clopen. We put $\widetilde{f} = f\rfloor_{\Omega\setminus\overline{U_f}}$. Observe that $\widetilde{f}$ has only infinite values; in particular, it is purely infinite.

CLAIM 1. *Let $f \in S$. Then the following are equivalent:*

(i) *f is cancellable in S.*
(ii) *f is directly finite in S.*
(iii) *$\widetilde{f} = 0$.*

PROOF OF CLAIM. (i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(iii) Suppose that $\widetilde{f} > 0$. Then the clopen set $V = \Omega \setminus \overline{U_f}$ is nonempty, and $\aleph_0\rfloor_V + f = f$, so f is not directly finite.

(iii)⇒(i) If $\widetilde{f} = 0$, then U_f is dense in Ω. Now let $g, h \in S$ such that $f+g = f+h$. Then $g(x) = h(x)$ for all $x \in U_f$ (because the values of f on U_f are finite), thus, since U_f is dense in Ω, $g = h$. Hence f is cancellable. □ Claim 1.

We thus obtain Axiom (M5).

Claim 2. *Every element of S can be written under the form $g + h$, where g is cancellable and h is purely infinite.*

Proof of Claim. Let $f \in S$. By Claim 1, $g = f\rfloor_{\overline{U_f}}$ is cancellable. We put $h = \widetilde{f}$. Since $f = g + h$ and h is purely infinite, the conclusion of the claim holds. □ Claim 2.

Now a weak form of pseudo-cancellation.

Claim 3. *S satisfies the following statement:*

$$\forall a, b, c,\ a + c \leq b + c \Rightarrow \exists x,\ x + c = c \text{ and } a \leq b + x.$$

Proof of Claim. This follows immediately from Claims 1 and 2, by putting $x = \tilde{c}$. □ Claim 3.

As a consequence, the equality $\widetilde{f} = \dfrac{f}{\infty}$ holds (see Corollary 2-6.2), for any $f \in S$, that is, $\widetilde{f}$ is, indeed, the largest $g \in S$ such that $g \ll f$.

Furthermore, we observe that for f, $g \in S$, the componentwise meet $f \wedge g$ of $\{f, g\}$ belongs to S, and it is the meet of $\{f, g\}$ in S. Obviously, $(f + h) \wedge (g + h) = (f \wedge g) + h$, for all f, g, $h \in S$. By using Proposition 1.23 of [**53**], it follows that S satisfies the refinement property. So, S satisfies Axiom (M1).

Now we characterize the projections of S. For any clopen set U of Ω, the map $f \mapsto f\rfloor_U$ defines obviously a projection of S.

Claim 4. *S has general comparability.*

Proof of Claim. Let f, $g \in S$. Put $V = \overline{[\![f < g]\!]}$. Since Ω is extremally disconnected, V is clopen. It is obvious that $p_V(f) \leq p_V(g)$ and that $p_{\Omega \setminus V}(g) \leq p_{\Omega \setminus V}(f)$. □ Claim 4.

Claim 5. *The projections of S are exactly the p_U, where U is a clopen subset of Ω.*

Proof of Claim. Let p be a projection of S. Put $u = p(\aleph_\gamma)$ and $v = p^\perp(\aleph_\gamma)$. Then $\aleph_\gamma = u + v$ and $u \wedge v = 0$, thus there exists a clopen subset U of Ω such that $u = \aleph_\gamma\rfloor_U$. So $p(f) \leq f\rfloor_U$, for all $f \in S$. Conversely, $f = p(f) + p^\perp(f) \leq p(f) + f\rfloor_{\Omega \setminus U}$, thus $f\rfloor_U \leq p(f)$. So $p(f) = f\rfloor_U$, for all $f \in S$. □ Claim 5.

Claim 6. *S satisfies Axiom* (M4).

Proof of Claim. Let $f, g \in S$, we prove that there exists a largest projection p of S such that $p(f) \leq p(g)$. For U a clopen subset of Ω, $p_U(f) \leq p_U(g)$ if and only if $U \subseteq F$, where we put $F = [\![f \leq g]\!]$. Since Ω is extremally disconnected, $\mathring{F}$ is clopen, hence, by Claim 5, $p_{\mathring{F}}$ is the largest projection p of S such that $p(f) \leq p(g)$. □ Claim 6.

By Claims 5 and 6, it follows that for any $f \in S$, $\mathrm{cc}(f) = p_U$, where we put $U = \overline{[\![0 < f]\!]}$.

It remains to verify (M6).

Claim 7. *For all $f, g \in S|_\infty$, $f \ll_{\mathrm{rem}} g$ if and only if $f \leq g$ and $[\![0 < g]\!] \cap [\![f = g]\!]$ is nowhere dense.*

Proof of Claim. We use the alternate form of $\ll_{\mathrm{rem}}$ given in Lemma 2-5.7(ii), justified by Lemma 2-4.5 and Claim 4.

We put $A = [\![0 < g]\!] \cap [\![f = g]\!]$. Observe that $[\![0 < g]\!] = [\![\aleph_0 \leq g]\!]$ (because $g \in S|_\infty$), hence it is clopen. Hence A is closed.

Suppose first that $f \ll_{\mathrm{rem}} g$. Of course, it follows that $f \leq g$. Towards a contradiction, suppose that A is not nowhere dense. Since Ω is extremally disconnected, $U = \overset{\circ}{A}$ is nonempty and clopen. Put $h = g\rfloor_{\Omega \setminus U}$. Then $f + h = g$, thus, since $f \ll_{\mathrm{rem}} g$, $g = h$, so $g\rfloor_U = 0$, a contradiction since U is nonempty and contained in $[\![0 < g]\!]$.

Conversely, suppose that $f \leq g$ and that A is nowhere dense. Let $h \in S$ such that $f + h = g$. Let $x \in [\![0 < g]\!] \setminus A$. Since $f(x) + h(x) = g(x)$ with $f(x) < g(x)$ and $g(x) \geq \aleph_0$, $h(x) = g(x)$. So g and h agree on $[\![0 < g]\!] \setminus A$, with A nowhere dense and $[\![0 < g]\!]$ clopen, hence g and h agree on $[\![0 < g]\!]$. It is obvious that both g and h are zero on $[\![g = 0]\!]$, so, finally, $g = h$. This proves that $f \ll_{\mathrm{rem}} g$. □ Claim 7.

Towards a proof of (M6), let $f, g \in S|_\infty$ with $f \ll_{\mathrm{rem}} g$, and set $A = [\![0 < g]\!] \cap [\![f = g]\!]$. We define a map $\overline{f} \colon \Omega \to K$ by the rule

$$\overline{f}(x) = \begin{cases} 0 & (\text{if } x \in [\![g = 0]\!]) \\ f(x) & (\text{if } x \in A) \\ f(x)^+ & (\text{if } x \in [\![0 < g]\!] \setminus A), \end{cases} \qquad \text{for any } x \in \Omega.$$

In the display above, $f(x)$ is an element of $\mathbf{2}_\infty$, and $f(x)^+$ denotes the successor of $f(x)$ in $\mathbf{2}_\infty$, see Notation 3-3.3.

At this point, we observe the obvious inequality $f \leq \overline{f} \leq g$.

Claim 8. *The map $\overline{f}$ is upper semicontinuous.*

Proof of Claim. Let $x \in \Omega$, and put $\alpha = f(x)$. Since $\overline{f}|_{[\![g=0]\!]} = 0$ and $[\![g = 0]\!]$ is clopen, $\overline{f}$ is continuous at every point of $[\![g = 0]\!]$. Now suppose that $g(x) > 0$. Suppose first that $x \in A$. So $g(x) = f(x) = \alpha$, thus $V = [\![0 < g]\!] \cap [\![g = \alpha]\!]$ is an open neighborhood of x. Let $y \in V$. If $y \in A$, then $\overline{f}(y) = f(y) = g(y) \leq \alpha$. If $y \notin A$, then $f(y) < g(y) \leq \alpha$, thus $\overline{f}(y) \leq f(y)^+ \leq \alpha$. Therefore, $f\rfloor_V \leq \overline{f}\rfloor_V \leq \alpha$. Since $f(x) = \alpha$, it follows that $\overline{f}$ is continuous at x.

Now suppose that $x \notin A$. Then $V = [\![f < \alpha^+]\!]$ is an open neighborhood of x, and $f(y) \leq \alpha$ and $\overline{f}(y) \leq \alpha^+$ for all $y \in V$. So, if $\beta \in K$ such that $\overline{f}(x) < \beta$, that is, $\alpha^+ < \beta$, then $\overline{f}(y) < \beta$ for any $y \in V$. Hence $\overline{f}$ is upper semicontinuous at x. □ Claim 8.

By Claim 8 and Proposition 3-3.2 (used for the dual partially ordered set of K), for any upper semicontinuous map $k \colon \Omega \to K$, there exists a largest element k_* of S such that $k_* \leq k$, and k_* is given by the formula

$$k_*(x) = \bigvee_{V \in \mathcal{N}(x)} \bigwedge k[V], \quad \text{for all } x \in \Omega.$$

By Claim 8, we can apply this to $k = \overline{f}$, thus obtaining an element $h = \overline{f}_*$ of S. Since the range of $\overline{f}$ is contained in $\mathbf{2}_\gamma$, so is the range of h, whence $h \in S|_\infty$. Furthermore, $f \leq \overline{f} \leq g$ and, since f is continuous, $f = f_*$ (see Proposition 3-3.2), thus

$$f = f_* \leq \overline{f}_* = h \leq \overline{f} \leq g.$$

It follows from the definition of $\overline{f}$ that $\overline{f}(x) > 0$ if and only if $g(x) > 0$, for any $x \in \Omega$. Since $[\![0 < g]\!]$ is clopen, it follows that $[\![0 < g]\!] = [\![0 < h]\!]$, whence $\mathrm{cc}(g) = \mathrm{cc}(h)$.

Claim 9. *The relation $f \ll_{\mathrm{rem}} h$ holds.*

Proof of Claim. We have seen that $f \leq h$. Put $B = [\![0 < h]\!] \cap [\![f = h]\!]$. Towards a contradiction, assume that $\overset{\circ}{B} \neq \varnothing$. Since A is closed and nowhere dense, $U = \overset{\circ}{B} \setminus A$ is a nonempty open subset of $[\![0 < g]\!]$. Pick $x \in U$ such that $\alpha = f(x)$ is minimum. Then $V = U \cap [\![f < \alpha^+]\!]$ is an open neighborhood of x. For any $y \in V$, $f(y) \leq \alpha$ and $y \in U$, thus, by minimality of α, $f(y) = \alpha$. This holds for any $y \in V$, and $V \subseteq [\![0 < g]\!] \setminus A$, thus $\overline{f}(y) = \alpha^+$ for any $y \in V$. Since V is an open neighborhood of x, $h(x) = \alpha^+ > \alpha = f(x)$, which contradicts $x \in B$. By Claim 7, $f \ll_{\mathrm{rem}} h$. □ Claim 9.

To conclude the proof of Theorem 3-3.6, it suffices to prove that if k is any element of $S|_\infty$ such that $f \ll_{\mathrm{rem}} k$ and $k^\perp = g^\perp$, then $h \leq k$. Observe first that from the assumption $k^\perp = g^\perp$, Lemma 2-5.3(i) implies that $\mathrm{cc}(k) = \mathrm{cc}(g)$, and so $[\![0 < g]\!] = [\![0 < k]\!]$.

Since $f \ll_{\mathrm{rem}} k$, $f \leq k$ and $B = [\![0 < k]\!] \cap [\![f = k]\!]$ is nowhere dense. For any $x \in [\![0 < f]\!] \setminus B$, the inequality $f(x) < k(x)$ holds, thus $h(x) \leq \overline{f}(x) \leq f(x)^+ \leq k(x)$. Since B is nowhere dense and $[\![0 < f]\!]$ is open, $h|_{[\![0<f]\!]} \leq k|_{[\![0<f]\!]}$. Now let $x \in [\![f = 0]\!]$. If $g(x) = 0$, then $h(x) = 0 \leq k(x)$. If $g(x) > 0$, then, since $[\![0 < g]\!] = [\![0 < k]\!]$, $k(x) > 0$, so $h(x) = \aleph_0 \leq k(x)$. Therefore, we have proved that $h \leq k$. □

3-4. Completeness of the Boolean algebra of projections

Standing hypothesis: S is a continuous dimension scale.

We first prove the following lemma.

Lemma 3-4.1. *Let a, $b \in S$ and $X \subseteq S$ such that $b = \bigvee X$. If $a \perp X$, then $a \perp b$.*

Proof. The statement $a \perp X$ means that $X \subseteq a^\perp$, that is, by Lemma 2-5.3, $X \subseteq \|a = 0\|\, S$. But by Lemma 2-3.16(ii), the range of any projection of S is closed under suprema. In particular, $b \in \|a = 0\|\, S$. □

We prove here an important structural result about $\operatorname{Proj} S$.

Proposition 3-4.2. *The Boolean algebra $\operatorname{Proj} S$ is complete. If $(p_i)_{i\in I}$ is any family of projections of S, then, for all $x \in S$,*

(i) *If $I \neq \varnothing$, then $\left(\bigwedge_{i\in I} p_i\right)(x) = \bigwedge_{i\in I} p_i(x)$.*
(ii) *$\left(\bigvee_{i\in I} p_i\right)(x) = \bigvee_{i\in I} p_i(x)$.*

Furthermore, if $p = \bigwedge_{i\in I} p_i$, then $pS = \bigcap_{i\in I} p_i S$.

Proof. The cases of (i) and (ii) where I is finite follow from Proposition 2-3.15. Therefore, by replacing in (i) (resp., in (ii)) the family $(p_i)_{i\in I}$ by the family of all nonempty finite meets (resp., finite joins) of the p_i-s, we can assume without loss of generality that I is an upward directed partially ordered set and that $i \leq j$ in I implies $p_i \geq p_j$ (resp., $p_i \leq p_j$).

Let us suppose that

$$i \leq j \text{ implies that } p_i \leq p_j, \quad \text{for all } i,\ j \in I. \tag{3-4.1}$$

We consider only the nontrivial case where $I \neq \varnothing$. The supremum $p(x) = \bigvee_{i\in I} p_i(x)$ is defined and $p(x) \leq x$, for all $x \in S$. Since $I \neq \varnothing$, $q(x) = \bigwedge_{i\in I} p_i^{\perp}(x)$ is also defined.

Claim 1. *The maps p and q are endomorphisms of S.*

Proof of Claim. It is obvious that $p(0) = 0$. Let x, y, $z \in S$ such that $z = x + y$. We compute:

$$\begin{aligned}
p(x) + p(y) &= \bigvee_{i\in I} p_i(x) + \bigvee_{j\in I} p_j(y) \\
&= \bigvee_{(i,j)\in I\times J} (p_i(x) + p_j(y)) \qquad \text{(by Corollary 2-6.7(ii))} \\
&= \bigvee_{k\in I} (p_k(x) + p_k(y)) \qquad \text{(because } I \text{ is upward directed} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and by (3-4.1))} \\
&= \bigvee_{k\in I} p_k(x+y) \\
&= p(x+y).
\end{aligned}$$

A similar argument, based on Corollary 2-6.7(i), proves that q is an endomorphism of S. ☐ Claim 1.

Claim 2. *$pS \perp qS$.*

Proof of Claim. Let x, $y \in S$. For all $i \in I$, $p_i(x) \perp p_i^{\perp}(y)$, thus, since $p_i^{\perp}(y) \geq q(y)$, $p_i(x) \perp q(y)$. This holds for all $i \in I$, thus, by Lemma 3-4.1, $p(x) \perp q(y)$. ☐ Claim 2.

Claim 3. *p and q are projections of S, and $q = p^{\perp}$.*

Proof of Claim. By the definition of a projection (Definition 2-3.1) and by Claim 3, it remains to prove that $x = p(x) + q(x)$ for all $x \in S$.

For all $i \in I$, $x = p_i(x) + p_i^{\perp}(x)$. Since $q \leq p_i^{\perp}$, $p_i(x) + q(x)$ is defined and $p_i(x) + q(x) \leq x$. This holds for all i, thus, by Corollary 2-6.7(ii), $p(x) + q(x)$ is defined and $p(x) + q(x) \leq x$.

For all $i \in I$, $p_i(x) \leq p(x) \leq x$, thus $x \smallsetminus p(x) \leq x \smallsetminus p_i(x) \leq p_i^{\perp}(x)$. By taking the infimum over all i, we obtain that $x \smallsetminus p(x) \leq q(x)$; whence $x \leq p(x) + q(x)$.

Therefore, $x = p(x) + q(x)$. ☐ Claim 3.

It follows easily (use Lemma 2-3.8) that $p = \bigvee_{i\in I} p_i$ and that $q = \bigwedge_{i\in I} p_i^{\perp}$. This proves simultaneously (i) and (ii).

Finally, suppose that $p = \bigwedge_{i\in I} p_i$. We prove that $pS = \bigcap_{i\in I} p_i S$. For $I = \varnothing$, this is trivial ($p = 1$), so suppose that $I \neq \varnothing$. We have seen above that $p(x) =$

$\bigwedge_{i \in I} p_i(x)$, for all $x \in S$. Hence, $x \in pS$ if and only if $p(x) = x$, if and only if $p_i(x) = x$ for all i (because $p_i(x) \leq x$ for all i), if and only if $x \in \bigcap_{i \in I} p_i S$. □

PROPOSITION 3-4.3. *Let X be a subset of S. Then $S = X^{\perp} \oplus X^{\perp\perp}$.*

PROOF. By using Proposition 3-4.2, we define a projection p of S by the formula

$$p = \bigvee_{x \in X} \mathrm{cc}(x).$$

Then $p^{\perp} = \bigwedge_{x \in X} \mathrm{cc}(x)^{\perp} = \bigwedge_{x \in X} \|x = 0\|$, thus, by Proposition 3-4.2 and Lemma 2-5.3(i),

$$p^{\perp} S = \bigcap_{x \in X} \|x = 0\| \, S = \bigcap_{x \in X} x^{\perp} = X^{\perp}.$$

It follows from this that $pS = (p^{\perp} S)^{\perp} = X^{\perp\perp}$. □

3-5. The elements $\langle p \mid \alpha \rangle$

Standing hypothesis: S is a continuous dimension scale.

We shall now define a certain doubly indexed family of elements of $S|_{\infty}$. These elements represent in some sense the "layers" of $S|_{\infty}$, and a process of "measuring" $S|_{\infty}$ against these elements will allow us to pin down the dimension theory of $S|_{\infty}$.

NOTATION 3-5.1. For any $p \in \operatorname{Proj} S$, we define inductively a transfinite sequence of elements $\langle p \mid \kappa \rangle$ of $S|_{\infty}$, for certain elements κ of $\mathbf{2}_{\infty}$, as follows.

(i) $\langle p \mid 0 \rangle = 0$.

(ii) Let $\kappa \in \mathbf{2}_{\infty}$, and suppose that $\langle p \mid \kappa \rangle$ is defined, and that it is purely infinite. We put

$$X = \{x \in S|_{\infty} \mid \langle p \mid \kappa \rangle \ll_{\mathrm{rem}} x \text{ and } \mathrm{cc}(x) = p\}.$$

If X is nonempty, then, by Axiom (M6), it has a least element. We denote this element by $\langle p \mid \kappa^+ \rangle$. If $X = \varnothing$, then we say that $\langle p \mid \kappa^+ \rangle$ is undefined.

(iii) Let λ be a limit cardinal. Suppose that $\langle p \mid \alpha \rangle$ has been defined for all $\alpha < \lambda$ in $\mathbf{2}_{\infty}$, and that these elements form an increasing, majorized sequence of elements of $S|_{\infty}$. Then we put

$$\langle p \mid \lambda \rangle = \bigvee_{\alpha < \lambda} \langle p \mid \alpha \rangle.$$

Otherwise, we say that $\langle p \mid \lambda \rangle$ is undefined.

For any $p \in \operatorname{Proj} S$, we define Λ_p as the class of all $\alpha \in \mathbf{2}_{\infty}$ such that $\langle p \mid \alpha \rangle$ is defined.

Observe that $\langle 0 \mid \kappa \rangle = 0$ for all κ. In particular, it follows that $\Lambda_0 = \mathbf{2}_{\infty}$. The following lemma summarizes some elementary properties of the elements $\langle p \mid \alpha \rangle$.

LEMMA 3-5.2.

(i) *Λ_p is a proper initial segment of $\mathbf{2}_{\infty}$, for all $p \in \operatorname{Proj}^* S$.*

(ii) *$\langle p \mid \alpha \rangle < \langle p \mid \beta \rangle$ for all $p \in \operatorname{Proj}^* S$ and all $\alpha < \beta$ in Λ_p.*

(iii) *$\langle p \mid \alpha \rangle \ll_{\mathrm{rem}} \langle p \mid \beta \rangle$ for all $p \in \operatorname{Proj} S$ and all $\alpha < \beta$ in Λ_p.*

(iv) *$\langle p \mid 0 \rangle = 0$, and $\mathrm{cc}(\langle p \mid \alpha \rangle) = p$ for all $p \in \operatorname{Proj} S$ and all $\alpha \in \Lambda_p \setminus \{0\}$.*

(v) *Let* $p, q \in \operatorname{Proj} S$ *such that* $p \leq q$. *Then* $\Lambda_q \subseteq \Lambda_p$ *and* $\langle p \mid \alpha \rangle = p(\langle q \mid \alpha \rangle)$, *for all* $\alpha \in \Lambda_q$.

(vi) *Let* $p \in \operatorname{Proj} S$ *and let* α *be an infinite cardinal number such that* $\langle p \mid \alpha \rangle$ *is defined. Then* $\langle p \mid \alpha^+ \rangle$ *is defined if and only if there exists* $x \in S|_\infty$ *such that* $\langle p \mid \alpha \rangle \ll_{\mathrm{rem}} x$, *and then,* $\langle p \mid \alpha^+ \rangle$ *is the least such* x.

PROOF. (i) and (ii) are obvious.

(iii) is an easy consequence of Lemma 2-5.6.

(iv) By induction on $\alpha \in \Lambda_p \setminus \{0\}$. The assertion $\langle p \mid 0 \rangle = 0$ holds by definition. If $\alpha = \beta^+$ for some $\beta \in \mathbf{2}_\infty$, then, by the definition of $\langle p \mid \alpha \rangle$, $\mathrm{cc}(\langle p \mid \alpha \rangle) = p$.

The limit step is an easy consequence of Lemma 2-5.16.

(v) We prove the statement by induction on $\alpha \in \mathbf{2}_\infty$. It is trivial for $\alpha = 0$. The limit step is an easy consequence of Lemma 2-3.16(ii).

Now suppose that $\alpha = \beta^+$, for $\beta \in \mathbf{2}_\infty$. Since $\langle q \mid \beta^+ \rangle$ is defined, $\langle q \mid \beta \rangle$ is defined, thus, by the induction hypothesis,

$$\langle p \mid \beta \rangle \text{ is defined, and } \langle p \mid \beta \rangle = p(\langle q \mid \beta \rangle). \tag{3-5.1}$$

We put $e = p(\langle q \mid \alpha \rangle)$. Since $\langle q \mid \beta \rangle \ll_{\mathrm{rem}} \langle q \mid \alpha \rangle$, it follows from Lemma 2-5.8(i) that $p(\langle q \mid \beta \rangle) \ll_{\mathrm{rem}} p(\langle q \mid \alpha \rangle)$, that is, $\langle p \mid \beta \rangle \ll_{\mathrm{rem}} e$. Furthermore, by (iv) above and by Lemma 2-5.16(ii), $\mathrm{cc}(e) = p \wedge q = p$. Hence,

$$\langle p \mid \alpha \rangle \text{ is defined, and } \langle p \mid \alpha \rangle \leq e. \tag{3-5.2}$$

So, $\langle p \mid \alpha \rangle \leq p(\langle q \mid \alpha \rangle)$, thus, since $qp^\perp(\langle q \mid \alpha \rangle) \leq p^\perp(\langle q \mid \alpha \rangle)$, the element $e' = \langle p \mid \alpha \rangle + qp^\perp(\langle q \mid \alpha \rangle)$ is defined, and $e' \leq \langle q \mid \alpha \rangle$. Note, further, that e' is purely infinite. Furthermore, by Lemma 2-5.8(i), the following relations hold:

$$\langle p \mid \beta \rangle \ll_{\mathrm{rem}} \langle p \mid \alpha \rangle\,,$$
$$qp^\perp(\langle q \mid \beta \rangle) \ll_{\mathrm{rem}} qp^\perp(\langle q \mid \alpha \rangle).$$

Hence, by Lemma 2-5.8(ii), we obtain that

$$\langle p \mid \beta \rangle + qp^\perp(\langle q \mid \beta \rangle) \ll_{\mathrm{rem}} e'. \tag{3-5.3}$$

From $\langle q \mid \beta \rangle \in qS$, it follows that $qp^\perp(\langle q \mid \beta \rangle) = p^\perp q(\langle q \mid \beta \rangle) = p^\perp(\langle q \mid \beta \rangle)$. By using (3-5.1), we obtain that (3-5.3) can be written as

$$\langle q \mid \beta \rangle \ll_{\mathrm{rem}} e'. \tag{3-5.4}$$

By Lemma 2-5.16 and by (iv) above, $\mathrm{cc}(e') = q$, hence, (3-5.4) implies that $\langle q \mid \alpha \rangle \leq e'$. Hence, by taking the image under p of each side, we obtain that $e \leq \langle p \mid \alpha \rangle$. Therefore, by (3-5.2), $\langle p \mid \alpha \rangle = e = p(\langle q \mid \alpha \rangle)$.

(vi) We define subsets X and Y of $S|_\infty$ by

$$X = \{x \in S|_\infty \mid \langle p \mid \alpha \rangle \ll_{\mathrm{rem}} x \text{ and } \mathrm{cc}(x) = p\}\,,$$
$$Y = \{x \in S|_\infty \mid \langle p \mid \alpha \rangle \ll_{\mathrm{rem}} x\}\,.$$

So, $\langle p \mid \alpha^+ \rangle$ is defined if and only if X is nonempty, which implies that Y is nonempty.

Conversely, suppose that Y is nonempty. For all $x \in Y$, $\langle p \mid \alpha \rangle \leq x$, thus, since $\alpha > 0$ and by (iv), $p \leq \mathrm{cc}(x)$. Thus, by Lemma 2-5.16(ii), $\mathrm{cc}(p(x)) = p$. Furthermore, $\langle p \mid \alpha \rangle \in pS$, thus, by Lemma 2-5.8(i), $\langle p \mid \alpha \rangle \ll_{\mathrm{rem}} p(x)$, that is, $p(x) \in Y$. Therefore, $pY \subseteq X$. In particular, $X \neq \varnothing$, so $\langle p \mid \alpha^+ \rangle$ is defined. For all $x \in Y$, $p(x) \in X$, thus $\langle p \mid \alpha^+ \rangle \leq p(x)$; whence $\langle p \mid \alpha^+ \rangle \leq x$. Therefore, $\langle p \mid \alpha^+ \rangle$ is also the least element of Y. $\square$

It follows immediately from the definition of the $(p,\alpha)\mapsto\langle p\mid\alpha\rangle$ operation that $\left\langle p \mid \bigvee_{i\in I}\alpha_i\right\rangle=\bigvee_{i\in I}\langle p\mid\alpha_i\rangle$ provided the left hand side is defined. Our next lemma shows that a similar "linearity" with respect to the variable p holds, thus showing a "bilinearity" property of the operation $(p,\alpha)\mapsto\langle p\mid\alpha\rangle$.

Lemma 3-5.3. *Let $\alpha\in\mathbf{2}_\infty$, let $(p_i)_{i\in I}$ be a family of elements of* $\operatorname{Proj} S$. *Put $p=\bigvee_{i\in I}p_i$. We make the following assumptions:*

(i) *$\langle p_i\mid\alpha\rangle$ is defined for all $i\in I$.*
(ii) *$\{\langle p_i\mid\alpha\rangle\mid i\in I\}$ is majorized.*

Then $\langle p\mid\alpha\rangle$ is defined, and $\langle p\mid\alpha\rangle=\bigvee_{i\in I}\langle p_i\mid\alpha\rangle$.

Proof. We argue by induction on α. The supremum $x=\bigvee_{i\in I}\langle p_i\mid\alpha\rangle$ is, by assumption (ii), defined. Furthermore, the result of Lemma 3-5.3 is obvious for $\alpha=0$. Now suppose that $\alpha>0$.

Suppose first that α is a limit cardinal. By Lemma 3-5.2(ii), $\langle p_i\mid\beta\rangle\leq x$ for any cardinal number $\beta<\alpha$. Therefore, by the induction hypothesis, $\langle p\mid\beta\rangle$ is defined and $\langle p\mid\beta\rangle=\bigvee_{i\in I}\langle p_i\mid\beta\rangle\leq x$. This holds for all $\beta<\alpha$, thus, by definition, $\langle p\mid\alpha\rangle$ is defined and $\langle p\mid\alpha\rangle\leq x$, thus, since the converse inequality is obvious, $\langle p\mid\alpha\rangle=x$.

Now we assume that $\alpha=\beta^+$, for some $\beta\in\mathbf{2}_\infty$. Put $y=\frac{x}{\infty}$. For any $i\in I$, since $\langle p_i\mid\alpha\rangle$ is purely infinite, it follows from Corollary 2-6.2(i) that $\langle p_i\mid\alpha\rangle\leq y$. Furthermore, it follows from Lemma 3-5.2(v) that $p_i(\langle p_i\mid\alpha\rangle)=\langle p_i\mid\alpha\rangle$, thus $\langle p_i\mid\alpha\rangle\leq p_i(y)\leq p(y)$. Hence, by Lemma 3-5.2(iv), for all $i\in I$, $p_i=\operatorname{cc}(\langle p_i\mid\alpha\rangle)\leq\operatorname{cc}(p(y))$. Since this holds for all $i\in I$ and since $\operatorname{cc}(p(y))\leq p$, we obtain the equality

$$\operatorname{cc}(p(y))=p. \tag{3-5.5}$$

For all $i\in I$, since the relations $\langle p_i\mid\beta\rangle\ll_{\mathrm{rem}}\langle p_i\mid\alpha\rangle$ and $\langle p_i\mid\alpha\rangle\leq p(y)$ hold, we deduce from Lemma 2-5.6 that the following relation holds:

$$\langle p_i\mid\beta\rangle\ll_{\mathrm{rem}}p(y). \tag{3-5.6}$$

In particular, $\langle p_i\mid\beta\rangle\leq p(y)$, thus, by the induction hypothesis, $\langle p\mid\beta\rangle$ is defined and $\langle p\mid\beta\rangle\leq p(y)$. Furthermore, it follows from (3-5.6) and from Lemma 2-5.8(i) that the relation $p_i(\langle p_i\mid\beta\rangle)\ll_{\mathrm{rem}}p_i(y)$ holds for all $i\in I$. By Lemma 3-5.2(v), $p_i(\langle p_i\mid\beta\rangle)=p_i(\langle p\mid\beta\rangle)=\langle p_i\mid\beta\rangle$, hence $p_i(\langle p\mid\beta\rangle)\ll_{\mathrm{rem}}p_i(y)$. This holds for all $i\in I$, thus, by Lemma 2-5.8(ii), $p(\langle p\mid\beta\rangle)\ll_{\mathrm{rem}}p(y)$. Again by Lemma 3-5.2(v), $p(\langle p\mid\beta\rangle)=\langle p\mid\beta\rangle$, so that $\langle p\mid\beta\rangle\ll_{\mathrm{rem}}p(y)$. Therefore, by (3-5.5), $\langle p\mid\alpha\rangle$ is defined and $\langle p\mid\alpha\rangle\leq p(y)\leq y\leq x$. Since the inequality $x\leq\langle p\mid\alpha\rangle$ is obvious, $x=\langle p\mid\alpha\rangle$. □

3-6. The dimension function μ

Standing hypothesis: S is a continuous dimension scale.

For every $x\in S$, we put

$$U^{(x)}=\left\{p\in\operatorname{Proj}^* S\mid\exists\alpha\in\Lambda_p\text{ such that }p\left(\frac{x}{\infty}\right)=\langle p\mid\alpha\rangle\right\}. \tag{3-6.1}$$

The following result is the main fact about the dimension theory of $S|_\infty$.

Lemma 3-6.1. *The set $U^{(x)}$ is a coinitial lower subset of* $\operatorname{Proj}^* S$*, for all $x\in S$.*

Proof. By replacing x by $\frac{x}{\infty}$, we may assume without loss of generality that x is purely infinite. The fact that $U^{(x)}$ is a lower subset of $\operatorname{Proj}^* S$ is an obvious consequence of Lemma 3-5.2. Now let us prove that $U^{(x)}$ is coinitial in $\operatorname{Proj}^* S$.

So, let $p \in \operatorname{Proj}^* S$. We find $q \in (0, p]$ such that $q \in U^{(x)}$. First, if $p \not\leq \operatorname{cc}(x)$, then there exists $q \in (0, p]$ such that $q(x) = 0$, thus, obviously, $q \in U^{(x)}$.

So we consider now the case where $p \leq \operatorname{cc}(x)$. Since the sequence of all $\langle p \mid \alpha \rangle$, for $\alpha \in \Lambda_p$, is strictly increasing (see Lemma 3-5.2(ii)) and continuous at limits, there exists a largest element α of $\mathbf{2}_\infty$ such that $\langle p \mid \alpha \rangle$ is defined and $\langle p \mid \alpha \rangle \leq x$. Since $0 \ll_{\text{rem}} p(x)$ and $\operatorname{cc}(p(x)) = p$ (use Lemma 2-5.16), the element $\langle p \mid \aleph_0 \rangle$ is defined, and $\langle p \mid \aleph_0 \rangle \leq p(x)$. Hence, α is an infinite cardinal number.

Now we put $q = \| x \leq \langle p \mid \alpha \rangle \|$. By Lemma 2-5.13,

$$q^\perp(\langle p \mid \alpha \rangle) \ll_{\text{rem}} q^\perp(x). \tag{3-6.2}$$

If $p \leq q^\perp$, then we obtain that $q^\perp(\langle p \mid \alpha \rangle) = q^\perp p(\langle p \mid \alpha \rangle) = \langle p \mid \alpha \rangle$, whence, by (3-6.2), $\langle p \mid \alpha \rangle \ll_{\text{rem}} x$. By Lemma 3-5.2(vi), $\langle p \mid \alpha^+ \rangle$ is defined and $\langle p \mid \alpha^+ \rangle \leq x$, which contradicts the definition of α.

Hence, $p \not\leq q^\perp$, that is, $r = p \wedge q$ is nonzero. Furthermore,

$$\begin{aligned} r(x) &\leq r(\langle p \mid \alpha \rangle) && \text{(because } r \leq q\text{)} \\ &= \langle r \mid \alpha \rangle && \text{(because } r \leq p\text{).} \end{aligned}$$

Since, on the other hand, $\langle p \mid \alpha \rangle \leq x$, we obtain that $r(x) = \langle r \mid \alpha \rangle$. Therefore, $r \in U^{(x)}$. □

For the remainder of Section 3-6*, we denote by* Ω *the ultrafilter space of* $\operatorname{Proj} S$. *By Propositions* 3-4.2 *and* 3-3.1*,* Ω *is a complete Boolean space. The clopen sets of* Ω *are exactly the sets of the form*

$$\Omega_p = \{\mathfrak{a} \in \Omega \mid p \in \mathfrak{a}\},$$

for all $p \in \operatorname{Proj} S$. *Moreover, we shall fix an ordinal* γ *such that* $\langle p \mid \aleph_\alpha \rangle$ *defined implies that* $\alpha \leq \gamma$*, for all* $p \in \operatorname{Proj}^* S$. *The existence of such a* γ *is ensured by Lemma* 3-5.2(i).

We now define, for any $x \in S$ and any $\mathfrak{a} \in \Omega$,

$$\mu(x)(\mathfrak{a}) = \bigvee \{\alpha \in \mathbf{2}_\gamma \mid \exists p \in \mathfrak{a} \text{ such that } \langle p \mid \alpha \rangle \text{ is defined and } \langle p \mid \alpha \rangle \leq x\}. \tag{3-6.3}$$

The rather involved construction of the elements $\langle p \mid \alpha \rangle$ will give us more control over the function $\mu(x)$ just defined than one has over (analogues of) the infinite dimension functions on nonsingular injective modules constructed in [**18**, Chapter XIII] and [**17**, Chapter 12].

LEMMA 3-6.2. *The function* $\mu(x)$ *is a continuous map from* Ω *to* $\mathbf{2}_\gamma$*, for any* $x \in S$.

We recall here that $\mathbf{2}_\gamma$ is endowed with its interval topology.

PROOF. For any $\kappa \in \mathbf{2}_\gamma$, we define subsets U_κ and V_κ of Ω by the formulas

$$\begin{aligned} U_\kappa &= \{\mathfrak{a} \in \Omega \mid \mu(x)(\mathfrak{a}) \geq \kappa\}, \\ V_\kappa &= \{\mathfrak{a} \in \Omega \mid \mu(x)(\mathfrak{a}) > \kappa\}. \end{aligned}$$

CLAIM. V_κ *is open, for any* $\kappa \in \mathbf{2}_\gamma$.

Proof of Claim. Let $\mathfrak{a} \in V_\kappa$. By the definition of $\mu(x)$, there exist $\alpha > \kappa$ in $\mathbf{2}_\gamma$ and $p \in \mathfrak{a}$ such that $\langle p \mid \alpha \rangle$ is defined and $\langle p \mid \alpha \rangle \leq x$. Thus, for any $\mathfrak{b} \in \Omega_p$, $\mu(x)(\mathfrak{b}) \geq \alpha > \kappa$, that is, $\mathfrak{b} \in V_\kappa$. □ Claim.

To conclude the proof of Lemma 3-6.2, it suffices to prove that U_κ is closed, for any $\kappa \in \mathbf{2}_\gamma$. This is trivial for $\kappa = 0$. If κ is a limit cardinal, then the equality

$$U_\kappa = \bigcap_{\alpha < \kappa \text{ in } \mathbf{2}_\gamma} U_{\alpha^+}$$

holds, hence it is sufficient to prove that U_{α^+} is closed, for any $\alpha \in \mathbf{2}_\gamma$. Towards this goal, we first observe that $U_{\alpha^+} = V_\alpha$, thus, by the Claim above, U_{α^+} is open. Since Ω is extremally disconnected, the closure $\overline{U_{\alpha^+}}$ of U_{α^+} is clopen, thus it has the form Ω_p, for some $p \in \operatorname{Proj} S$. If $p = 0$ then $\overline{U_{\alpha^+}} = \varnothing$ and we are done, so suppose that $p > 0$. For any $q \in (0, p]$, Ω_q meets U_{α^+}, thus there exists $\mathfrak{a} \in \Omega_q$ such that $\mu(x)(\mathfrak{a}) \geq \alpha^+$. Hence there exists $r \in (0, q] \cap \mathfrak{a}$ such that $\langle r \mid \alpha^+ \rangle$ is defined and $\langle r \mid \alpha^+ \rangle \leq x$. Therefore, the set of all $r \in (0, p]$ such that $\langle r \mid \alpha^+ \rangle$ is defined and $\langle r \mid \alpha^+ \rangle \leq x$ is coinitial in $(0, p]$, which proves, by Lemma 3-5.3, that $\langle p \mid \alpha^+ \rangle$ is defined and $\langle p \mid \alpha^+ \rangle \leq x$. This means that $\Omega_p \subseteq U_{\alpha^+}$. Therefore, $U_{\alpha^+} = \Omega_p$ is clopen. □

For all $x \in S$, we put

$$\Omega^{(x)} = \bigcup \{\Omega_p \mid p \in U^{(x)}\}, \tag{3-6.4}$$

where $U^{(x)}$ has been defined in (3-6.1). It follows from Lemma 3-6.1 that $\Omega^{(x)}$ *is a dense, open subset of* Ω.

Lemma 3-6.3. *Let $x \in S$. For any $\mathfrak{a} \in \Omega^{(x)}$, $\mu(x)(\mathfrak{a})$ is the unique element α of $\mathbf{2}_\gamma$ such that*

$$\exists p \in \mathfrak{a} \text{ with } \alpha \in \Lambda_p \text{ and } p\left(\frac{x}{\infty}\right) = \langle p \mid \alpha \rangle .$$

Proof. Let $p \in U^{(x)}$ such that $\mathfrak{a} \in \Omega_p$. By the definition of $U^{(x)}$, there exists $\alpha \in \Lambda_p$ such that $p\left(\frac{x}{\infty}\right) = \langle p \mid \alpha \rangle$. In particular, $\mu(x)(\mathfrak{a}) \geq \alpha$.

Let $q \in \mathfrak{a}$ and $\beta \in \Lambda_q$ such that $\langle q \mid \beta \rangle \leq x$. Then $r = p \wedge q$ belongs to $\mathfrak{a}$ and $\beta \in \Lambda_r$, so $\langle r \mid \beta \rangle \leq \langle q \mid \beta \rangle \leq x$, thus $\langle r \mid \beta \rangle \leq \frac{x}{\infty}$, from which it follows that $\langle r \mid \beta \rangle = p(\langle r \mid \beta \rangle) \leq p\left(\frac{x}{\infty}\right) = \langle p \mid \alpha \rangle$. Hence $\langle r \mid \beta \rangle = r(\langle r \mid \beta \rangle) \leq r(\langle p \mid \beta \rangle) = \langle r \mid \alpha \rangle$, so $\beta \leq \alpha$. Hence $\mu(x)(\mathfrak{a}) \leq \alpha$, so, finally, $\mu(x)(\mathfrak{a}) = \alpha$. □

Proposition 3-6.4.

(i) *$\mu(x + y) = \mu(x) + \mu(y)$, for all $x, y \in S$ such that $x + y$ is defined.*

(ii) *$\mu(x) \leq \mu(y)$ if and only if $x \leq y$, for all $x, y \in S|_\infty$.*

(iii) *The set $\mu[S|_\infty]$ is a lower subset of $\mathbf{C}(\Omega, \mathbf{2}_\gamma)$.*

In particular, the restriction of μ to $S|_\infty$ is a lower embedding from $S|_\infty$ into $\mathbf{C}(\Omega, \mathbf{2}_\gamma)$.

Proof. (i) Put $\Omega' = \Omega^{(x)} \cap \Omega^{(y)}$. Let $\mathfrak{a} \in \Omega'$, and put $\alpha = \mu(x)(\mathfrak{a})$ and $\beta = \mu(y)(\mathfrak{a})$. By Lemma 3-6.3, there exists $p \in \mathfrak{a}$ such that

$$p\left(\frac{x}{\infty}\right) = \langle p \mid \alpha \rangle \quad \text{and} \quad p\left(\frac{y}{\infty}\right) = \langle p \mid \beta \rangle .$$

Hence,

$$\begin{aligned}
p\left(\frac{x+y}{\infty}\right) &= p\left(\frac{x}{\infty}+\frac{y}{\infty}\right) && \text{(by Lemma 2-6.4)}\\
&= p\left(\frac{x}{\infty}\right)+p\left(\frac{y}{\infty}\right)\\
&= \langle p \mid \alpha\rangle + \langle p \mid \beta\rangle\\
&= \langle p \mid \alpha+\beta\rangle && \text{(by Lemma 3-5.2(iii))}.
\end{aligned}$$

Hence, $p \in U^{(x+y)}$, and $\mu(x+y)(\mathfrak{a}) = \mu(x)(\mathfrak{a}) + \mu(y)(\mathfrak{a})$. Therefore, $\mu(x+y)$ and $\mu(x)+\mu(y)$ agree on an open dense subset of Ω, so, since they are continuous, they are equal.

(ii) By (i), $x \leq y$ implies that $\mu(x) \leq \mu(y)$. Conversely, for any $p \in U^{(x)} \cap U^{(y)}$, there exist α, $\beta \in \mathbf{2}_\gamma$ such that $\langle p \mid \alpha\rangle$ and $\langle p \mid \beta\rangle$ are defined and equal to $p(x)$ and $p(y)$, respectively. Since $p \neq 0$, there exists $\mathfrak{a} \in \Omega$ such that $p \in \mathfrak{a}$. Then, by Lemma 3-6.3, $\mu(x)(\mathfrak{a}) = \alpha$ and $\mu(y)(\mathfrak{a}) = \beta$, so $\mu(x) \leq \mu(y)$ implies, by Lemma 3-5.2(ii), that $\alpha \leq \beta$. Hence, $p(x) \leq p(y)$, that is, $p \leq \|x \leq y\|$. This holds for all p in the coinitial subset $U^{(x)} \cap U^{(y)}$ of $\operatorname{Proj}^* S$ (see Lemma 3-6.1), so $x \leq y$.

(iii) Let $x \in S|_\infty$, and let $f \in \mathbf{C}(\Omega, \mathbf{2}_\gamma)$ such that $f \leq \mu(x)$. We find $y \leq x$ in $S|_\infty$ such that $\mu(y) = f$.

We put $U = \{p \in \operatorname{Proj}^* S \mid f|_{\Omega_p} \text{ is constant}\}$.

Claim 1. *U is a coinitial lower subset of* $\operatorname{Proj}^* S$.

Proof of Claim. It is obvious that U is a lower subset of $\operatorname{Proj}^* S$.

Let $p \in \operatorname{Proj}^* S$, we find $q \in (0,p] \cap U$. Let α be the minimum value of f on Ω_p. Then $\Omega' = \{\mathfrak{a} \in \Omega \mid f(\mathfrak{a}) < \alpha^+\}$ is, by continuity of f, an open subset of Ω, and $\Omega' \cap \Omega_p \neq \varnothing$. Let $q \in (0,p]$ such that $\Omega_q \subseteq \Omega' \cap \Omega_p$. Then $f|_{\Omega_q}$ is constant with value α. □ Claim 1.

Now let $\{p_i \mid i \in I\}$ be a maximal antichain of $U \cap U^{(x)}$. By Claim 1 above, $\{p_i \mid i \in I\}$ is also a maximal antichain of $\operatorname{Proj}^* S$. Let α_i denote the constant value of f on Ω_{p_i}, for all $i \in I$. If $\mathfrak{a} \in \Omega_{p_i}$, then $\alpha_i = f(\mathfrak{a}) \leq \mu(x)(\mathfrak{a})$. Since $p_i \in U^{(x)}$, the equality $p_i(x) = \langle p_i \mid \mu(x)(\mathfrak{a})\rangle$ holds. Hence, $\langle p_i \mid \alpha_i\rangle$ is defined and $\langle p_i \mid \alpha_i\rangle \leq p_i(x)$.

It follows that $\{\langle p_i \mid \alpha_i\rangle \mid i \in I\}$ is a majorized (by x) subset of $S|_\infty$, thus it has a supremum, say, y. Note that $y \leq x$. Furthermore, $\mu(y)(\mathfrak{a}) = \alpha_i = f(\mathfrak{a})$, for all $i \in I$ and all $\mathfrak{a} \in \Omega_{p_i}$. Hence, $\mu(y)$ and f agree on a dense open subset of Ω, so, since they are continuous, $\mu(y) = f$. □

The following trivial property of μ will later prove very useful.

Proposition 3-6.5. *The equality $\mu(p(x)) = \mu(x)\rfloor_{\Omega_p}$ holds, for all $x \in S$ and all $p \in \operatorname{Proj} S$.*

3-7. Projections on the directly finite elements

We start with the following easy but fundamental result.

Lemma 3-7.1. *Let S be a continuous dimension scale, let T be a lower subset of S, viewed as a partial submonoid of S (see Definition 2-1.4). Then the following assertions hold:*

(i) *T is closed under all projections of S, and $p|_T \in \operatorname{Proj} T$ for all $p \in \operatorname{Proj} S$.*

(ii) *Every projection of* T *extends to a projection of* S.
(iii) *If* T *is dense in* S, *then every projection of* T *extends to a unique projection of* S.

Proof. We recall that by Lemma 3-1.9, T is a continuous dimension scale.

Furthermore, $p(x) \leq x$ for any projection p of S and any $x \in S$. Since T is a lower subset of S and $p(x) \leq x$ for any projection p of S and any $x \in S$, (i) holds.

Now we prove (ii). So let $p \in \operatorname{Proj} T$. By the definition of a projection, $T = pT \oplus p^{\perp}T$. Moreover, it follows from Proposition 3-4.3 that $S = (pT)^{\perp} \oplus (pT)^{\perp\perp}$, where orthogonals are computed in S. Therefore, there exists a projection q of S such that $qS = (pT)^{\perp\perp}$ and $q^{\perp}S = (pT)^{\perp}$. From $pT \subseteq qS$ it follows that

$$q(x) = x, \quad \text{for all } x \in pT. \tag{3-7.1}$$

Since $p^{\perp}T$ and pT are orthogonal in T and T is a lower subset of S, $p^{\perp}T$ and pT are orthogonal in S. Hence $p^{\perp}T \subseteq (pT)^{\perp} = q^{\perp}S$, which implies that

$$q(x) = 0, \quad \text{for all } x \in p^{\perp}T. \tag{3-7.2}$$

From (3-7.1) and (3-7.2) it follows that $q|_T = p$.

Now we prove (iii). It suffices to prove that if p, $q \in \operatorname{Proj} S$ and $p(x) \leq q(x)$ for all $x \in T$, then $p \leq q$. Suppose otherwise. Then $pq^{\perp} > 0$, thus there exists a nonzero element a in $pq^{\perp}S$. By the assumption of (iii), we can suppose without loss of generality that $a \in T$. So $p(a) > 0$ while $q(a) = 0$ with $a \in T$, a contradiction. $\square$

The following series of results allows to relate the structure of the lower subset of directly finite elements of a continuous dimension scale to the dimension function μ introduced in Section 3-6. We first introduce a definition.

Definition 3-7.2. Let S be a partial commutative monoid. An element a of S is *multiple-free* (or, in some references, *abelian*), if $2x \leq a$ implies that $x = 0$, for all $x \in S$. We denote by S_{mf} the subset of S consisting of all multiple-free elements.

It is not hard to verify that in any partial refinement monoid, multiple-free elements are cancellable (see Definition 2-4.7). In continuous dimension scales, this is also an immediate consequence of Axiom (M5) and Lemma 2-4.8. Multiple-free elements of a lattice-ordered group are called *singular* in [**5**].

We recall that $\mathbb{Z}_0 = \mathbb{Z}^+ \cup \{\aleph_0\}$ and $\mathbb{R}_0 = \mathbb{R}^+ \cup \{\aleph_0\}$, see Section 3-1.

Notation 3-7.3. Let Ω be a topological space, and let K be either $\mathbb{Z}_0$ or $\mathbb{R}_0$, endowed with the interval topology. We denote by $\mathbf{C}_{\mathrm{fin}}(\Omega, K)$ the set of all continuous maps $f\colon \Omega \to K$ such that $f^{-1}\{\aleph_0\}$ is nowhere dense. We extend Notation 3-3.5 to $\mathbf{C}_{\mathrm{fin}}$, thus defining, for topological spaces Ω_{I} and Ω_{II},

$$\mathbf{C}_{\mathrm{fin}}(\Omega_{\mathrm{I}}, \mathbb{Z}_0; \Omega_{\mathrm{II}}, \mathbb{R}_0) = \{f \in \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_0; \Omega_{\mathrm{II}}, \mathbb{R}_0) \mid f^{-1}\{\aleph_0\} \text{ is nowhere dense}\}.$$

Proposition 3-7.4. *Let* S *be a stably finite continuous dimension scale* (*see Definition* 2-4.7). *We suppose that* S_{mf} *is dense in* S. *Let* Ω *be the the ultrafilter space of* $\operatorname{Proj} S$. *Then there exists a map* $\delta\colon S \to \mathbf{C}_{\mathrm{fin}}(\Omega, \mathbb{Z}_0)$ *that satisfies the following conditions:*

(i) δ *is a lower embedding* (*see Definition* 2-1.6).
(ii) $\delta(p(x)) = \delta(x)\rfloor_{\Omega_p}$, *for all* $x \in S$ *and all* $p \in \operatorname{Proj} S$.

Outline of proof. Observe that S is cancellative (see Lemma 2-4.8). By Lemma 3-2.2, $\widetilde{S}$ is the positive cone of a Dedekind complete lattice-ordered group, say, G. By Proposition 3-2.1, G^+ is a continuous dimension scale, and by Lemma 3-7.1(i, iii), the restriction map $\operatorname{Proj} G^+ \to \operatorname{Proj} S$, $p \mapsto p|_S$ is an isomorphism. Hence it suffices to prove that the conclusion of Proposition 3-7.4 holds in case $S = G^+$, that is, S is a *total* monoid.

By Théorème 13.5.6 of [**5**], there exist a complete Boolean space Ω' and a lower embedding δ of G^+ into $\mathbf{C}_{\text{fin}}(\Omega', \mathbb{Z}_0)$. This map δ is defined as an "evaluation map" on the Stone space Ω', which implies that the condition (ii) above is satisfied (see p. 272 in [**5**] for the definition of δ). Furthermore, $\Omega' = \sigma G$ is the ultrafilter space of the complete Boolean algebra of polar subsets of G, thus, $\Omega' \cong \Omega$ canonically, so we may replace Ω' by Ω. $\square$

In the case where there are no nontrivial multiple-free elements, we get the following.

Proposition 3-7.5. *Let S be a stably finite continuous dimension scale with no nontrivial multiple-free element. Then there exists a map $\delta\colon S \to \mathbf{C}_{\text{fin}}(\Omega, \mathbb{R}_0)$ that satisfies the following conditions:*

(i) *δ is a lower embedding.*
(ii) *$\delta(p(x)) = \delta(x)\rfloor_{\Omega_p}$, for all $x \in S$ and all $p \in \operatorname{Proj} S$.*

Outline of proof. The proof of Proposition 3-7.5 follows the lines of the proof of Proposition 3-7.4, with the following changes. The Dedekind complete lattice-ordered group G has no nontrivial multiple-free element, thus, by Theorem 11.2.13 of [**5**], it is divisible. The rest of the proof is the same as for Proposition 3-7.4, by using Corollaire 13.4.2 of [**5**] instead of Théorème 13.5.6 of [**5**]. $\square$

As experience proves, it is often useful to state explicitly the definition of the embedding δ of Propositions 3-7.4 and 3-7.5. The definition that we present here is equivalent to the one given by S. J. Bernau's embedding theorem for Archimedean lattice-ordered groups, see [**4**], or [**1**, Theorem 2.4]. It is convenient to first define the concept of a *finitary unit* in a continuous dimension scale.

Definition 3-7.6. Let S be a continuous dimension scale. A *finitary unit* of S is a dense antichain E of S_{fin} such that for any $e \in E$, either e is multiple-free or there is no nonzero multiple-free element below e.

Lemma 3-7.7. *Every continuous dimension scale has a finitary unit.*

Proof. Let U denote the set of all elements $x \in S_{\text{fin}} \setminus \{0\}$ that are either multiple-free or without nonzero multiple-free element below. Let $a \in S_{\text{fin}} \setminus \{0\}$. If there is no nonzero multiple-free element below a, then $a \in U$. If there exists a nonzero multiple-free element $e \leq a$, observe that $e \in U$. So U is dense in S_{fin}, and the finitary units of S are exactly the maximal antichains of U. $\square$

Now let G be a Dedekind complete lattice-ordered group. Every polar subset of G is an orthogonal direct summand of G, thus $p \mapsto pG^+ + (-pG^+)$ defines an isomorphism from $\operatorname{Proj} G^+$ onto the Boolean algebra of polar subsets of G. We denote again by Ω the ultrafilter space of $\operatorname{Proj} G^+$. Let E be a finitary unit of the continuous dimension scale G^+ (see Definition 3-7.6). We put

$$\delta(x)(\mathfrak{a}) = \bigvee \left\{m/n \mid (m,n) \in \mathbb{Z}^+ \times \mathbb{N} \text{ and } \|me \leq nx\| \in \mathfrak{a} \text{ for all } e \in E\right\}, \tag{3-7.3}$$

for all $x \in G^+$ and $\mathfrak{a} \in \Omega$. Then $\delta(x)$ is a continuous map from Ω to $\mathbb{R}_0$, for all $x \in \Omega$, and δ is the desired embedding.

Unlike the map μ given in (3-6.3), the map δ is not intrinsic, for it depends on the choice of a finitary unit of G^+. Furthermore, it is not apparent through (3-7.3) that under the assumptions of Proposition 3-7.4, the map δ takes its values in $\mathbf{C}_{\mathrm{fin}}(\Omega, \mathbb{Z}_0)$ (rather than just in $\mathbf{C}_{\mathrm{fin}}(\Omega, \mathbb{R}_0)$). However, it is possible to prove that under those assumptions, since E is a finitary unit of G^+, the following equality holds,

$$\delta(x)(\mathfrak{a}) = \bigvee \left\{n \in \mathbb{Z}^+ \mid \|ne \leq x\| \in \mathfrak{a} \text{ for all } e \in E\right\}, \tag{3-7.4}$$

for all $x \in G^+$ and all $\mathfrak{a} \in \Omega$. Hence the map $\delta(x)$ is integer-valued. Similarly, the proof that the range of δ is, in the context of Proposition 3-7.4, a lower embedding, uses the assumption that E is a finitary unit of G^+.

Definition 3-7.8. For a general continuous dimension scale S, we define ideals S_{I}, S_{II}, and S_{III} of S as follows:

$$S_{\mathrm{I}} = S_{\mathrm{mf}}^{\perp\perp};$$
$$S_{\mathrm{II}} = S_{\mathrm{mf}}^{\perp} \cap S_{\mathrm{fin}}^{\perp\perp};$$
$$S_{\mathrm{III}} = S_{\mathrm{fin}}^{\perp}.$$

We say that S is *Type* I (resp., *Type* II, *Type* III), if $S_{\mathrm{II}} = S_{\mathrm{III}} = \{0\}$ (resp., $S_{\mathrm{I}} = S_{\mathrm{III}} = \{0\}$, $S_{\mathrm{I}} = S_{\mathrm{II}} = \{0\}$).

It follows from Proposition 3-4.3 that the equality

$$S = S_{\mathrm{I}} \oplus S_{\mathrm{II}} \oplus S_{\mathrm{III}}$$

holds (see Notation 2-2.3). Observe, in particular, that $S_{\mathrm{fin}} \subseteq S_{\mathrm{fin}}^{\perp\perp} = S_{\mathrm{I}} \oplus S_{\mathrm{II}}$. We denote by p_{I} (resp., p_{II}, p_{III}) the projection of S on S_{I} (resp., S_{II}, S_{III}). So $p_{\mathrm{I}} \oplus p_{\mathrm{II}} \oplus p_{\mathrm{III}} = 1$ in $\operatorname{Proj} S$, hence $\Omega = \Omega_{\mathrm{I}} \sqcup \Omega_{\mathrm{II}} \sqcup \Omega_{\mathrm{III}}$, where we put

$$\Omega_{\mathrm{I}} = \Omega_{p_{\mathrm{I}}}, \qquad \Omega_{\mathrm{II}} = \Omega_{p_{\mathrm{II}}}, \qquad \Omega_{\mathrm{III}} = \Omega_{p_{\mathrm{III}}}.$$

Observe that Ω_{I}, Ω_{II}, and Ω_{III} are clopen subsets of Ω.

By using Propositions 3-7.4 and 3-7.5, we obtain lower embeddings

$$\delta_{\mathrm{I}} \colon S_{\mathrm{I}} \cap S_{\mathrm{fin}} \hookrightarrow \mathbf{C}_{\mathrm{fin}}(\Omega_{\mathrm{I}}, \mathbb{Z}_0) \quad \text{and} \quad \delta_{\mathrm{II}} \colon S_{\mathrm{II}} \cap S_{\mathrm{fin}} \hookrightarrow \mathbf{C}_{\mathrm{fin}}(\Omega_{\mathrm{II}}, \mathbb{R}_0)$$

such that $\delta_i(p(x)) = \delta_i(x)\rfloor_{\Omega_p}$, for all $i \in \{\mathrm{I}, \mathrm{II}\}$, all $x \in S_i$, and all $p \in \operatorname{Proj} S_i$. Now we identify $\operatorname{Proj}(S_{\mathrm{I}} \oplus S_{\mathrm{II}})$ with $\operatorname{Proj}(S_{\mathrm{fin}})$, *via* Lemma 3-7.1. Hence, by combining δ_{I} and δ_{II}, we obtain the following result.

Proposition 3-7.9. *Let S be a continuous dimension scale. Then there exists a map $\delta \colon S_{\mathrm{fin}} \to \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_0; \Omega_{\mathrm{II}}, \mathbb{R}_0; \Omega_{\mathrm{III}}, \{0\})$ (see Definition 3-7.8) that satisfies the following conditions:*

(i) *δ is a lower embedding.*
(ii) *The values of $\delta(x)$ are finite on an open dense subset of Ω, for every $x \in S_{\mathrm{fin}}$.*
(iii) *$\delta(p(x)) = \delta(x)\rfloor_{\Omega_p}$, for all $x \in S_{\mathrm{fin}}$ and all $p \in \operatorname{Proj} S$.*

The map δ depends on the choice of a finitary unit E of S_{fin}, and it is then given by the formula (3-7.3), for all $x \in S_{\mathrm{fin}}$ and all $\mathfrak{a} \in \Omega_{\mathrm{I}} \cup \Omega_{\mathrm{II}}$.

3-8. Embedding arbitrary continuous dimension scales

Standing hypotheses: S is a continuous dimension scale, S_{I}, S_{II}, S_{III}, Ω_{I}, Ω_{II}, Ω_{III}, p_{I}, p_{II}, p_{III} are as in Section 3-7.

Let γ be the ordinal and $\mu\colon S \to \mathbf{C}(\Omega, \mathbf{2}_\gamma)$ the dimension function defined in Section 3-6.

We put $\overline{S} = \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma; \Omega_{\mathrm{II}}, \mathbb{R}_\gamma; \Omega_{\mathrm{III}}, \mathbf{2}_\gamma)$. We pick a lower embedding $\delta\colon S_{\mathrm{fin}} \hookrightarrow \overline{S}$ as in Proposition 3-7.9.

LEMMA 3-8.1. *Let $(a_i)_{i\in I}$ be a majorized family of pairwise orthogonal directly finite elements of S. Then $a = \oplus_{i\in I} a_i$ is directly finite.*

PROOF. Put $p_i = \mathrm{cc}(a_i)$, for all $i \in I$. Observe that $a_i = p_i(a)$, for all i. Let $x \in S$ such that $a + x = a$. It follows that for any $i \in I$, $a_i + p_i(x) = a_i$, thus, since a_i is directly finite, $p_i(x) = 0$, that is, $x \in p_i^{\perp} S$. Put $p = \bigvee_{i\in I} p_i$. By Lemma 2-5.16, $p = \mathrm{cc}(a)$, thus, by using Lemma 2-5.3 and Proposition 3-4.2,

$$x \in \bigcap_{i\in I} p_i^{\perp} S = p^{\perp} S = a^{\perp}.$$

But $x \leq a$, therefore, $x = 0$. □

LEMMA 3-8.2. *Let $x \in S|_\infty$ and $\mathfrak{a} \in \Omega$. If $\mathrm{cc}(x) \in \mathfrak{a}$, then $\mu(x)(\mathfrak{a}) \geq \aleph_0$.*

PROOF. For all $p \in (0, \mathrm{cc}(x)]$, there exist, by Lemma 3-6.1, an infinite cardinal number α and $q \in (0, p]$ such that $q(x) = \alpha \cdot q$. So, for any $\mathfrak{a} \in \Omega_q$, $\mu(x)(\mathfrak{a}) = \alpha \geq \aleph_0$. Therefore, the set of all $\mathfrak{a} \in \Omega_{\mathrm{cc}(x)}$ such that $\mu(x)(\mathfrak{a}) \geq \aleph_0$ is dense in $\Omega_{\mathrm{cc}(x)}$. Since $\mu(x)$ is continuous, the conclusion of Lemma 3-8.2 follows. □

LEMMA 3-8.3. *Let $a \in S_{\mathrm{fin}}$ and $b \in S|_\infty$. If $\mathrm{cc}(a) \leq \mathrm{cc}(b)$, then $a \leq b$ and $\delta(a) \leq \mu(b)$.*

PROOF. Put $p = \|b \leq a\|$ and $q = \|a \leq b\|$. Then $p(b) \leq p(a)$, with $p(a)$ directly finite and $p(b)$ purely infinite, thus $p(b) = 0$, that is, $p \wedge \mathrm{cc}(b) = 0$. By assumption, $p \wedge \mathrm{cc}(a) = 0$, that is, $p(a) = 0$, thus $p \leq q$. By general comparability, $p \vee q = 1$, so, in fact, $q = 1$. Therefore, $a \leq b$.

Now put $r = \mathrm{cc}(b)$. For any $\mathfrak{a} \in \Omega_r$, it follows from Lemma 3-8.2 that $\delta(a)(\mathfrak{a}) \leq \aleph_0 \leq \mu(b)(\mathfrak{a})$. For any $\mathfrak{a} \in \Omega_{r^\perp}$,

$$\delta(a)(\mathfrak{a}) = \delta(a)\rfloor_{\Omega_{r^\perp}}(\mathfrak{a}) = \delta(r^\perp(a))(\mathfrak{a}) = 0,$$

because $r^\perp(a) = 0$. Therefore, $\delta(a)(\mathfrak{a}) \leq \mu(b)(\mathfrak{a})$, for any $\mathfrak{a} \in \Omega$. □

LEMMA 3-8.4. *Let $f \in \overline{S}$ be directly finite, let $b \in S|_\infty$ such that $f \leq \mu(b)$. Then there exists a directly finite $a \leq b$ in S such that $f = \delta(a)$.*

PROOF. We put $p = \mathrm{cc}(b) \wedge (p_{\mathrm{I}} \oplus p_{\mathrm{II}})$, and we define a subset U of $\mathrm{Proj}^* S$ by

$$U = \{q \in (0, p] \mid \exists x \in S_{\mathrm{fin}},\ \mathrm{cc}(x) = q \text{ and } f\rfloor_{\Omega_q} \leq \delta(x)\}.$$

CLAIM 1. *U is coinitial in $(0, p]$.*

Proof of Claim. By Lemma 3-8.3, every directly finite element of pS lies below b, thus, since b is purely infinite and by Lemma 2-1.9, pS_{fin} is a total monoid.

Let $q \in (0, p]$, we prove that $U \cap (0, q]$ is nonempty. Since $0 < q \leq p_{\mathrm{I}} \oplus p_{\mathrm{II}}$ and $(p_{\mathrm{I}} \oplus p_{\mathrm{II}})S = S_{\mathrm{fin}}^{\perp\perp}$, qS has a directly finite, nonzero element y. Observe that $\delta(y)$ is a nonzero element of $\mathbf{C}_{\mathrm{fin}}(\Omega, \mathbb{R}_0)$, thus, since $\delta(y)$ is continuous, there exists $q' \in (0, q]$ such that $\delta(y)(\mathfrak{a}) > 0$ for all $\mathfrak{a} \in \Omega_{q'}$. Without loss of generality, we may assume that $\mathrm{cc}(y) = q'$.

Suppose that $n\delta(y) \leq f$ for all $n \in \mathbb{Z}^+$. Then $f(\mathfrak{a}) \geq \aleph_0$, for any $\mathfrak{a} \in \Omega_{q'}$, so $f + \aleph_0\rfloor_{\Omega_{q'}} = f$, which contradicts the assumption that f is directly finite. Hence, there exists a largest nonnegative integer n such that $n\delta(y) \leq f$. Since $\delta(y)$ vanishes outside $\Omega_{q'}$, there exists $r \in (0, q']$ such that $f\rfloor_{\Omega_r} \leq (n+1)\delta(y)$. Therefore, r belongs to U (with witness $x = (n+1)r(y)$). □ Claim 1.

By Claim 1, there exists a maximal antichain W of $[0, p]$ such that $W \subseteq U$. For each $q \in W$, pick a directly finite $x_q \in S$ such that $\mathrm{cc}(x_q) = q$ and $f\rfloor_{\Omega_q} \leq \delta(x_q)$. Observe that $x_q \leq b$ for all q, thus $x = \oplus_{q \in W} x_q$ is defined and $x \leq b$. Furthermore, by Lemma 3-8.1, x is directly finite. By Lemma 2-3.16(ii), $q(x) = x_q$ for all $q \in W$. For any $q \in W$ and any $\mathfrak{a} \in \Omega_q$,

$$f(\mathfrak{a}) = f\rfloor_{\Omega_q}(\mathfrak{a}) \leq \delta(x_q)(\mathfrak{a}) = \delta(q(x))(\mathfrak{a}) = \delta(x)\rfloor_{\Omega_q}(\mathfrak{a}) = \delta(x)(\mathfrak{a}).$$

Since $\bigcup_{q \in W} \Omega_q$ is dense in Ω_p and both f and $\delta(x)$ are continuous and vanish outside Ω_p, it follows that $f \leq \delta(x)$. Since δ is a lower embedding, there exists $a \leq x$ in S_{fin} such that $f = \delta(a)$. □

We now wish to define a homomorphism of partial monoids $\varepsilon \colon S \to \overline{S}$ by the rule

$$\varepsilon(x + y) = \delta(x) + \mu(y), \qquad \text{for all } x \in S_{\mathrm{fin}} \text{ and all } y \in S|_\infty. \tag{3-8.1}$$

Since $\mu(y)$ is purely infinite, $\delta(x) + \mu(y)$ is the maximum of $\delta(x)$ and $\mu(y)$. It follows that the existence of ε is ensured by the following Lemmas 3-8.5 and 3-8.6.

Lemma 3-8.5. *Let a, $b \in S_{\mathrm{fin}}$ and $c \in S|_\infty$. If $a \leq b + c$, then $\delta(a) \leq \delta(b) + \mu(c)$.*

Proof. By the refinement property, $a = b' + c'$ for some $b' \leq b$ and $c' \leq c$. In particular, b' and c' are directly finite, and, of course, $\mathrm{cc}(c') \leq \mathrm{cc}(c)$. By Lemma 3-8.3, $\delta(c') \leq \mu(c)$. It follows that $\delta(a) = \delta(b') + \delta(c') \leq \delta(b) + \mu(c)$. □

Lemma 3-8.6. *Let a, $c \in S|_\infty$ and $b \in S_{\mathrm{fin}}$. If $a \leq b + c$, then $\mu(a) \leq \delta(b) + \mu(c)$.*

Proof. Since b is directly finite and c is purely infinite, $\dfrac{b}{\infty} = 0$ and $\dfrac{c}{\infty} = c$. By Lemma 2-6.4, $a = \dfrac{a}{\infty} \leq \dfrac{b}{\infty} + \dfrac{c}{\infty} = c$. Therefore, $\mu(a) \leq \mu(c) \leq \delta(b) + \mu(c)$. □

This shows the existence of a unique homomorphism of partial monoids $\varepsilon \colon S \to \overline{S}$ satisfying the condition (3-8.1). Observe that the following additional condition is satisfied by ε (because it is satisfied by μ and by δ, see Propositions 3-6.5 and 3-7.9):

$$\varepsilon(p(x)) = \varepsilon(x)\rfloor_{\Omega_p}, \qquad \text{for all } x \in S \text{ and all } p \in \operatorname{Proj} S. \tag{3-8.2}$$

The purpose of the following Lemmas 3-8.7 and 3-8.8 is to prove that ε is a lower embedding.

Lemma 3-8.7. *The map ε is an order-embedding.*

PROOF. Let a, $b \in S$ such that $\varepsilon(a) \leq \varepsilon(b)$, we prove that $a \leq b$. By Corollary 2-6.2(iii,iv), there exists a directly finite $b' \in S$ such that $b = b' + \frac{b}{\infty}$.

Suppose now that a is directly finite. We put $p = \mathrm{cc}\left(\frac{b}{\infty}\right)$. Then $\mathrm{cc}(p(a)) \leq p = \mathrm{cc}\left(\frac{b}{\infty}\right)$, with $p(a)$ directly finite and $\frac{b}{\infty}$ purely infinite, thus, by Lemma 3-8.3, we obtain that

$$p(a) \leq \frac{b}{\infty}. \tag{3-8.3}$$

Furthermore, $p^{\perp}\left(\frac{b}{\infty}\right) = 0$, hence, by using (3-8.2) and the definition of ε,

$$\delta(p^{\perp}(a)) = \varepsilon(p^{\perp}(a)) \leq \varepsilon(p^{\perp}(b')) + \varepsilon\left(p^{\perp}\left(\frac{b}{\infty}\right)\right) = \delta(p^{\perp}(b')),$$

thus, since δ is an embedding, we obtain that

$$p^{\perp}(a) \leq p^{\perp}(b') \leq b'. \tag{3-8.4}$$

It follows from (3-8.3) and (3-8.4) that $a \leq b$.

In the general case, there exists, by Corollary 2-6.2(iii,iv), a directly finite $a' \in S$ such that $a = a' + \frac{a}{\infty}$. It follows from the previous paragraph that the inequality

$$a' \leq b. \tag{3-8.5}$$

holds. Furthermore, dividing by ∞ the inequality $\delta(a') + \mu\left(\frac{a}{\infty}\right) \leq \delta(b') + \mu\left(\frac{b}{\infty}\right)$ and using Proposition 3-6.4 yields, since both $\delta(a')$ and $\delta(b')$ are finite-valued on a dense subset of Ω and both $\mu\left(\frac{a}{\infty}\right)$ and $\mu\left(\frac{b}{\infty}\right)$ are purely infinite, that

$$\frac{a}{\infty} \leq \frac{b}{\infty}. \tag{3-8.6}$$

By (3-8.5) and (3-8.6), $a = a' + \frac{a}{\infty} \leq b + \frac{b}{\infty} = b$. $\square$

LEMMA 3-8.8. *The range of ε is a lower subset of $\overline{S}$.*

PROOF. Let $b \in S$ and let $f \in \overline{S}$ such that $f \leq \varepsilon(b)$. We find $a \leq b$ in S such that $f = \varepsilon(a)$. By Corollary 2-6.2(iii,iv), there exists a directly finite $b' \in S$ such that $b = b' + \frac{b}{\infty}$.

We start with the case where f is directly finite. Since $f \leq \delta(b') + \mu\left(\frac{b}{\infty}\right)$ and since $\overline{S}$ satisfies the refinement property, there are g, $h \in \overline{S}$ such that $f = g + h$, $g \leq \delta(b')$, and $h \leq \mu\left(\frac{b}{\infty}\right)$. Since δ is a lower embedding, there exists $x \leq b'$ in S_{fin} such that $g = \delta(x)$. Since f is directly finite and $h \leq f$, h is directly finite, thus, by Lemma 3-8.4, there exists a directly finite $y \leq \frac{b}{\infty}$ such that $h = \delta(y)$. Since $x \leq b'$, $y \leq \frac{b}{\infty}$, and $b' + \frac{b}{\infty} = b$, $x + y$ is defined, $x + y \leq b$, and $\delta(x + y) = g + h = f$.

Now the general case. Since $\overline{S}$ is a continuous dimension scale, there exists a directly finite $f' \in \overline{S}$ such that $f = f' + \frac{f}{\infty}$. By the previous paragraph, $f' = \delta(a')$ for some directly finite $a' \leq b$. Furthermore, by dividing by ∞ the inequality $f' + \frac{f}{\infty} \leq \delta(b') + \mu\left(\frac{b}{\infty}\right)$, we obtain that $\frac{f}{\infty} \leq \mu\left(\frac{b}{\infty}\right)$, thus, by Proposition 3-6.4, there exists $\overline{a} \leq \frac{b}{\infty}$ in $S|_{\infty}$ such that $\frac{f}{\infty} = \mu(\overline{a})$. Since $a' \leq b$, $\overline{a} \leq \frac{b}{\infty}$, and $b + \frac{b}{\infty} = b$, $a' + \overline{a}$ is defined, $a' + \overline{a} \leq b$, and $\varepsilon(a' + \overline{a}) = \delta(a') + \mu(\overline{a}) = f' + \frac{f}{\infty} = f$. $\square$

We finally arrive at the following more precise version of Theorem C.

THEOREM 3-8.9. *Let S be a continuous dimension scale, let* $\operatorname{Proj} S$ *be the complete Boolean algebra of projections of S, and let Ω be the ultrafilter space of* $\operatorname{Proj} S$*, with the decomposition $\Omega = \Omega_{\mathrm{I}} \sqcup \Omega_{\mathrm{II}} \sqcup \Omega_{\mathrm{III}}$ as given in Section* 3-7*.*

Then there exist an ordinal γ and a lower embedding

$$\varepsilon\colon S \hookrightarrow \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma; \Omega_{\mathrm{II}}, \mathbb{R}_\gamma; \Omega_{\mathrm{III}}, \mathbf{2}_\gamma)$$

such that $\varepsilon(p(x)) = \varepsilon(x)\rfloor_{\Omega_p}$, for all $x \in S$ and all $p \in \operatorname{Proj} S$.

Conversely, for every ordinal γ, every complete Boolean space Ω, decomposed as $\Omega = \Omega_{\mathrm{I}} \sqcup \Omega_{\mathrm{II}} \sqcup \Omega_{\mathrm{III}}$ with Ω_{I}, Ω_{II}, and Ω_{III} clopen, every lower subset of the space

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma; \Omega_{\mathrm{II}}, \mathbb{R}_\gamma; \Omega_{\mathrm{III}}, \mathbf{2}_\gamma),$$

endowed with its canonical structure of partial commutative monoid, is a continuous dimension scale.

REMARK 3-8.10. Of course, as observed earlier, the following isomorphism

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma; \Omega_{\mathrm{II}}, \mathbb{R}_\gamma; \Omega_{\mathrm{III}}, \mathbf{2}_\gamma) \cong \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma) \tag{3-8.7}$$

holds, so we could have formulated part of Theorem 3-8.9 by using the right hand side of (3-8.7) instead of its left hand side. However, the formulation of the relation $\varepsilon(p(x)) = \varepsilon(x)\rfloor_{\Omega_p}$ would have then been more cumbersome.

PROOF. The first statement (existence of ε) follows from the construction of ε discussed in all previous results of Section 3-8.

Conversely, by Theorem 3-3.6, every monoid of the form

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma; \Omega_{\mathrm{II}}, \mathbb{R}_\gamma; \Omega_{\mathrm{III}}, \mathbf{2}_\gamma)$$

is a continuous dimension scale, and, by Lemma 3-1.9, every lower subset of a continuous dimension scale, viewed as a partial submonoid, is a continuous dimension scale. Theorem 3-8.9 follows. ☐

Any continuous dimension scale is a *partial* monoid, which sometimes makes computations cumbersome. However, the following corollary makes it possible to reduce most problems about continuous dimension scales to *total* monoids.

COROLLARY 3-8.11. *Let S be a continuous dimension scale. Then the universal monoid $\widetilde{S}$ of S is a continuous dimension scale, and $p \mapsto p|_S$ defines an isomorphism from* $\operatorname{Proj} \widetilde{S}$ *onto* $\operatorname{Proj} S$*.*

PROOF. By universality, the map ε extends to a unique monoid homomorphism $\tilde{\varepsilon}$ from $\widetilde{S}$ to $\overline{S}$. Since S is a lower subset of $\widetilde{S}$ and ε is one-to-one, it follows from [**56**, Lemma 3.9] that $\tilde{\varepsilon}$ is one-to-one. Since $\varepsilon[S]$ is a lower subset of the refinement monoid $\overline{S}$, the monoid $\tilde{\varepsilon}[\widetilde{S}]$, which is equal to the submonoid of $\overline{S}$ generated by $\varepsilon[S]$, is also a lower subset of $\overline{S}$. In particular, $\widetilde{S}$ is isomorphic to a lower subset of $\overline{S}$. The conclusion follows from Theorem 3-3.6 and Lemma 3-7.1. ☐

For the reader's convenience, we restate explicitly the construction of the map $\varepsilon\colon S \hookrightarrow \overline{S}$ of Theorem 3-8.9, with $\overline{S} = \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma; \Omega_{\mathrm{II}}, \mathbb{R}_\gamma; \Omega_{\mathrm{III}}, \mathbf{2}_\gamma)$, for some large enough ordinal γ. We first pick a finitary unit E of S (see Definition 3-7.6). Then we embed S into the universal monoid $\widetilde{S}$ of S, see Proposition 2-1.13. We observe that S is a lower subset of $\widetilde{S}$ (Proposition 2-1.13), and that $p \mapsto p|_S$ defines an isomorphism from $\operatorname{Proj}\widetilde{S}$ onto $\operatorname{Proj} S$ (Lemma 3-7.1). The definition of the map $\delta\colon \widetilde{S}_{\mathrm{fin}} \to \overline{S}$ has the parameter E, and it is given by the formula (3-7.3), that is,

$$\delta(x)(\mathfrak{a}) = \bigvee \left\{m/n \mid (m,n) \in \mathbb{Z}^{+} \times \mathbb{N} \text{ and } \|me \leq nx\| \in \mathfrak{a} \text{ for all } e \in E\right\},$$

for all $x \in \widetilde{S}_{\mathrm{fin}}$ and all $\mathfrak{a} \in \Omega_{\mathrm{I}} \cup \Omega_{\mathrm{II}}$.

The definition of μ is intrinsic (it does not depend on the finitary unit E), but it requires the ordinal γ to be chosen large enough. It is given by the formula (3-6.3), that is,

$$\mu(x)(\mathfrak{a}) = \bigvee\{\alpha \in \mathbf{2}_{\gamma} \mid \exists p \in \mathfrak{a} \text{ such that } \langle p \mid \alpha\rangle \text{ is defined and } \langle p \mid \alpha\rangle \leq x\}.$$

for all $x \in S$ and all $\mathfrak{a} \in \Omega$.

Finally, $\varepsilon(x+y) = \delta(x) + \mu(y)$, for all $(x,y) \in S_{\mathrm{fin}} \times S|_{\infty}$. We will call this map ε the *canonical embedding from S into $\overline{S}$, relatively to the finitary unit E.*

3-9. Uniqueness of the canonical embedding

Standing hypotheses: S is a continuous dimension scale, S_{I}, S_{II}, S_{III}, Ω_{I}, Ω_{II}, Ω_{III}, p_{I}, p_{II}, p_{III} are as in Section 3-7.

Let γ be an ordinal, large enough for the dimension function $\mu\colon S \to \mathbf{C}(\Omega, \mathbf{2}_{\gamma})$ introduced in Section 3-6 *to be defined.*

We put $\overline{S} = \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_{\gamma}; \Omega_{\mathrm{II}}, \mathbb{R}_{\gamma}; \Omega_{\mathrm{III}}, \mathbf{2}_{\gamma})$. We fix a finitary unit E of S, and we let $\delta\colon S_{\mathrm{fin}} \hookrightarrow \overline{S}$ and $\varepsilon\colon S \hookrightarrow \overline{S}$ be the canonical maps defined from E in Sections 3-7 *and* 3-8.

Throughout the present section until Theorem 3-8.9, we let $\varepsilon'\colon S \hookrightarrow \overline{S}$ be a lower embedding satisfying the conditions

$$\varepsilon'(p(x)) = \varepsilon'(x)\rfloor_{\Omega_p}, \qquad \text{for all } (x,p) \in S \times \operatorname{Proj} S, \tag{3-9.1}$$

$$\varepsilon'(e) = \varepsilon(e), \qquad \text{for all } e \in E \cap S_{\mathrm{II}}. \tag{3-9.2}$$

We shall prove that $\varepsilon' = \varepsilon$.

We first embed S into its universal monoid $\widetilde{S}$. By Corollary 3-8.11, $\widetilde{S}$ is a continuous dimension scale. Furthermore, the argument of the proof of Corollary 3-8.11 shows that the unique extension of ε' to a map from $\widetilde{S}$ to $\overline{S}$ is a lower embedding. Since $p \mapsto p|_S$ defines an isomorphism from $\operatorname{Proj}\widetilde{S}$ onto $\operatorname{Proj} S$ (Lemma 3-7.1), $\widetilde{\varepsilon'}$ satisfies the condition (3-9.1). Of course, it obviously satisfies (3-9.2). *Therefore, we may assume, without loss of generality, that S is a* total *monoid, that is, $S = \widetilde{S}$.*

Next, it follows from Theorem 3-3.6 that $\overline{S}$ is a continuous dimension scale. Furthermore, by Claim 5 of the proof of Theorem 3-3.6, the projections of $\overline{S}$ are exactly the maps $f \mapsto f|_{\Omega_p}$, for $p \in \operatorname{Proj} S$, in particular, $\operatorname{Proj} S \cong \operatorname{Proj} \overline{S}$. Thus we shall identify every projection p of S with the associated projection of $\overline{S}$. Modulo this identification, the central cover $\mathrm{cc}(f)$ of any $f \in \overline{S}$ is exactly the topological closure of the set $[\![0 < f]\!] = \{\mathfrak{a} \in \Omega \mid f(\mathfrak{a}) > 0\}$.

Lemma 3-9.1. *The equality $\mathrm{cc}(\varepsilon'(x)) = \mathrm{cc}(x)$ holds, for all $x \in S$.*

Proof. Put $p = \mathrm{cc}(x)$. From $x = p(x)$ and (3-9.1) it follows that $\varepsilon'(x) = \varepsilon'(x)\rfloor_{\Omega_p}$, whence $\mathrm{cc}(\varepsilon'(x)) \leq p$. Put $q = p \wedge \neg\,\mathrm{cc}(\varepsilon'(x))$. Then $\varepsilon'(q(x)) = \varepsilon'(x)\rfloor_{\Omega_q} = 0$, thus, since ε' is an embedding, $q(x) = 0$. However, $q \leq p = \mathrm{cc}(x)$, whence $q = 0$, that is, $p = \mathrm{cc}(\varepsilon'(x))$. $\square$

3-9.1. Uniqueness on the directly finite elements. We compute the values of ε on the elements of E. Let $\chi(p)$ denote the characteristic function of Ω_p, for any $p \in \operatorname{Proj} S$.

LEMMA 3-9.2. *The equality $\varepsilon(e) = \chi(\mathrm{cc}(e))$ holds, for all $e \in E$.*

PROOF. Since e is directly finite, $\varepsilon(e) = \delta(e)$, and it is given by (3-7.3). Put $p = \mathrm{cc}(e)$. From $p(e) = e$ and (3-9.1) it follows that $\varepsilon'(e)$ vanishes outside Ω_p. Now let $\mathfrak{a} \in \Omega_p$. For $e' \in E$ and $(m,n) \in \mathbb{Z}^+ \times \mathbb{N}$, the relation $\|me' \leq ne\| \in \mathfrak{a}$ always holds for $e' \neq e$ (because then, $p(e') = 0$, thus $\|me' = 0\| \in \mathfrak{a}$), while for $e' = e$, it is equivalent to the existence of $q \in \mathfrak{a}$ such that $mq(e) \leq nq(e)$. However, for any $q \in \mathfrak{a}$, $q \wedge \mathrm{cc}(e)$ is nonzero, thus $q(e)$ is nonzero, but it is directly finite, thus $mq(e) \leq nq(e)$ if and only if $m \leq n$. The conclusion follows immediately. $\square$

LEMMA 3-9.3. *The equality $\varepsilon'(e) = \varepsilon(e)$ holds, for all $e \in E$.*

PROOF. The conclusion holds by assumption for $e \in E \cap S_{\mathrm{II}}$. Now let $e \in E \cap S_{\mathrm{I}}$, so e is multiple-free and $\Omega_{\mathrm{cc}(e)} \subseteq \Omega_{\mathrm{I}}$. Moreover, it follows from Lemma 3-9.1 that $\mathrm{cc}(\varepsilon'(e)) = \mathrm{cc}(e)$, therefore, since $\varepsilon'(e) \in \overline{S}$ and $\Omega_{\mathrm{cc}(e)} \subseteq \Omega_{\mathrm{I}}$, we obtain the inequality

$$\varepsilon'(e) \geq \chi(\mathrm{cc}(e)). \tag{3-9.3}$$

Let $p \in [0, \mathrm{cc}(e)]$ such that $2\chi(p) \leq \varepsilon'(e)$. Since ε' is a lower embedding, there exists $x \leq e$ such that $\varepsilon'(x) = \chi(p)$, thus (we recall that S is a total monoid) $\varepsilon'(2x) = 2\chi(p) \leq \varepsilon'(e)$, thus, since ε' is an embedding, $2x \leq e$, thus, since e is multiple-free, $x = 0$, whence $p = 0$. This holds for all $p \in [0, \mathrm{cc}(e)]$ such that $2\chi(p) \leq \varepsilon'(e)$, thus, since $\varepsilon'(e)$ vanishes outside $\Omega_{\mathrm{cc}(e)}$, we get $\varepsilon'(e) \leq \chi(\mathrm{cc}(e))$. Therefore, by (3-9.3) and Lemma 3-9.2, $\varepsilon'(e) = \chi(\mathrm{cc}(e)) = \varepsilon(e)$. $\square$

LEMMA 3-9.4. *The equality $\varepsilon'(x) = \delta(x)$ holds, for all $x \in S_{\mathrm{fin}}$.*

PROOF. If the result has been established for S_{fin}, then it obviously holds for S. Hence we may assume that $S = S_{\mathrm{fin}}$, that is, since S is total, S is the positive cone of some Dedekind complete lattice-ordered group (Lemma 3-2.2). Now put $\Omega' = \bigcup_{e \in E} \Omega_{\mathrm{cc}(e)}$, an open subset of Ω. Since every element of S meets some element of E, it follows from Proposition 3-4.3 that Ω' is dense in Ω.

Now let $x \in S$. Since $\delta(x)$ belongs to $\mathbf{C}_{\mathrm{fin}}(\Omega_{\mathrm{I}}, \mathbb{Z}_0; \Omega_{\mathrm{II}}, \mathbb{R}_0)$ (see Notation 3-7.3), there exists an open dense subset Ω'' of Ω' such that $\delta(x)(\mathfrak{a})$ is finite, for all $\mathfrak{a} \in \Omega''$. Since both maps $\varepsilon'(x)$ and $\delta(x)$ are continuous and Ω'' is dense, in order to conclude the proof, it suffices to establish the equality $\varepsilon'(x)(\mathfrak{a}) = \delta(x)(\mathfrak{a})$, for all $\mathfrak{a} \in \Omega''$. Since $\mathfrak{a} \in \Omega'$, there exists a unique $e \in E$ such that $e \in \mathfrak{a}$, hence

$$\delta(x)(\mathfrak{a}) = \bigvee \{m/n \mid (m,n) \in \mathbb{Z}^+ \times \mathbb{N} \text{ and } \|me \leq nx\| \in \mathfrak{a}\}.$$

Let $(m,n) \in \mathbb{Z}^+ \times \mathbb{N}$ such that $p = \|me \leq nx\|$ belongs to $\mathfrak{a}$. Applying ε' to the inequality $mp(e) \leq np(x)$ and using (3-9.1), we obtain the inequalities

$$m\varepsilon'(e)\rfloor_{\Omega_p} = m\varepsilon'(p(e)) \leq n\varepsilon'(p(x)) \leq n\varepsilon'(x),$$

thus, by Lemmas 3-9.2 and 3-9.3, $m\chi(p \wedge \mathrm{cc}(e)) \leq n\varepsilon'(x)$. Evaluate at $\mathfrak{a}$. Since $p \wedge \mathrm{cc}(e)$ belongs to $\mathfrak{a}$, we obtain that $m/n \leq \varepsilon'(x)(\mathfrak{a})$. This holds for all (m,n) such that $\|me \leq nx\| \in \mathfrak{a}$, whence

$$\delta(x)(\mathfrak{a}) \leq \varepsilon'(x)(\mathfrak{a}). \tag{3-9.4}$$

Now the converse. From $\mathfrak{a} \in \Omega''$ it follows that there exists $h \in \mathbb{N}$ such that $\|he \leq x\| \notin \mathfrak{a}$. Let $n \in \mathbb{N}$. There exists a largest $m \in \mathbb{Z}^+$ such that $\|me \leq nx\| \in \mathfrak{a}$, in fact $m < hn$. Suppose that $(m+1)/n < \delta(x)(\mathfrak{a})$. There exists $(m', n') \in \mathbb{Z}^+ \times \mathbb{N}$ such that $(m+1)/n \leq m'/n'$ and $\|m'e \leq n'x\| \in \mathfrak{a}$. On the other hand, from $(m+1)n'e \leq m'ne$ it follows that

$$\|(m+1)e \leq nx\| = \|(m+1)n'e \leq nn'x\| \geq \|m'ne \leq n'nx\| = \|m'e \leq n'x\| \in \mathfrak{a},$$

whence $\|(m+1)e \leq nx\| \in \mathfrak{a}$, a contradiction; so we have proved that $\delta(x)(\mathfrak{a}) \leq (m+1)/n$.

From $\|(m+1)e \leq nx\| \notin \mathfrak{a}$ and general comparability it follows that the projection $q = \|nx \leq (m+1)e\|$ belongs to $\mathfrak{a}$. Thus, applying ε' to the inequality $nq(x) \leq (m+1)q(e)$ and using (3-9.1) together with Lemmas 3-9.2 and 3-9.3, we obtain the inequalities

$$n\varepsilon'(x)\rfloor_{\Omega_q} = n\varepsilon'(q(x)) \leq (m+1)\varepsilon'(q(e)) \leq (m+1)\varepsilon'(e) = (m+1)\chi(\mathrm{cc}(e)),$$

whence, evaluating at $\mathfrak{a}$, $n\varepsilon'(x)(\mathfrak{a}) \leq m+1$, so

$$\varepsilon'(x)(\mathfrak{a}) \leq (m+1)/n \leq \delta(x)(\mathfrak{a}) + 1/n.$$

This holds for all $n \in \mathbb{N}$, thus $\varepsilon'(x)(\mathfrak{a}) \leq \delta(x)(\mathfrak{a})$. By (3-9.4), the conclusion follows. $\square$

3-9.2. Uniqueness on the purely infinite elements. We need to prove that $\varepsilon'(x) = \mu(x)$, for all $x \in S|_\infty$. We recall that we have identified the projections of S and those of $\overline{S}$. For $\alpha \in \mathbf{2}_\infty$ and $p \in \operatorname{Proj} S$, we shall denote by $\langle p \mid \alpha\rangle_S$ (resp., $\langle p \mid \alpha\rangle_{\overline{S}}$) the value of $\langle p \mid \alpha\rangle$ in S (resp., in $\overline{S}$), if defined. By $\alpha\cdot\chi(p)$, we denote the function defined on Ω sending any element of Ω_p to α and any element of $\Omega \setminus \Omega_p$ to 0.

Lemma 3-9.5. *The value $\langle p \mid \alpha\rangle_{\overline{S}}$ is defined and equal to $\alpha\cdot\chi(p)$, for all $\alpha \in \mathbf{2}_\gamma$ and $p \in \operatorname{Proj} S$.*

Proof. By induction on α. The case $\alpha = 0$ and the limit step are obvious. Suppose that $\alpha = \beta^+$, for some $\beta \in \mathbf{2}_\gamma$. It is easy to verify that $\beta\cdot\chi(p) \ll_{\mathrm{rem}} \alpha\cdot\chi(p)$ and $\mathrm{cc}(\alpha\cdot\chi(p)) = p$; thus, by the induction hypothesis, $\langle p \mid \alpha\rangle_{\overline{S}}$ is defined and lies above $\alpha\cdot\chi(p)$. Now suppose that $\langle p \mid \alpha\rangle_{\overline{S}} < \alpha\cdot\chi(p)$. There exists $q \in (0, p]$ such that

$$q(\langle p \mid \alpha\rangle_{\overline{S}}) \leq \beta\cdot\chi(q). \tag{3-9.5}$$

In particular, from $\mathrm{cc}(\langle p \mid \alpha\rangle_{\overline{S}}) = p$ it follows that $\beta > 0$. However, by applying q to the relation $\langle p \mid \beta\rangle_{\overline{S}} \ll_{\mathrm{rem}} \langle p \mid \alpha\rangle_{\overline{S}}$ and using Lemma 2-5.8(i), we obtain that $\beta\cdot\chi(q) \ll_{\mathrm{rem}} q(\langle p \mid \alpha\rangle_{\overline{S}})$. Hence, by (3-9.5) and Lemma 2-5.6, $\beta\cdot\chi(q) \ll_{\mathrm{rem}} \beta\cdot\chi(q)$, whence, since $\beta > 0$, we obtain that $q = 0$, a contradiction. $\square$

Since ε' is a lower embedding, the following lemma is obvious.

Lemma 3-9.6. *$a \ll_{\mathrm{rem}} b$ if and only if $\varepsilon'(a) \ll_{\mathrm{rem}} \varepsilon'(b)$, for all a, $b \in S$.*

Lemma 3-9.7. *The equality $\varepsilon'(\langle p \mid \alpha\rangle_S) = \alpha\cdot\chi(p)$ holds, for all $p \in \operatorname{Proj} S$ and all $\alpha \in \mathbf{2}_\infty$ such that $\langle p \mid \alpha\rangle_S$ is defined.*

Proof. By induction on α. For $\alpha = 0$ it is trivial. Suppose that $\alpha > 0$ and $\langle p \mid \alpha \rangle_S$ is defined. It follows from the induction hypothesis that $\beta \cdot \chi(p) < \varepsilon'(\langle p \mid \alpha \rangle_S)$, for all $\beta < \alpha$ in $\mathbf{2}_\infty$; whence $\alpha \cdot \chi(p) \leq \varepsilon'(\langle p \mid \alpha \rangle_S)$. Since ε' is a lower embedding, there exists $x \leq \langle p \mid \alpha \rangle_S$ in $S|_\infty$ such that $\varepsilon'(x) = \alpha \cdot \chi(p)$. The relation $\varepsilon'(\langle p \mid \beta \rangle_S) \ll_{\mathrm{rem}} \varepsilon'(x)$ holds, for all $\beta < \alpha$, thus, by Lemma 3-9.6, $\langle p \mid \beta \rangle_S \ll_{\mathrm{rem}} x$. Furthermore, by Lemma 3-9.1,

$$\mathrm{cc}(x) = \mathrm{cc}(\varepsilon'(x)) = \mathrm{cc}(\alpha \cdot \chi(p)) = p,$$

hence, by the definition of $\langle p \mid \alpha \rangle$, we get that $\langle p \mid \alpha \rangle_S \leq x$, so, finally, $x = \langle p \mid \alpha \rangle_S$ and $\varepsilon'(\langle p \mid \alpha \rangle_S) = \alpha \cdot \chi(p)$. □

Lemma 3-9.8. *The equality $\varepsilon'(x) = \mu(x)$ holds, for all $x \in S|_\infty$.*

Proof. We let $\Omega^{(x)}$ denote the open dense subset of Ω defined in (3-6.4). It suffices to prove that the equality $\varepsilon'(x)(\mathfrak{a}) = \mu(x)(\mathfrak{a})$ holds, for all $\mathfrak{a} \in \Omega^{(x)}$.

Since x is purely infinite and $\mathfrak{a} \in \Omega^{(x)}$, the value $\alpha = \mu(x)(\mathfrak{a})$ is the unique element of $\mathbf{2}_\infty$ such that $\langle p \mid \alpha \rangle_S$ is defined and equal to $p(x)$, for some $p \in \mathfrak{a}$. Therefore, we can compute:

$$\begin{aligned}
\varepsilon'(x)(\mathfrak{a}) &= \varepsilon'(p(x))(\mathfrak{a}) && \text{(by (3-9.1))} \\
&= \varepsilon'(\langle p \mid \alpha \rangle_S)(\mathfrak{a}) \\
&= (\alpha \cdot \chi(p))(\mathfrak{a}) && \text{(by Lemma 3-9.7)} \\
&= \alpha \\
&= \mu(x)(\mathfrak{a}).
\end{aligned}$$

□

3-9.3. Uniqueness of ε. By putting together Lemmas 3-9.4 and 3-9.8, we obtain the following.

Corollary 3-9.9. *The equality $\varepsilon'(x) = \varepsilon(x)$ holds, for all $x \in S$.*

In order to formulate concisely the corresponding uniqueness result, it is convenient to extend the usual definition of a continuous dimension scale, as follows. We endow each of the proper classes $\mathbb{Z}_\infty$, $\mathbb{R}_\infty$, and $\mathbf{2}_\infty$ introduced in Notation 3-3.3 with its interval topology. The latter consists, for example, of all open sub*sets* of the corresponding class, the essential fact being that for a topological space Ω (we emphasize that Ω is a *set*), the spaces of continuous functions $\mathbf{C}(\Omega, \mathbb{Z}_\infty)$, $\mathbf{C}(\Omega, \mathbb{R}_\infty)$, and $\mathbf{C}(\Omega, \mathbf{2}_\infty)$ are well-understood (anyway, any map from a set to $\mathbb{R}_\infty$ is majorized by some $\aleph_\alpha$). Then we naturally extend Notation 3-3.5 to the case where the K_i-s may also be $\mathbb{Z}_\infty$, $\mathbb{R}_\infty$, or $\mathbf{2}_\infty$.

By putting this together with Lemma 3-9.2 and Theorem 3-8.9, we have obtained the following structure theorem for continuous dimension scales.

Theorem 3-9.10. *Let S be a continuous dimension scale, let $\operatorname{Proj} S$ be the complete Boolean algebra of projections of S, and let Ω be the ultrafilter space of $\operatorname{Proj} S$, with the decomposition $\Omega = \Omega_{\mathrm{I}} \sqcup \Omega_{\mathrm{II}} \sqcup \Omega_{\mathrm{III}}$ as given in Section* 3-7. *Let E be a finitary unit of S* (*see Definition* 3-7.6).

Then there exists a unique lower embedding

$$\varepsilon \colon S \hookrightarrow \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\infty; \Omega_{\mathrm{II}}, \mathbb{R}_\infty; \Omega_{\mathrm{III}}, \mathbf{2}_\infty)$$

such that $\varepsilon(p(x)) = \varepsilon(x)\rfloor_{\Omega_p}$, for all $x \in S$ and all $p \in \operatorname{Proj} S$, and $\varepsilon(e)$ takes its values in $\{0, 1\}$, for all $e \in E \cap S_{\mathrm{II}}$. Furthermore, this embedding satisfies that $\varepsilon(e)$ takes its values in $\{0, 1\}$, for all $e \in E$.

3-10. Continuous dimension scales which are proper classes

All the forthcoming section can easily be formulated in such a standard class theory as the Bernays-Gödel system with choice, BGC. An alternative formulation consists of working in classical set theory ZFC and identifying any statement (with parameters) with one free variable, say, $\varphi(x)$, with the "class" that it represents, namely, $\{x \mid \varphi(x)\}$; this way, the mention to classes becomes a mere expendable commodity.

We shall encounter in Chapter 5 situations where it may appear as artificial to restrict continuous dimension scales to be *sets*, as opposed to *proper classes*. For example, with every right self-injective regular ring R, we associate the category $\mathsf{NSI}\text{-}R$ of all nonsingular injective right R-modules. With the class $\mathsf{NSI}\text{-}R$ is associated a class that meets all attributes of a continuous dimension scale, except that it is not a set (see Section 5-3). We shall call such objects *Continuous Dimension Scales* (with capitals), and we shall define them shortly. We first do this for monoids.

Definition 3-10.1. A *Monoid* is a class M, endowed with an associative binary operation $+$ and a zero element 0. A *Partial Commutative Monoid* is a class S, endowed with a commutative, associative (in the sense of Definition 2-1.2) partial binary operation $+$, with a zero element 0.

Hence the definition of a Monoid (resp., Partial Commutative Monoid) extends the one of a monoid (resp., partial commutative monoid), by allowing proper classes.

The problem in defining Continuous Dimension Scales is not that easy to solve. Indeed, we wish our "Continuous Dimension Scales" to satisfy a version of the main embedding theorem, Theorem 3-8.9. More precisely, we wish every "Continuous Dimension Scale" to embed as a lower subclass into a (proper class) monoid of the form

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\infty; \Omega_{\mathrm{II}}, \mathbb{R}_\infty; \Omega_{\mathrm{III}}, \mathbf{2}_\infty), \tag{3-10.1}$$

(see Notation 3-3.3), for pairwise disjoint complete Boolean spaces Ω_{I}, Ω_{II}, and Ω_{III}. We shall now state the new axioms defining Continuous Dimension Scales. Of course, our definition is modeled on Definition 3-1.1 and Corollary 3-1.3.

Definition 3-10.2. A *Continuous Dimension Scale* is a Partial Commutative Monoid S which satisfies the following axioms.

- (M1) S has refinement, and the algebraic preordering on S is antisymmetric.
- (M2) Every nonempty subset of S admits an infimum.
- (N1) $\forall a, b, \exists c, x, y$ such that $a = c + x$, $b = c + y$, and $x \perp y$.
- (N2) $S = a^\perp + a^{\perp\perp}$, for all $a \in S$ (*where $x \in a^\perp$ means, of course, that $x \wedge a = 0$, while $x \in a^{\perp\perp}$ means that $x \in u^\perp$ for any $u \in a^\perp$*).
- (N3) $b \smallsetminus a$ exists, for all a, $b \in S$ such that $a \leq b$.
- (M5) Every element a of S can be written $a = x + y$, where x is directly finite and y is purely infinite.
- (M6) Let a, b be purely infinite elements of S. If $a \ll_{\mathrm{rem}} b$, then the set of all purely infinite elements x of S such that $a \ll_{\mathrm{rem}} x$ and $x^\perp = b^\perp$ has a least element.
- ($\mathrm{M_{ht}}$) The class $(a] = \{x \in S \mid x \leq a\}$ is a set, for all $a \in S$.
- ($\mathrm{M_{lh}}$) There exists a dense lower *subset* U of S. (*We will call U a* generating lower subset *of S.*)

Axiom $(\mathrm{M_{ht}})$ is there to ensure that the "infinity" in (3-10.1) does not exceed the class of all ordinals. Axiom $(\mathrm{M_{lh}})$ is there to ensure that the "base spaces" Ω_{I}, Ω_{II}, Ω_{III} in (3-10.1) are sets (as opposed to proper classes). We emphasize that we require no condition on *subclasses* of S, lest this might pave the way to undesirable set-theoretical paradoxes. In fact, since the axioms defining Continuous Dimension Scales are requirements on either elements or subsets of S, we obtain the following result.

PROPOSITION 3-10.3. *Let S be a Partial Commutative Monoid. Then S is a Continuous Dimension Scale if and only if every lower subset of S is a continuous dimension scale and S satisfies both* $(\mathrm{M_{ht}})$ *and* $(\mathrm{M_{lh}})$.

PROOF. If S is a Continuous Dimension Scale, then every lower subset of S is a continuous dimension scale: the proof is *mutatis mutandis* the same as for Lemma 3-1.9.

Conversely, suppose that every lower subset of S is a continuous dimension scale and S satisfies both $(\mathrm{M_{ht}})$ and $(\mathrm{M_{lh}})$. Every subset X of S is contained in a generating lower subset $\overline{X}$ of S: namely, take

$$\overline{X} = \bigcup \{(x] \mid x \in X\}.$$

The rest of the proof is similar to the proof of Lemma 3-1.10. $\square$

Observe, in particular, that *every generating lower subset of a continuous dimension scale is a continuous dimension scale.* We also obtain the following extension of Theorem 3-3.6.

COROLLARY 3-10.4. *Let Ω be a complete Boolean space, written as a disjoint union $\Omega = \Omega_{\mathrm{I}} \sqcup \Omega_{\mathrm{II}} \sqcup \Omega_{\mathrm{III}}$, for clopen subsets Ω_{I}, Ω_{II}, Ω_{III} of Ω. Then the Monoid*

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\infty; \Omega_{\mathrm{II}}, \mathbb{R}_\infty; \Omega_{\mathrm{III}}, \mathbf{2}_\infty)$$

is a Continuous Dimension Scale.

Everything is now ready for the proof of our general embedding theorem for Continuous Dimension Scales.

THEOREM 3-10.5. *Let S be a Continuous Dimension Scale, let E be a finitary unit of S. Let U be a generating lower subset of S containing E, let Ω be the ultrafilter space of* $\operatorname{Proj} U$, *with the decomposition $\Omega = \Omega_{\mathrm{I}} \sqcup \Omega_{\mathrm{II}} \sqcup \Omega_{\mathrm{III}}$ as given in Section* 3-7. *Then there exists a unique lower embedding*

$$\varepsilon \colon S \hookrightarrow \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\infty; \Omega_{\mathrm{II}}, \mathbb{R}_\infty; \Omega_{\mathrm{III}}, \mathbf{2}_\infty)$$

such that $\varepsilon(p(x)) = \varepsilon(x)\rfloor_{\Omega_p}$, for all $x \in S$ and all $p \in \operatorname{Proj} S$, and $\varepsilon(e)$ takes its values in $\{0, 1\}$, for all $e \in E$ (or, which is equivalent, for all $e \in E \cap U_{\mathrm{II}}$).

PROOF. Let $\mathcal{C}$ be the class of all lower subsets T of S containing U. In particular, for all $T \in \mathcal{C}$, U is a generating lower subset of T, thus, by Lemma 3-7.1, $p \mapsto p|_U$ defines an isomorphism from $\operatorname{Proj} T$ onto $\operatorname{Proj} U$. Let $p \mapsto p^T$ denote its inverse. Therefore, the ultrafilter space Ω^T of $\operatorname{Proj} T$ is homeomorphic to Ω, *via* the map

$$\mathfrak{a} \mapsto \mathfrak{a}|_U = \{p|_U \mid p \in \mathfrak{a}\}, \text{ for all } \mathfrak{a} \in \Omega^T.$$

Let the projections of U act on T, by defining $p(x) = p^T(x)$, for any $x \in T$ and $p \in \operatorname{Proj} U$. Hence, by carrying the structure of Ω^T to Ω *via* this isomorphism and then applying Theorem 3-9.10 to the continuous dimension scale T with the finitary unit E, we obtain that there exists a unique lower embedding

$$\varepsilon_T \colon T \hookrightarrow \overline{S},$$

where $\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\infty; \Omega_{\mathrm{II}}, \mathbb{R}_\infty; \Omega_{\mathrm{III}}, \mathbf{2}_\infty)$, such that $\varepsilon_T(p(x)) = \varepsilon_T(x)\rfloor_{\Omega_p}$, for all $x \in T$ and all $p \in \operatorname{Proj} U$, and $\varepsilon_T(e)$ takes its values in $\{0, 1\}$, for any $e \in E$.

Furthermore, for elements T_1, T_2 of $\mathcal{C}$ such that $T_1 \subseteq T_2$, the restriction of ε_{T_2} to T_1 satisfies the requirements of ε_{T_1}. Hence, by the uniqueness statement of Theorem 3-9.10, ε_{T_2} extends ε_{T_1}.

Let ε denote the union of all the maps ε_T, for $T \in \mathcal{C}$. It follows from Axiom $(\mathrm{M_{lh}})$ that the union of all the elements of $\mathcal{C}$ is S, thus ε is a map from S to $\overline{S}$. It obviously satisfies the required conditions. This concludes the "existence" part.

If $\varepsilon' \colon S \hookrightarrow \overline{S}$ is another lower embedding satisfying the conditions of the conclusion of Theorem 3-10.5, then, by the uniqueness statement of Theorem 3-9.10 applied to T, the restriction of ε' to T equals ε_T, for all $T \in \mathcal{C}$; whence $\varepsilon' = \varepsilon$. This concludes the "uniqueness" part. $\square$

CHAPTER 4

Espaliers

4-1. The axioms

We shall now give the fundamental lattice-theoretical definition underlying the whole paper, the definition of an *espalier*. This definition will consist of a list of simple axioms, numbered from (L1) to (L8). Interspersed between these axioms, we shall also list some very elementary properties of espaliers. The role of each of these comments will also be to prepare for the formulation of the axioms that follow them.

Definition 4-1.1. An *espalier* is a structure $(L, \leq, \perp, \sim)$, where $(L, \leq)$ is a partially ordered set, $\perp$ is a binary relation on L, and $\sim$ is an equivalence relation on L, subject to the following axioms:

(L1) Every nonempty subset of L has an infimum. Equivalently, every majorized subset of L has a supremum.

In particular, L has a smallest element, that we shall denote by 0. For a, $b \in L$, the meet $a \wedge b$ is always defined, while the join $a \vee b$ is defined if and only if the pair $\{a, b\}$ is majorized.

(L2) For all a, b, $c \in L$, the following statements hold:
- (i) $a \perp 0$.
- (ii) if $a \perp b$, then $b \perp a$.
- (iii) if $a \leq b$ and $b \perp c$, then $a \perp c$.
- (iv) if $\{a, b, c\}$ is majorized, $a \perp b$, and $(a \vee b) \perp c$, then $a \perp (b \vee c)$.
- (v) if $a \perp a$, then $a = 0$.

We can then define in L a partial binary operation $\oplus$, by putting $c = a \oplus b$ if and only if $c = a \vee b$ and $a \perp b$. So, (i)–(iv) *above means exactly that $(L, \oplus, 0)$ is a partial commutative monoid.*
We say that a family $(a_i)_{i \in I}$ of elements of L is orthogonal, *if it is majorized and $\oplus_{i \in J} a_i$ is defined for every finite subset J of I. We then define $\oplus_{i \in I} a_i = \bigvee_{i \in I} a_i$.*

(L3) For all a, $b \in L$, if $a \leq b$, then there exists $x \in L$ such that $a \oplus x = b$.

Since $a \oplus x = a \vee x$, the converse of Axiom (L3) *is, of course, trivial.*

(L4) Let $a \in L$, let $(b_i)_{i \in I}$ be an orthogonal family of elements of L. If $a \perp (\oplus_{i \in J} b_i)$, for all finite $J \subseteq I$, then $a \perp (\oplus_{i \in I} b_i)$.

(L5) $x \sim 0$ implies that $x = 0$, for all $x \in L$.

(L6) The relation $\sim$ is *unrestrictedly refining*, that is, for every $a \in L$ and every orthogonal family $(b_i)_{i \in I}$ of elements of L, if $a \sim \oplus_{i \in I} b_i$, then there exists a decomposition $a = \oplus_{i \in I} a_i$ such that $a_i \sim b_i$ for all $i \in I$.

(L7) The relation $\sim$ is *unrestrictedly additive*, that is, for all orthogonal families $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ of elements of L, if $a_i \sim b_i$ for all $i \in I$, then $\oplus_{i\in I} a_i \sim \oplus_{i\in I} b_i$.

(L8) (the *parallelogram rule*) For all a, b, x, $y \in L$ such that $a \vee b$ is defined,

$$\big((a \wedge b) \oplus x = a \text{ and } b \oplus y = a \vee b\big) \Longrightarrow x \sim y.$$

An espalier is *bounded*, if it has a largest element.

The following result makes it possible to create new espaliers from old ones. We leave the straightforward proof to the reader.

PROPOSITION 4-1.2.

(i) *For any espalier $(L, \leq, \perp, \sim)$, any lower subset K of L, endowed with the restrictions of $\leq$, $\perp$, and $\sim$, is an espalier.*

(ii) *Let $(L_i, \leq_i, \perp_i, \sim_i)_{i\in I}$ be a family of espaliers. Then the product $L = \prod_{i\in I} L_i$, endowed with the componentwise $\leq$, $\perp$, $\sim$, is an espalier.*

In the context of Proposition 4-1.2(i), we shall say that K is a *lower subespalier* of L. In the context of Proposition 4-1.2(ii), we shall say that L is the *direct product* of the family $(L_i)_{i\in I}$ of espaliers. If K and L are espaliers, we shall say that a map $\varphi\colon K \to L$ is a *lower embedding* of espaliers, if it is an isomorphism from K onto a lower subespalier of L. The verification of the following lemma is straightforward.

LEMMA 4-1.3. *Let K and L be espaliers, let $\varphi\colon K \to L$ be a map. Then φ is a lower embedding if and only if the following conditions hold:*

(i) *the range of φ is a lower subset of L.*

(ii) *$x \leq_K y$ if and only if $\varphi(x) \leq_L \varphi(y)$, for all x, $y \in K$.*

(iii) *$x \perp_K y$ if and only if $\varphi(x) \perp_L \varphi(y)$, for all x, $y \in K$.*

(iv) *$x \sim_K y$ if and only if $\varphi(x) \sim_L \varphi(y)$, for all x, $y \in K$.*

For the remainder of Section 4-1, *we shall fix an espalier $(L, \leq, \perp, \sim)$.*

We start up with elementary properties of orthogonal families.

LEMMA 4-1.4. *For all a, $b \in L$ such that $a \perp b$, the following holds:*

(i) $a \wedge b = 0$.

(ii) $x = (x \oplus b) \wedge a$, *for all $x \leq a$. (Here, this means that (a, b) is a* modular pair*, see* [**19**] *for the general definition of those.)*

PROOF. As in [**39**, Lemma 1.1]. ☐

COROLLARY 4-1.5. *Let a, b, $c \in L$.*

(i) *If (a, b, c) is orthogonal, then $(a \oplus c) \wedge (b \oplus c) = c$.*

(ii) *If $a \leq b \leq c$, then there exists $x \in L$ such that $b \wedge x = a$ and $b \vee x = c$. (That is, every closed interval of L is a* relatively complemented *lattice.)*

PROOF. (i) Apply Lemma 4-1.4 to the pair $(b \oplus c, a)$ and to $c \leq b \oplus c$.

(ii) By Axiom (L3), there are u, v such that $a \oplus u = b$ and $b \oplus v = c$. So $c = a \oplus u \oplus v$. By (i), $x = a \oplus v$ satisfies the required conditions. ☐

By using Axiom (L4), it is easy to prove the following result (see also Theorem 1.2 of [**39**]).

LEMMA 4-1.6. *Let I and J be sets, let $\pi\colon I \twoheadrightarrow J$ be a surjective map, let $(a_i)_{i\in I}$ be a family of elements of L, and let $a \in S$. Then the following are equivalent:*

(i) $a = \oplus_{i\in I} a_i$.

(ii) *For all $j \in J$, the family $(a_i)_{i\in\pi^{-1}\{j\}}$ is orthogonal, and, if we denote its join by b_j, then $a = \oplus_{j\in J} b_j$.*

Another useful elementary orthogonality property of espaliers is the following.

LEMMA 4-1.7. *Let $a \in L$, let X be a majorized subset of L. If $a \perp \bigvee Y$ for all finite $Y \subseteq X$, then $a \perp \bigvee X$.*

PROOF. For finite X, this is trivial. Now suppose that X is infinite. Write $X = \{b_\xi \mid \xi < \kappa\}$, where κ is the cardinality of X, and put $b = \bigvee X$. We argue by induction on κ. Put $\overline{b}_\xi = \bigvee_{\eta<\xi} b_\eta$, for all $\xi < \kappa$. Observe that $\overline{b}_0 = 0$. For all $\xi < \kappa$, there exists, by Axiom (L3), $c_\xi \in L$ such that $\overline{b}_\xi \oplus c_\xi = \overline{b}_{\xi+1}$. It follows easily that $\overline{b}_\xi = \oplus_{\eta<\xi} c_\eta$ for all $\xi < \kappa$, while $b = \oplus_{\eta<\kappa} c_\eta$. Furthermore, it follows from the induction hypothesis that $a \perp \overline{b}_\xi$ for all $\xi < \kappa$, whence $a \perp \oplus_{\eta\in J} c_\eta$ for every finite subset J of κ. By Axiom (L4), it follows that $a \perp \oplus_{\eta<\kappa} c_\eta$, that is, $a \perp b$. $\square$

NOTATION 4-1.8. For a, $b \in L$, let $a \lesssim b$ hold, if $a \sim x$ for some $x \leq b$.

LEMMA 4-1.9. *Let a, b, $c \in L$.*

(i) *If $a \vee c = b \vee c$ and $a \wedge c = b \wedge c$, then $a \sim b$.* (That is, if a and b are perspective, then $a \sim b$.)

(ii) *If $a \oplus c = b \oplus c$, then $a \sim b$.*

(iii) *If $a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$, then $a \lesssim b$.*

PROOF. (i) We put $u = a \wedge c = b \wedge c$, $v = a \vee c = b \vee c$. Let x, y, $z \in L$ such that

$$u \oplus x = a; \qquad u \oplus y = b; \qquad c \oplus z = v.$$

By the parallelogram rule, $x \sim z$ and $y \sim z$. Thus $x \sim y$, so

$$a = u \oplus x \sim u \oplus y = b.$$

(ii) This is, by Lemma 4-1.4(i), a particular case of (i).

(iii) By using Axiom (L3), there are u, v, x, y, $z \in L$ such that

$$a = (a \wedge c) \oplus u, \quad a \vee c = c \oplus v,$$
$$b \wedge c = (a \wedge c) \oplus x, \quad b = (b \wedge c) \oplus y, \quad b \vee c = (a \vee c) \oplus z.$$

It follows that $b \vee c = c \oplus v \oplus z$, thus, by Axiom (L8), $y \sim v \oplus z$, thus, by Axiom (L6), there are $v' \sim v$ and $z' \sim z$ such that $y = v' \oplus z'$. Moreover, it also follows from Axiom (L8) that $u \sim v$, thus $u \sim v'$. From $v' \leq y$, $a \wedge c \leq b \wedge c$, and $y \perp (b \wedge c)$ it follows (by Axiom (L2)) that $(a \wedge c) \perp v'$, thus, since $a \wedge c$, $v' \leq b$, $(a \wedge c) \oplus v'$ is defined and below b. Therefore, by using Axiom (L7), $a = (a \wedge c) \oplus u \sim (a \wedge c) \oplus v' \leq b$. $\square$

We observe that Lemma 4-1.9(iii) is stronger than Axiom (iii) in Definition 2.2 of [**12**].

> *From now on, we denote by $\Delta(a)$, or $\Delta_L(a)$ if there is any ambiguity on L, the $\sim$-equivalence class of a, for every $a \in L$. Furthermore, we denote by S the range of Δ, and we call it the* dimension range *of L, in notation, $S = \operatorname{Drng} L$.*

We endow S with the partial binary operation $+$ defined by

$$\gamma = \alpha + \beta, \text{ if there are } a,\ b,\ c \in L \text{ such that}$$
$$\alpha = \Delta(a),\ \beta = \Delta(b),\ \gamma = \Delta(c), \text{ and } c = a \oplus b,$$

for all α, β, $\gamma \in L$. The fact that $+$ is indeed well-defined follows from the finite case of Axiom (L7), namely, if $a \oplus b$ and $a' \oplus b'$ are defined and $a \sim a'$ and $b \sim b'$, then $a \oplus b \sim a' \oplus b'$.

We denote $\Delta(0)$ by 0.

Proposition 4-1.10. $(S, +, 0)$ *is a partial commutative monoid.*

Proof. Only the verification of the associativity of $+$ is not completely trivial. Given a, b, $c \in L$ such that $(\Delta(a) + \Delta(b)) + \Delta(c)$ is defined, there exist x, $y \in L$ such that $\Delta(a) + \Delta(b) = \Delta(x)$ and $\Delta(x) + \Delta(c) = \Delta(y)$. Then $x = a' \oplus b'$ for some $a' \sim a$ and $b' \sim b$ in L, while $y = x' \oplus c'$ for some $x' \sim x$ and $c' \sim c$ in L. By the finite case of Axiom (L6), $x' = a'' \oplus b''$ for some $a'' \sim a'$ and $b'' \sim b'$ in L. Axiom (L2) then implies that $y = a'' \oplus (b'' \oplus c')$, and therefore, $\Delta(a) + (\Delta(b) + \Delta(c))$ is defined, and equal to

$$\Delta(a) + \Delta(b'' \oplus c'') = \Delta(y) = (\Delta(a) + \Delta(b)) + \Delta(c).$$

Since $+$ is obviously commutative, this implies that $+$ is associative as well. $\square$

> *The dimension range S will always be endowed with its* algebraic *preordering $\leq$, see Definition* 2-1.3. *Hence* $\Delta(a) \leq \Delta(b)$ *if and only if $a \lesssim b$, for all a, $b \in L$.*

Lemma 4-1.11. *The following assertions hold:*

(i) *Let α, $\beta \in S$ and let $c \in L$ such that $\Delta(c) = \alpha + \beta$. Then there are a, $b \in L$ such that $c = a \oplus b$ while $\Delta(a) = \alpha$ and $\Delta(b) = \beta$.*

(ii) *Let $a \in L$ and let $\xi \in S$. If $\xi \leq \Delta(a)$, then there exists $x \leq a$ in L such that $\Delta(x) = \xi$.*

Proof. (i) By the definition of $\alpha + \beta$, there are orthogonal u, $v \in L$ such that $\Delta(c) = \Delta(u \oplus v)$ while $\Delta(u) = \alpha$ and $\Delta(v) = \beta$. So $c \sim u \oplus v$, thus, by the finite case of Axiom (L6), there are a, $b \in L$ such that $a \sim u$, $b \sim v$, and $c = a \oplus b$. Observe that $\Delta(a) = \Delta(u) = \alpha$ and $\Delta(b) = \Delta(v) = \beta$.

(ii) By the definition of the algebraic preordering of S, there exists $\eta \in S$ such that $\Delta(a) = \xi + \eta$. The conclusion follows from (i). $\square$

Proposition 4-1.10 and Lemma 4-1.11 are particular cases of more general results in Chapter 4 of [**56**].

Proposition 4-1.12. *The algebraic preordering on S is antisymmetric. That is, if $a \lesssim b$ and $b \lesssim a$, then $a \sim b$, for all a, $b \in L$.*

Proof. Similar to the proof of Theorem 41 of [**30**], see also Lemma 6.1.3 of [**42**]. We give the details here for convenience.

Put $a_0 = a$ and $b_0 = b$. By induction on $n < \omega$, we construct a_{n+1}, b_{n+1}, x_n, $y_n \in L$ such that the following relations hold:

$$a_n = b_{n+1} \oplus x_n, \quad b_n = a_{n+1} \oplus y_n, \tag{4-1.1}$$
$$a_n \sim a_{n+1}, \quad b_n \sim b_{n+1}. \tag{4-1.2}$$

Since $a_0 \lesssim b_0$ and $b_0 \lesssim a_0$, this is easy to satisfy for $n = 0$. Now the induction step. Since $b_{n+1} \sim b_n = a_{n+1} \oplus y_n$, there are a_{n+2} and y_{n+1} in L such that

$$b_{n+1} = a_{n+2} \oplus y_{n+1}, \quad a_{n+2} \sim a_{n+1}, \quad y_{n+1} \sim y_n.$$

Similarly, we obtain elements b_{n+2}, x_{n+1} in L such that

$$a_{n+1} = b_{n+2} \oplus x_{n+1}, \quad b_{n+2} \sim b_{n+1}, \quad x_{n+1} \sim x_n.$$

This completes the induction step for (4-1.1) and (4-1.2). Observe that all the x_n (resp., all the y_n) are mutually $\sim$-equivalent. But then, the family consisting of all the x_{2n+1} and y_{2n+2}, for $n < \omega$, is orthogonal and majorized by a_1. Thus there exists $c \in L$ such that

$$a_1 = c \oplus (\oplus_{n<\omega} x_{2n+1}) \oplus (\oplus_{n<\omega} y_{2n+2}). \tag{4-1.3}$$

Since $b = a_1 \oplus y_0$, we obtain the equality

$$b = c \oplus (\oplus_{n<\omega} x_{2n+1}) \oplus (\oplus_{n<\omega} y_{2n}). \tag{4-1.4}$$

Since $y_{2n} \sim y_{2n+2}$ for all n, it follows from (4-1.3), (4-1.4), and Axiom (L7) that $a_1 \sim b$. But $a \sim a_1$; whence $a \sim b$. □

Of course, it follows that S is *conical*. However, a direct proof of the conicality of S is immediate from Axiom (L5).

PROPOSITION 4-1.13. *The partial commutative monoid* $(S, +, 0)$ *satisfies the refinement property* (*see Definition* 2-1.12).

PROOF. Let α, α', β, $\beta' \in S$ such that $\alpha + \alpha' = \beta + \beta'$. By the definition of the addition of S and the finite version of Axiom (L6), there are a, a', b, $b' \in L$ such that $\Delta(a) = \alpha$, $\Delta(a') = \alpha'$, $\Delta(b) = \beta$, $\Delta(b') = \beta'$, and $a \oplus a' = b \oplus b'$. We put $e = a \oplus a' = b \oplus b'$. We observe that $a \vee b$ is defined. There are elements u, u', v, v', w in L such that

$$(a \wedge b) \oplus u = a; \qquad b \oplus u' = a \vee b;$$
$$(a \wedge b) \oplus v = b; \qquad a \oplus v' = a \vee b;$$
$$(a \vee b) \oplus w = e.$$

By Axiom (L8), $u \sim u'$ and $v \sim v'$. Furthermore,

$$b \oplus b' = e = (a \vee b) \oplus w = b \oplus u' \oplus w,$$

thus, by Lemma 4-1.9(ii), $b' \sim u' \oplus w$. Hence $b' = \overline{u} \oplus w_2$, for some $\overline{u} \sim u'$ (thus $\overline{u} \sim u$) and some $w_2 \sim w$. Similarly, $a' = \overline{v} \oplus w_1$, for some $\overline{v} \sim v$ and some $w_1 \sim w$. Hence we have obtained the following refinement matrix:

	β	β'
α	$\Delta(a \wedge b)$	$\Delta(u) = \Delta(\overline{u})$
α'	$\Delta(\overline{v}) = \Delta(v)$	$\Delta(w_1) = \Delta(w_2)$

This completes the proof. □

From Propositions 4-1.13 and 4-1.12, we deduce immediately the following.

COROLLARY 4-1.14. *The partial commutative monoid S satisfies Axiom* (M1).

In particular, by Proposition 2-3.11, the set $\operatorname{Proj} S$ of all projections of S, endowed with the ordering $\leq$ given by $p \leq q$ if and only if $pS \subseteq qS$ (see Lemma 2-3.8), is a Boolean algebra. We shall also refer to the projections of S as the *projections of* L. They operate also on L in a projection-like manner (up to equivalence)—see Lemmas 4-1.25 and 4-1.27 below.

For the remainder of Section 4-1, we shall analyze in some detail the pairs (a, b) of elements of L such that $\Delta(a) \perp \Delta(b)$. Observe that the conditions $a \perp b$ (in L) and $\Delta(a) \perp \Delta(b)$ (in S) are *a priori* unrelated. For a, $b \in L$, $\Delta(a) \perp \Delta(b)$ if and only if the only element $x \in L$ such that $x \lesssim a, b$ is $x = 0$.

Corollary 4-1.15. *Let a, b, $c \in L$ such that $\Delta(a) \perp \Delta(c)$ and $\Delta(b) \perp \Delta(c)$. If $\{a, b\}$ is majorized, then $\Delta(a \vee b) \perp \Delta(c)$.*

Proof. Let $b' \in L$ such that $a \oplus b' = a \vee b$. In particular, $\Delta(a \vee b) = \Delta(a) + \Delta(b')$. It follows from Lemma 4-1.9(iii) that $b' \lesssim b$, thus, by assumption, $\Delta(b') \perp \Delta(c)$. Therefore, by Proposition 4-1.13 and Lemma 2-2.2(i), $\Delta(a) + \Delta(b') \perp \Delta(c)$, that is, $\Delta(a \vee b) \perp \Delta(c)$. $\square$

Notation 4-1.16. For a, b, $c \in L$, we define $c = a \boxplus b$ to mean that $c = a \vee b$ and $\Delta(a) \perp \Delta(b)$.

Note, in particular, that $c = a \boxplus b$ implies that $a \wedge b = 0$. Much more is true, see Proposition 4-1.18.

Lemma 4-1.17. *Let a, a', b, $c \in L$. If $c = a \boxplus b = a' \oplus b$ and if $a' \leq a$, then $a' = a$, so $a \perp b$.*

Proof. Let $b' \in L$ such that $c = a \oplus b'$. Since $c = a \vee b$ with $a \wedge b = 0$, it follows from Lemma 4-1.9(i) that $b \sim b'$. Let v, $v' \in L$ such that

$$b = (b \wedge b') \oplus v \quad \text{and} \quad b' = (b \wedge b') \oplus v'.$$

Let $u \in L$ such that $a' \oplus u = a$. Then $c = a' \oplus b = a' \oplus (b \wedge b') \oplus v$, so that $c = a \oplus b' = a' \oplus (b \wedge b') \oplus u \oplus v'$. Therefore, by Lemma 4-1.9(ii), $u \oplus v' \sim v$, thus $u \lesssim v$. So $u \leq a$ and $u \lesssim v \leq b$, hence, by assumption, $u = 0$. It follows that $a = a'$. In particular, $a \perp b$. $\square$

We obtain the following important tool, Proposition 4-1.18. It is an analogue of Axiom (D) in [**35**] and of Axiom $(2, \varepsilon)$ in [**39**]. It also holds in the "cardinal lattices" considered in [**12**], as Lemma 2.7 of [**12**] shows. However, the proof of Lemma 2.7 of [**12**] cannot be applied here, because there is no "orthocomplement" in our axiom system for espaliers.

Proposition 4-1.18. *Let a, $b \in L$. If $\Delta(a) \perp \Delta(b)$ and $\{a, b\}$ is majorized, then $a \perp b$; so $a \boxplus b = a \oplus b$.*

Proof. By Axiom (L1), $c = a \vee b$ exists. Let $a' \in L$ such that $c = a' \oplus b$. By Lemma 4-1.9(i), $a \sim a'$. Since a, $a' \leq c$, there exists $a_1 = a \vee a'$ in L, but $\Delta(a) = \Delta(a') \perp \Delta(b)$, so, by Corollary 4-1.15, $\Delta(a_1) \perp \Delta(b)$. So, $c = a_1 \boxplus b = a' \oplus b$, with $a' \leq a_1$. Hence, by Lemma 4-1.17, $a_1 \perp b$. Since $a \leq a_1$, it follows from Axiom (L2)(iii) that $a \perp b$. $\square$

We now extend Proposition 4-1.18 to arbitrary families of elements of L.

Definition 4-1.19. A family $(a_i)_{i \in I}$ is *strongly orthogonal*, if it is majorized and $\Delta(a_i) \perp \Delta(a_j)$, for all $i \neq j$ in I.

COROLLARY 4-1.20. *Every strongly orthogonal family of elements of L is orthogonal.*

PROOF. It suffices to prove the result for I finite. We argue by induction on the cardinality of I. Pick $i \in I$. By the induction hypothesis, $(a_j)_{j \neq i}$ is orthogonal, and, by Corollary 4-1.15, $\Delta(a_i) \perp \Delta(\oplus_{j \neq i} a_j)$. Hence, by Proposition 4-1.18, $a_i \perp \oplus_{j \neq i} a_j$ in L, that is, $(a_j)_{j \in I}$ is orthogonal. □

COROLLARY 4-1.21. *Let a, b, x, y, $c \in L$. If $c = a \boxplus b = x \oplus y$ with $\Delta(x) \perp \Delta(b)$ and $\Delta(y) \perp \Delta(a)$, then $x = a$ and $y = b$.*

PROOF. Since $a \vee x$ is defined (and $a \vee x \leq c$) and $\Delta(a)$, $\Delta(x) \perp \Delta(b)$, it follows from Corollary 4-1.15 that $\Delta(a \vee x) \perp \Delta(b)$. So, $c = (a \vee x) \boxplus b$. Since $c = a \oplus b$ (by Proposition 4-1.18) and $a \leq a \vee x$, it follows from Lemma 4-1.17 that $a = a \vee x$, that is, $x \leq a$. Similarly, $y \leq b$. Let x', $y' \in L$ such that $a = x \oplus x'$ and $b = y \oplus y'$. Then

$$c = a \oplus b = x \oplus y \oplus x' \oplus y' = c \oplus x' \oplus y',$$

whence $x' = y' = 0$. Therefore, $x = a$ and $y = b$. □

COROLLARY 4-1.22. *Let $a \in L$, let X be a majorized subset of L. If $\Delta(a) \perp \Delta(x)$ for all $x \in X$, then $\Delta(a) \perp \Delta(\bigvee X)$.*

PROOF. Put $b = \bigvee X$. It follows from Corollary 4-1.15 that

$$\Delta(a) \perp \Delta\left(\bigvee Y\right), \quad \text{for all finite } Y \subseteq X. \tag{4-1.5}$$

Let $t \in (0, b]$. Then $t \not\perp b$, thus, by Lemma 4-1.7, $t \not\perp \bigvee Y$ for some finite $Y \subseteq X$. But $\{t, \bigvee Y\}$ is majorized (by b), thus, by Proposition 4-1.18, $\Delta(t) \not\perp \Delta(\bigvee Y)$. Therefore, by (4-1.5), $\Delta(t) \nleq \Delta(a)$, that is, $t \not\lesssim a$. So we have proved that $\Delta(a) \perp \Delta(b)$. □

COROLLARY 4-1.23. *Let $a \in L$ and let $p \in \operatorname{Proj} S$. Then there exists a largest element u of $[0, a]$ such that $\Delta(u) \in pS$. Furthermore, $\Delta(u) = p(\Delta(a))$.*

PROOF. By Lemma 2-2.4, $pS = (pS)^{\perp\perp}$. Hence, by Corollary 4-1.22, the supremum u of the set X of all elements x of $[0, a]$ such that $\Delta(x) \in pS$ belongs to X.

From $u \leq a$ it follows that $\Delta(u) \leq \Delta(a)$, thus, since $\Delta(u) \in pS$, $\Delta(u) = p(\Delta(u)) \leq p(\Delta(a))$. Conversely, $p(\Delta(a)) \leq \Delta(a)$, thus, by Lemma 4-1.11(ii), there exists $v \leq a$ such that $p(\Delta(a)) = \Delta(v)$. But $\Delta(v) \in pS$, so $v \leq u$, and thus $p(\Delta(a)) = \Delta(v) \leq \Delta(u)$. Finally, by Proposition 4-1.12, $p(\Delta(a)) = \Delta(u)$. □

We shall denote by $p \cdot a$ the element u of Corollary 4-1.23, and we shall repeatedly use the properties $p \cdot a \leq a$ and $\Delta(p \cdot a) = p(\Delta(a))$, for all $a \in L$ and all $p \in \operatorname{Proj} S$.

We shall also put $p \cdot L = \{p \cdot x \mid x \in L\}$. Note that $p \cdot L$ is a lower subset of L: indeed, if $a \in L$ and $b \leq p \cdot a$, then $\Delta(b) \in pS$, so $p \cdot b = b$.

We gather up various elementary properties of the map $(p, a) \mapsto p \cdot a$ in the following Lemmas 4-1.24 and 4-1.25.

LEMMA 4-1.24. *Let $p \in \operatorname{Proj} S$. Then the following assertions hold:*

(i) $p \cdot L = \Delta^{-1}[pS]$.

(ii) *$p \cdot L$ is a lower subset of $(L, \lesssim)$.*

(iii) *$p \cdot L$ is a lower subset of $(L, \leq)$.*
(iv) *$p \cdot L$ is closed under majorized suprema.*

PROOF. (i) is an immediate consequence of Corollary 4-1.23. The assertions (ii), (iii) follow immediately.

(iv) follows immediately from (i), Corollary 4-1.22, and the fact that $pS = (p^{\perp}S)^{\perp}$. $\square$

LEMMA 4-1.25. *Let a, $b \in L$, let p, $q \in \operatorname{Proj} S$. Then the following assertions hold:*

(i) *$a \leq b$ implies that $p \cdot a \leq p \cdot b$.*
(ii) *$a \lesssim b$ implies that $p \cdot a \lesssim p \cdot b$.*
(iii) *$a \sim b$ implies that $p \cdot a \sim p \cdot b$.*
(iv) *$p \leq q$ implies that $p \cdot a \leq q \cdot a$.*

PROOF. (i) $p \cdot a \leq a \leq b$ and $\Delta(p \cdot a) \in pS$, thus $p \cdot a \leq p \cdot b$ by the definition of $p \cdot b$.

(ii) It follows from $a \lesssim b$ that $\Delta(a) \leq \Delta(b)$, thus $\Delta(p \cdot a) = p(\Delta(a)) \leq p(\Delta(b)) = \Delta(p \cdot b)$, that is, $p \cdot a \lesssim p \cdot b$.

(iii) follows immediately from (ii) and from Proposition 4-1.12.

(iv) Since $p \leq q$, $\Delta(p \cdot a) \in pS \subseteq qS$, and so, since $p \cdot a \leq a$, it follows that $p \cdot a \leq q \cdot a$. $\square$

PROPOSITION 4-1.26. *Let $(p_i)_{i \in I}$ be an orthogonal family of elements of $\operatorname{Proj} S$ and let $a \in L$. Then the family $(p_i \cdot a)_{i \in I}$ is orthogonal in L.*

PROOF. For $i \neq j$ in I, $\Delta(p_i \cdot a) \in p_i S$ and $\Delta(p_j \cdot a) \in p_j S$, thus, since $p_i p_j = 0$, $\Delta(p_i \cdot a) \perp \Delta(p_j \cdot a)$. The result follows then from Corollary 4-1.20. $\square$

LEMMA 4-1.27. *Let $a \in L$ and let $p \in \operatorname{Proj} S$. Then the following assertions hold:*

(i) *$a = p \cdot a \boxplus p^{\perp} \cdot a$ (thus, by Proposition 4-1.18, $a = p \cdot a \oplus p^{\perp} \cdot a$).*
(ii) *Let $x \in p \cdot L$ and $y \in p^{\perp} \cdot L$ such that $a = x \vee y$. Then $x = p \cdot a$ and $y = p^{\perp} \cdot a$.*

PROOF. (i) Since the join $a' = p \cdot a \vee p^{\perp} \cdot a$ is defined (and $a' \leq a$) and since $\Delta(p \cdot a) \perp \Delta(p^{\perp} \cdot a)$, it follows from Proposition 4-1.18 that $a' = p \cdot a \oplus p^{\perp} \cdot a$. Since

$$\Delta(a) = p(\Delta(a)) + p^{\perp}(\Delta(a)) = \Delta(p \cdot a) + \Delta(p^{\perp} \cdot a) = \Delta(p \cdot a \oplus p^{\perp} \cdot a)$$

and by Axiom (L6), there are $u \sim p \cdot a$ and $v \sim p^{\perp} \cdot a$ such that $a = u \oplus v$. Since $\Delta(u) = \Delta(p \cdot a) \in pS$ and $u \leq a$, we have $u \leq p \cdot a$. Likewise, $v \leq p^{\perp} \cdot a$, thus $a = u \oplus v \leq a'$, whence $a = a' = p \cdot a \boxplus p^{\perp} \cdot a$.

(ii) By assumption, $a = x \boxplus y$. By Proposition 4-1.18, $a = x \oplus y$. By Corollary 4-1.21 and by (i), $x = p \cdot a$ and $y = p^{\perp} \cdot a$. $\square$

PROPOSITION 4-1.28. *S has general comparability.*

PROOF. We prove the two following claims.

CLAIM 1. *$S = \boldsymbol{a}^{\perp} + \boldsymbol{a}^{\perp\perp}$, for all $\boldsymbol{a} \in S$.*

PROOF OF CLAIM. Let $\boldsymbol{a}$, $\boldsymbol{x} \in S$. Pick a, $x \in L$ such that $\Delta(a) = \boldsymbol{a}$ and $\Delta(x) = \boldsymbol{x}$. By Corollary 4-1.22, there exists a largest element $u \leq x$ such that $\Delta(u) \perp \boldsymbol{a}$. Let $v \in L$ such that $u \oplus v = x$. So $\boldsymbol{x} = \Delta(u) + \Delta(v)$, with $\Delta(u) \in \boldsymbol{a}^{\perp}$.

For any $t \leq v$ such that $\Delta(t) \in \boldsymbol{a}^{\perp}$, the inequality $t \leq u$ holds by the definition of u, but $t \leq v$, so $t = 0$ since $u \perp v$. Hence $\Delta(v) \in \boldsymbol{a}^{\perp\perp}$. □ Claim 1.

Claim 2. *For all a, $b \in L$, there are u, v, x, $y \in L$ such that $a = u \oplus x$ and $b = v \oplus y$ while $u \sim v$ and $\Delta(x) \perp \Delta(y)$.*

Proof of Claim. An easy application of Zorn's Lemma yields a subset $X = \{(a_i, b_i) \mid i \in I\}$ of $(0, a] \times (0, b]$ which is maximal with respect to the following properties:

(i) Both families $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ are orthogonal.
(ii) $a_i \sim b_i$ for all $i \in I$.

Put $u = \bigvee_{i\in I} a_i$ and $v = \bigvee_{i\in I} b_i$. By Axiom (L7), $u \sim v$. Pick x and y such that $a = u \oplus x$ and $b = v \oplus y$. If $\Delta(x) \not\perp \Delta(y)$, then there are nonzero $x' \leq x$ and $y' \leq y$ such that $x' \sim y'$. But then, $X \cup \{(x', y')\}$ still satisfies (i) and (ii) above, which contradicts the maximality of X. Hence $\Delta(x) \perp \Delta(y)$. □ Claim 2.

By Lemma 2-4.2, general comparability follows from Claims 1 and 2. □

We recall that general comparability is also Axiom (M3). Note that general comparability in S can be stated as the following property of L, which we also refer to as *general comparability*: for any a, $b \in L$, there exists $p \in \operatorname{Proj} S$ such that $p \cdot a \lesssim p \cdot b$ and $p^{\perp} \cdot b \lesssim p^{\perp} \cdot a$.

Lemma 4-1.29. *Let $a \in L$, let $(a_i)_{i\in I}$ be a family of elements of L, let $p \in \operatorname{Proj} S$.*

(i) *If $I \neq \varnothing$ and $a = \bigwedge_{i\in I} a_i$, then $p \cdot a = \bigwedge_{i\in I}(p \cdot a_i)$.*
(ii) *If $a = \bigvee_{i\in I} a_i$, then $p \cdot a = \bigvee_{i\in I}(p \cdot a_i)$.*

Proof. (i) It is clear that $p \cdot a \leq p \cdot a_i$ for all i. Let $b \in L$ such that $b \leq p \cdot a_i$ for all $i \in I$. In particular, $b \leq a_i$ for all i, thus $b \leq a$. In addition, since $I \neq \varnothing$, it follows from Lemma 4-1.24(iii) that $b \in p \cdot L$, so $b = p \cdot b \leq p \cdot a$.

(ii) The equality $a_i = (p \cdot a_i) \vee (p^{\perp} \cdot a_i)$ holds for all $i \in I$, so

$$a = \bigvee_{i\in I}(p \cdot a_i) \vee \bigvee_{i\in I}(p^{\perp} \cdot a_i).$$

By Lemma 4-1.24(iv), $\bigvee_{i\in I}(p \cdot a_i) \in p \cdot L$ and $\bigvee_{i\in I}(p^{\perp} \cdot a_i) \in p^{\perp} \cdot L$. Therefore, by Lemma 4-1.27, $p \cdot a = \bigvee_{i\in I}(p \cdot a_i)$ and $p^{\perp} \cdot a = \bigvee_{i\in I}(p^{\perp} \cdot a_i)$. □

Proposition 4-1.30. *The Boolean algebra $\operatorname{Proj} S$ is complete.*

Proof. It suffices to prove that every orthogonal family $(p_i)_{i\in I}$ of elements of $\operatorname{Proj} S$ admits a supremum. By Proposition 4-1.26, the family $(p_i \cdot a)_{i\in I}$ is orthogonal, for all $a \in L$. Furthermore, if $b \in L$ such that $a \sim b$, then, by Lemma 4-1.25(iii), $p_i \cdot a \sim p_i \cdot b$ for all $i \in I$, thus, by Axiom (L7),

$$\oplus_{i\in I}(p_i \cdot a) \sim \oplus_{i\in I}(p_i \cdot b).$$

Hence, we can define a map $p\colon S \to S$ by the rule

$$p(\Delta(a)) = \Delta(\oplus_{i\in I}(p_i \cdot a)), \quad \text{for all } a \in L.$$

It is obvious that $p(0) = 0$. Let $\boldsymbol{a}$, $\boldsymbol{b} \in S$ such that $\boldsymbol{a} + \boldsymbol{b}$ is defined. So $\boldsymbol{a} + \boldsymbol{b} = \Delta(a \oplus b)$, for some $a \in \boldsymbol{a}$ and $b \in \boldsymbol{b}$ such that $a \perp b$. So $p(\boldsymbol{a}) = \Delta(a')$ and $p(\boldsymbol{b}) = \Delta(b')$, where a' and b' are defined by

$$a' = \oplus_{i\in I}(p_i \cdot a) \quad \text{and} \quad b' = \oplus_{i\in I}(p_i \cdot b).$$

In particular, $a' \leq a$ and $b' \leq b$, so $a' \perp b'$, and, by using Lemma 4-1.29(ii),

$$a' \oplus b' = \oplus_{i \in I}(p_i \cdot a \oplus p_i \cdot b) = \oplus_{i \in I}(p_i \cdot (a \oplus b)),$$

whence

$$p(\boldsymbol{a}) + p(\boldsymbol{b}) = \Delta(a' \oplus b') = \Delta(\oplus_{i \in I}(p_i \cdot (a \oplus b))) = p(\boldsymbol{a} + \boldsymbol{b}).$$

So p is an endomorphism of $(S, +, 0)$.

Now let $\boldsymbol{a} \in S$. Pick $a \in \boldsymbol{a}$, and pick $u \in L$ such that $a = \oplus_{i \in I}(p_i \cdot a) \oplus u$. For all $v \leq u$, if $\Delta(v) \in p_i S$, then $v \leq p_i \cdot a$, so $v \leq u \wedge (p_i \cdot a) = 0$. Hence $\Delta(u) \in (p_i S)^{\perp}$. We infer that $\Delta(u) \in (pS)^{\perp}$. Indeed, let $\boldsymbol{x} \in S$. Pick x such that $\Delta(x) = \boldsymbol{x}$. So $p(\boldsymbol{x}) = \Delta(\oplus_{i \in I}(p_i \cdot x))$ by the definition of p. But $\Delta(p_i \cdot x) \perp \Delta(u)$ for all $i \in I$, thus, by Corollary 4-1.22, $p(\boldsymbol{x}) \perp \Delta(u)$. Hence $\Delta(u) \in (pS)^{\perp}$, with $\boldsymbol{a} = p(\boldsymbol{a}) + \Delta(u)$. It follows that p is a projection of S.

It is clear that $p_i \leq p$ for all $i \in I$. Let $q \in \operatorname{Proj} S$ such that $p_i \leq q$ for all $i \in I$. Let $a \in L$. Then $p_i \cdot a \leq q \cdot a$, for all $i \in I$, thus $\oplus_{i \in I}(p_i \cdot a) \leq q \cdot a$. Taking the image under Δ of both sides yields that $p(\Delta(a)) \leq q(\Delta(a))$. This holds for all $a \in L$, whence $p \leq q$. So we have verified that $p = \bigvee_{i \in I} p_i$. □

As a consequence, we obtain that S satisfies Axiom (M4) (observe that $p \cdot a \lesssim b$ if and only if $p(\Delta(a)) \leq p(\Delta(b))$).

PROPOSITION 4-1.31. *For all a, $b \in L$, there exists a largest $p \in \operatorname{Proj} S$ such that $p \cdot a \lesssim b$.*

PROOF. Put $X = \{q \in \operatorname{Proj} S \mid q \cdot a \lesssim b\}$, and put $p = \bigvee X$. Let $(p_i)_{i \in I}$ be a maximal orthogonal family of elements of X. For all $i \in I$, let $b_i \leq b$ such that $p_i \cdot a \sim b_i$. In particular, $\Delta(b_i) = \Delta(p_i \cdot a) \in p_i S$, so $b_i \leq p_i \cdot b$. By Corollary 4-1.20, the family $(b_i)_{i \in I}$ is orthogonal, and by Proposition 4-1.26, the family $(p_i \cdot a)_{i \in I}$ is orthogonal. Hence,

$$\begin{aligned}
\Delta(p \cdot a) &= p(\Delta(a)) && \text{(by the definition of } p \cdot a\text{)}\\
&= \Delta(\oplus_{i \in I}(p_i \cdot a)) && \text{(by the proof of Proposition 4-1.30)}\\
&= \Delta(\oplus_{i \in I} b_i) && \text{(by Axiom (L7))}\\
&\leq \Delta(b).
\end{aligned}$$

□

NOTATION 4-1.32. For a, $b \in L$, we put $\|a \lesssim b\| = \|\Delta(a) \leq \Delta(b)\|$. That is, in accordance to Definition 2-5.1, $\|a \lesssim b\|$ is the largest projection p of S such that $p \cdot a \lesssim b$.

In a similar spirit as for Notation 2-5.2, we put $\|a \sim b\| = \|a \lesssim b\| \wedge \|b \lesssim a\|$, which by Proposition 4-1.12 is the largest $p \in \operatorname{Proj} S$ such that $p \cdot a \sim p \cdot b$.

4-2. Purely infinite elements; trim sequences

Standing hypotheses: L is an espalier, S is the dimension range of L, and $\Delta \colon L \twoheadrightarrow S$ is the canonical map.

Following Definition 2-5.10, we say that an element a of L is *purely infinite*, if $\Delta(a)$ is purely infinite in S. This occurs if and only if $a = a' \oplus a''$ for some a', $a'' \sim a$ in L.

Purely infinite elements are also, in some references, called *idempotent*, or, as in [**49**], *idem-multiple*.

Similarly, following Definition 2-4.7, we say that an element a of L is *directly finite*, if $\Delta(a)$ is directly finite in S. Observe that a is directly finite if and only if $a \sim b \leq a$ implies that $b = a$, for any $b \in L$.

DEFINITION 4-2.1. A family $(a_i)_{i\in I}$ of elements of L is *homogeneous*, if it is orthogonal and $a_i \sim a_j$, for all $i, j \in I$.

A homogeneous family $(a_i)_{i\in I}$ is *trivial*, if $a_i = 0$, for all i; equivalently, $a_i = 0$ for *some* $i \in I$.

LEMMA 4-2.2. *Let $a \in L$. Then the following are equivalent:*

(i) *a is purely infinite.*
(ii) *There exists a homogeneous sequence $(x_n)_{n<\omega}$ such that $a = \oplus_{n<\omega} x_n$.*

PROOF. (i)⇒(ii) There are a' and a'' such that $a = a' \oplus a''$ and $a \sim a' \sim a''$. By using Axiom (L6), it is then easy to construct inductively sequences $(a_n)_{n<\omega}$ and $(a'_n)_{n<\omega}$ such that $a_0 = a$, $a_n = a_{n+1} \oplus a'_n$, and $a_n \sim a_{n+1} \sim a'_n$, for all n. So $(a'_n)_{n<\omega}$ is a homogeneous sequence whose join, a', belongs to $[0, a]$. Since $a \sim a'_0 \leq a' \leq a$, we obtain that $a \sim a'$ by Proposition 4-1.12. Hence, the conclusion follows from Axiom (L6).

(ii)⇒(i) Put $a' = \oplus_n x_{2n}$ and $a'' = \oplus_n x_{2n+1}$. By Axiom (L7), $a \sim a' \sim a''$. Since $a = a' \oplus a''$, a is purely infinite. □

LEMMA 4-2.3. *Let $a \in L$. The following are equivalent:*

(i) *a is directly finite.*
(ii) *There is no nontrivial purely infinite element below a.*
(iii) *The interval $[0, a]$ has no infinite nontrivial homogeneous sequence.*

PROOF. (i)⇒(ii) Let $b \in [0, a]$ be a purely infinite element, and let $x \in L$ such that $a = b \oplus x$. Then $\Delta(a) = \Delta(b) + \Delta(x) = 2\Delta(b) + \Delta(x) = \Delta(b) + \Delta(a)$. Since a is directly finite, $\Delta(b) = 0$, whence $b = 0$.

(ii)⇔(iii) follows immediately from Lemma 4-2.2.

(iii)⇒(i) Let $x \in L$ such that $\Delta(a) + \Delta(x) = \Delta(a)$. So $a = a' \oplus x'$, for some $a' \sim a$ and $x' \sim x$. It is then easy to construct, by induction (and Axiom (L6)), sequences $(a_n)_{n<\omega}$ and $(x_n)_{n<\omega}$ of elements of L such that $a_0 = 0$, $a_n \sim a$, $x_n \sim x$, and $a_n = a_{n+1} \oplus x_n$ for all n. In particular, the sequence $(x_n)_{n<\omega}$ is homogeneous, thus, by assumption, $x_n = 0$ for all n. Therefore, $x = 0$. So a is directly finite. □

We deduce from this that S satisfies Axiom (M5).

PROPOSITION 4-2.4. *For all $a \in L$, there are $b, c \in L$ such that b is purely infinite, c is directly finite, and $a = b \oplus c$.*

PROOF. Let $(x_i)_{i\in I}$ be a maximal orthogonal family of nonzero purely infinite elements of $[0, a]$. We put $b = \oplus_{i\in I} x_i$. For all $i \in I$, there exists a decomposition $x_i = x'_i \oplus x''_i$ where $x'_i \sim x''_i \sim x_i$. Put $b' = \oplus_{i\in I} x'_i$ and $b'' = \oplus_{i\in I} x''_i$. By Axiom (L7), $b' \sim b'' \sim b$. Since $b = b' \oplus b''$, b is purely infinite.

Let $c \in L$ such that $a = b \oplus c$. Suppose that c is not directly finite. Then, by Lemma 4-2.3, there exists a purely infinite element x such that $0 < x \leq c$. But then, enlarging the family $(x_i)_{i\in I}$ by x yields an orthogonal family of nonzero purely infinite elements of $[0, a]$, which contradicts the maximality of $(x_i)_{i\in I}$. So, c is directly finite. □

We can then reformulate Proposition 2-6.5 in the language of lattices.

Proposition 4-2.5. *Let $a,\ b \in L$ such that $a \lesssim b$. Then there exists $c \in L$ such that $c \leq b$ and $\Delta(c) = \Delta(b) \smallsetminus \Delta(a)$.*

Proof. Propositions 4-1.12, 4-1.13, 4-1.28, 4-1.31, and 4-2.4 establish the hypotheses of Proposition 2-6.5. Thus, $\Delta(b) \smallsetminus \Delta(c)$ exists in S. Since this element lies below $\Delta(b)$, there exists $c \in [0, b]$ such that $\Delta(c) = \Delta(b) \smallsetminus \Delta(a)$. $\square$

The following important definition involves both the lattice structure and the dimension function. It is the key to proving the existence of majorized suprema in S.

Definition 4-2.6.

(i) Let $a,\ b \in L$. We write $a \leq_{\mathrm{trim}} b$, if there exists $c \in L$ such that $a \oplus c = b$ and $\Delta(c) = \Delta(b) \smallsetminus \Delta(a)$.
(ii) Let κ be an ordinal. A κ-sequence $(a_\xi)_{\xi<\kappa}$ of elements of L is *trim*, if the following conditions hold:
   (a) $a_\xi \leq_{\mathrm{trim}} a_{\xi+1}$ for all ξ such that $\xi + 1 < \kappa$.
   (b) For any limit ordinal $\lambda < \kappa$, the sequence $\{a_\xi \mid \xi < \lambda\}$ is majorized, and $\bigvee_{\xi<\lambda} a_\xi \leq_{\mathrm{trim}} a_\lambda$.

Lemma 4-2.7. *Let $a,\ b \in L$ such that $a \leq b$, let $\boldsymbol{x} \in S$ such that $\Delta(a) \leq \boldsymbol{x} \leq \Delta(b)$. Then there exists $x \in L$ such that $a \leq_{\mathrm{trim}} x \leq b$ and $\Delta(x) = \boldsymbol{x}$.*

Proof. Let $c \in L$ such that $a \oplus c = b$. So $\Delta(a) \leq \boldsymbol{x} \leq \Delta(b) = \Delta(a) + \Delta(c)$, thus $\boldsymbol{x} \smallsetminus \Delta(a) \leq \Delta(c)$. Hence there exists $y \leq c$ such that $\Delta(y) = \boldsymbol{x} \smallsetminus \Delta(a)$. Now we put $x = a \oplus y$. Then $\Delta(x) = \Delta(a) + (\boldsymbol{x} \smallsetminus \Delta(a)) = \boldsymbol{x}$, $a \leq x \leq b$, and $\Delta(x) \smallsetminus \Delta(a) = \boldsymbol{x} \smallsetminus \Delta(a) = \Delta(y)$. So, $a \leq_{\mathrm{trim}} x$. $\square$

In the statement of the following Lemma 4-2.8, a *lifting* of a family $(\boldsymbol{a}_i)_{i\in I}$ of elements of S is a family $(a_i)_{i\in I}$ of elements of L such that $\Delta(a_i) = \boldsymbol{a}_i$ for all $i \in I$.

Lemma 4-2.8. *Let κ be an ordinal.*

(i) *For all $b \in L$, every increasing κ-sequence of elements of $[0, \Delta(b)]$ has a trim lifting in $[0, b]$.*
(ii) *For any majorized trim sequences $(x_\xi)_{\xi<\kappa}$ and $(y_\xi)_{\xi<\kappa}$ of elements of L,*
$$x_\xi \sim y_\xi \text{ for all } \xi < \kappa \quad \text{implies that} \quad \bigvee_{\xi<\kappa} x_\xi \sim \bigvee_{\xi<\kappa} y_\xi.$$
(iii) *For every majorized trim lifting $(a_\xi)_{\xi<\kappa}$ of a κ-sequence $(\boldsymbol{a}_\xi)_{\xi<\kappa}$ of elements of S,*
$$\Delta\Bigl(\bigvee_{\xi<\kappa} a_\xi\Bigr) = \bigvee_{\xi<\kappa} \boldsymbol{a}_\xi.$$

Proof. We argue by transfinite induction on κ. The result is vacuous for $\kappa = 0$. Suppose that we have proved the lemma for all ordinals $\kappa' < \kappa$, with $\kappa > 0$.

(i) Let $(\boldsymbol{a}_\xi)_{\xi<\kappa}$ be an increasing κ-sequence of elements of $[0, \Delta(b)]$. We construct inductively elements a_ξ of $[0, b]$, for $\xi < \kappa$.

For $\xi = 0$, pick any element a_0 of $[0, b]$ such that $\Delta(a_0) = \boldsymbol{a}_0$.

Suppose we have constructed $a_\xi \leq b$ such that $\Delta(a_\xi) = \boldsymbol{a}_\xi$, with $\xi + 1 < \kappa$. By Lemma 4-2.7, there exists $a_{\xi+1} \leq b$ such that $a_\xi \leq_{\mathrm{trim}} a_{\xi+1}$ and $\Delta(a_{\xi+1}) = \boldsymbol{a}_{\xi+1}$.

Suppose finally that $\lambda < \kappa$ is a limit ordinal and that $(a_\xi)_{\xi<\lambda}$ is a trim lifting of $(\boldsymbol{a}_\xi)_{\xi<\lambda}$ in $[0, b]$. We put

$$\overline{a}_\lambda = \bigvee_{\xi<\lambda} a_\xi.$$

Now, we observe that $\Delta(a_\xi) = \boldsymbol{a}_\xi \leq \boldsymbol{a}_\lambda$ for all $\xi < \lambda$. Since $(a_\xi)_{\xi<\lambda}$ is trim and majorized, it follows from (iii) of the induction hypothesis that $\Delta(\overline{a}_\lambda) \leq \boldsymbol{a}_\lambda$. By applying once again Lemma 4-2.7, we obtain $a_\lambda \leq b$ such that $\overline{a}_\lambda \leq_{\mathrm{trim}} a_\lambda$ and $\Delta(a_\lambda) = \boldsymbol{a}_\lambda$.

By the definition of a trim sequence, $(a_\xi)_{\xi<\kappa}$ is a trim lifting of $(\boldsymbol{a}_\xi)_{\xi<\kappa}$ in $[0, b]$.

(ii) We construct inductively elements $x'_\xi \in L$, for $\xi < \kappa$, of L, as follows. We put $x'_0 = x_0$. If $\xi + 1 < \kappa$, then $x_\xi \leq_{\mathrm{trim}} x_{\xi+1}$, so there exists $x'_{\xi+1}$ such that $x_{\xi+1} = x_\xi \oplus x'_{\xi+1}$ and $\Delta(x'_{\xi+1}) = \Delta(x_{\xi+1}) \smallsetminus \Delta(x_\xi)$. If $\lambda < \kappa$ is a limit ordinal, we put $\overline{x}_\lambda = \bigvee_{\xi<\lambda} x_\xi$. Since $\overline{x}_\lambda \leq_{\mathrm{trim}} x_\lambda$, there exists x'_λ such that $\overline{x}_\lambda \oplus x'_\lambda = x_\lambda$ and $\Delta(x'_\lambda) = \Delta(x_\lambda) \smallsetminus \Delta(\overline{x}_\lambda)$. It follows that $x_\xi = \oplus_{\eta\leq\xi} x'_\eta$ for all $\xi < \kappa$. In particular, $\bigvee_{\xi<\kappa} x_\xi = \oplus_{\xi<\kappa} x'_\xi$.

Let $(y'_\xi)_{\xi<\kappa}$ be constructed from $(y_\xi)_{\xi<\kappa}$ the same way $(x'_\xi)_{\xi<\kappa}$ is constructed from $(x_\xi)_{\xi<\kappa}$. So $x'_0 = x_0 \sim y_0 = y'_0$. Let ξ such that $\xi + 1 < \kappa$. Since $x_\xi \sim y_\xi$ and $x_{\xi+1} \sim y_{\xi+1}$,

$$\Delta(x'_{\xi+1}) = \Delta(x_{\xi+1}) \smallsetminus \Delta(x_\xi) = \Delta(y_{\xi+1}) \smallsetminus \Delta(y_\xi) = \Delta(y'_{\xi+1}).$$

Let $\lambda < \kappa$ be a limit ordinal. By (iii) of the induction hypothesis, $\overline{x}_\lambda \sim \overline{y}_\lambda$. Since $x_\lambda \sim y_\lambda$, we obtain that

$$\Delta(x'_\lambda) = \Delta(x_\lambda) \smallsetminus \Delta(\overline{x}_\lambda) = \Delta(y_\lambda) \smallsetminus \Delta(\overline{y}_\lambda) = \Delta(y'_\lambda).$$

Hence we have proved that $x'_\xi \sim y'_\xi$ for all $\xi < \kappa$. Hence, by Axiom (L7), $\oplus_{\xi<\kappa} x'_\xi \sim \oplus_{\xi<\kappa} y'_\xi$, that is, $\bigvee_{\xi<\kappa} x_\xi \sim \bigvee_{\xi<\kappa} y_\xi$.

(iii) Let $(a_\xi)_{\xi<\kappa}$ be a majorized trim lifting of $(\boldsymbol{a}_\xi)_{\xi<\kappa}$. We put $a = \bigvee_{\xi<\kappa} a_\xi$. So, $\boldsymbol{a}_\xi = \Delta(a_\xi) \leq \Delta(a)$ for all $\xi < \kappa$. Now let $\boldsymbol{b} \in S$ such that $\boldsymbol{a}_\xi \leq \boldsymbol{b}$ for all $\xi < \kappa$, and let $b \in L$ such that $\Delta(b) = \boldsymbol{b}$. By (i), $(\boldsymbol{a}_\xi)_{\xi<\kappa}$ has a trim lifting $(a'_\xi)_{\xi<\kappa}$ in $[0, b]$. In particular, $\Delta\Big(\bigvee_{\xi<\kappa} a'_\xi\Big) \leq \Delta(b)$. However, by (ii), $\Delta(a) = \Delta\Big(\bigvee_{\xi<\kappa} a'_\xi\Big)$; whence $\Delta(a) \leq \boldsymbol{b}$. So $\Delta(a) = \bigvee_{\xi<\kappa} \boldsymbol{a}_\xi$. $\square$

Corollary 4-2.9. *S satisfies Axiom* (M2).

Proof. We prove that every majorized subset X of S has a supremum. By Proposition 4-1.12, Proposition 4-1.28, and Lemma 2-4.3, every majorized finite subset of S has a supremum.

So it remains to conclude in case X is infinite. We argue by induction on the cardinality of X. Write $X = \{\boldsymbol{a}_\xi \mid \xi < \kappa\}$, where κ is the cardinality of X. By the finite case and the induction hypothesis, for all $\xi < \kappa$, the set $\{\boldsymbol{a}_\eta \mid \eta \leq \xi\}$ has a supremum, say, $\boldsymbol{b}_\xi$. Since X is majorized, so is $\{\boldsymbol{b}_\xi \mid \xi < \kappa\}$, that is, there exists $b \in L$ such that $\boldsymbol{b}_\xi \leq \Delta(b)$ for all $\xi < \kappa$. By Lemma 4-2.8(i), the family $(\boldsymbol{b}_\xi)_{\xi<\kappa}$ has a trim lifting in $[0, b]$, say, $(b_\xi)_{\xi<\kappa}$. Put $c = \bigvee_{\xi<\kappa} b_\xi$. By Lemma 4-2.8(iii), $\Delta(b)$ is the supremum of $\{\boldsymbol{b}_\xi \mid \xi < \kappa\}$, that is, the supremum of X. $\square$

4-3. Axiom (M6)

Standing hypotheses: L is an espalier, S is the dimension range of L, and $\Delta\colon L \twoheadrightarrow S$ is the canonical map.

At this point, what remains to do in order to conclude the proof of Theorem A is to establish that S satisfies Axiom (M6). We shall devote Section 4-3 to this.

In accordance with Definition 2-5.5, we state the following definition.

Definition 4-3.1. Let a, $b \in L$. We say that a is *removable* from b, in notation $a \lesssim_{\mathrm{rem}} b$, if $\Delta(a) \ll_{\mathrm{rem}} \Delta(b)$ in S. Equivalently, $a \lesssim_{\mathrm{rem}} b$, if $a \lesssim b$, and $b \lesssim a \oplus x$ implies that $b \lesssim x$, for all $x \in S$.

Notation 4-3.2. Let κ be a cardinal number, let a, $b \in L$.

(i) Let $\kappa \cdot a \sim b$ be the statement that there exists a homogeneous κ-sequence $(a_\xi)_{\xi<\kappa}$ such that

$$\oplus_{\xi<\kappa} a_\xi = b \quad \text{and} \quad a \sim a_0.$$

(ii) Let $\kappa \cdot a \lesssim b$ be the statement that there exists a homogeneous κ-sequence $(a_\xi)_{\xi<\kappa}$ such that

$$\oplus_{\xi<\kappa} a_\xi \leq b \quad \text{and} \quad a \sim a_0.$$

For example, $1 \cdot a \sim b$ (resp., $1 \cdot a \lesssim b$) means that $a \sim b$ (resp., $a \lesssim b$). Another example is that $2 \cdot a \sim a$ if and only if a is purely infinite.

Lemma 4-3.3. *Let a, $b \in L \setminus \{0\}$, let β be an infinite cardinal number. If $\beta \cdot a \lesssim b$, then there exist an infinite cardinal number $\gamma \geq \beta$ and a projection p of S such that $p \cdot a > 0$ and $\gamma \cdot (p \cdot a) \sim p \cdot b$.*

Proof. We start with a homogeneous family of β elements of $[0, b]$ all equivalent to a (modulo $\sim$), and enlarge it to a maximal such family, say, $\vec{a} = (a_\xi)_{\xi<\gamma}$, where $\gamma \geq \beta$ is an infinite cardinal number. Let $b' \in L$ such that

$$b = b' \oplus (\oplus_{\xi<\gamma} a_\xi).$$

By general comparability, there exists $p \in \operatorname{Proj} S$ such that $p \cdot b' \lesssim p \cdot a$ and $p^\perp \cdot a \lesssim p^\perp \cdot b'$. By the maximality of $\vec{a}$, $a \not\lesssim b'$, hence $p \cdot a > 0$. Now we put

$$b^* = b' \oplus (\oplus_{0<\xi<\gamma} a_\xi).$$

Since γ is an infinite cardinal and by Axiom (L7), $b \sim b^*$. Moreover, $p \cdot b' \lesssim p \cdot a_0$, so, by Lemmas 4-1.25 and 4-1.29,

$$p \cdot b \sim p \cdot b^* \lesssim p \cdot (\oplus_{\xi<\gamma} a_\xi) \leq p \cdot b.$$

Hence, by using Proposition 4-1.12 and Lemma 4-1.29,

$$p \cdot b \sim p \cdot (\oplus_{\xi<\gamma} a_\xi) = \oplus_{\xi<\gamma} (p \cdot a_\xi). \qquad \square$$

Lemma 4-3.4. *$\aleph_0 \cdot a \sim a$, for all purely infinite $a \in L$.*

Proof. By Lemma 4-2.2, we have $a = \oplus_{n<\omega} x_n$ for some homogeneous sequence $(x_n)_{n<\omega}$. Let $\omega = \bigsqcup_{n<\omega} I_n$ be an infinite partition of ω, with all the I_n infinite. Put $a_n = \oplus_{k \in I_n} x_k$, for all $n < \omega$. By Axiom (L7), $a_n \sim a$ for all n. The proof is concluded by the observation that $a = \oplus_{n<\omega} a_n$. $\square$

By replacing a bijection from $\omega \times \omega$ onto ω by a bijection from $\kappa \times \kappa$ onto κ, for any infinite cardinal κ, in the proof above, we easily obtain the following result.

LEMMA 4-3.5. *Let a, $b \in L$, let κ be an infinite cardinal number. If $b \sim \kappa \cdot a$, then $b \sim \kappa \cdot b$.*

NOTATION 4-3.6. For $a \in L$, we put $\mathrm{cc}(a) = \mathrm{cc}(\Delta(a))$ (see Definition 2-5.14).

So, $\mathrm{cc}(a) = \|a \sim 0\|^{\perp}$, for all $a \in L$. In view of Lemma 4-1.27, $\mathrm{cc}(a)$ is the smallest projection p of S such that $p \cdot a = a$.

LEMMA 4-3.7. *Let a, $b \in L$ purely infinite such that $a \lesssim_{\mathrm{rem}} b$ and $b \neq 0$. Then there exists a purely infinite $e \in L$ such that*

(i) *$e \leq b$ and $e \not\lesssim a$.*
(ii) *$e \lesssim c$, for all purely infinite $c \leq b$ such that $a \lesssim_{\mathrm{rem}} c$ and $\mathrm{cc}(e) \leq \mathrm{cc}(c)$.*

PROOF. We put $F = \{x \leq b \mid x \text{ is purely infinite and } x \not\lesssim a\}$. If $b \notin F$, then $b \lesssim a$, thus, since $a \ll_{\mathrm{rem}} b$, $b = 0$, a contradiction. So, $b \in F$. For all $x \in F$, we denote by $\nu(x)$ the least infinite cardinal number α such that $\alpha \cdot y \not\lesssim x$ for all $y \in F$. By Lemma 4-3.4, $\nu(x) \geq \aleph_1$ for all $x \in F$. We pick $e \in F$ such that $\nu(e) = \alpha$ is minimum, and we prove that this e satisfies the required conditions. Of course, (i) holds since $e \in F$.

Let $c \in L$ be purely infinite such that $a \lesssim_{\mathrm{rem}} c \leq b$ and $\mathrm{cc}(e) \leq \mathrm{cc}(c)$. Note that $\mathrm{cc}(e) > 0$ (otherwise, $e = 0$, a contradiction).

CLAIM 1. *For all $p \in (0, \mathrm{cc}(c)]$ and for every infinite cardinal number $\beta < \alpha$, there exists $q \in (0, p]$ such that $q \cdot c \sim \beta \cdot (q \cdot c)$.*

PROOF OF CLAIM. If $p \cdot c \lesssim p \cdot a$, then, since $p \cdot a \lesssim_{\mathrm{rem}} p \cdot c$ (Lemma 2-5.8(i)), $p \cdot c = 0$, which is impossible since $0 < p \leq \mathrm{cc}(c)$. So, $p \cdot c \not\lesssim p \cdot a$, so $p \cdot c \in F$. In particular, $\nu(p \cdot c) \geq \alpha > \beta$, so, by the definition of $\nu(p \cdot c)$, there exists $d \in F$ such that $\beta \cdot d \lesssim p \cdot c$. Note that $p \cdot d = d$. Hence, by Lemma 4-3.3, there are $q \in \mathrm{Proj}^* S$ and an infinite cardinal number $\gamma \geq \beta$ such that $\gamma \cdot (q \cdot d) \sim qp \cdot c$ and $q \cdot d > 0$. In particular, $qp \cdot d = q \cdot d > 0$, so we may replace q by qp, and then $\gamma \cdot (q \cdot d) \sim q \cdot c$. Therefore, by Lemma 4-3.5, $\gamma \cdot (q \cdot c) \sim q \cdot c$, with $\gamma \geq \beta$, thus, since $\gamma \geq \beta$ and by Proposition 4-1.12, $\beta \cdot (q \cdot c) \sim q \cdot c$. □ Claim 1.

CLAIM 2. *For all $p \in (0, \mathrm{cc}(c)]$, there exists $q \in (0, p]$ such that $q \cdot e \lesssim q \cdot c$.*

PROOF OF CLAIM. By general comparability, there exists a decomposition $p = p' \oplus p''$ (in $\mathrm{Proj}\, S$) such that $p' \cdot c \lesssim p' \cdot e$ and $p'' \cdot e \lesssim p'' \cdot c$. If $p'' > 0$, then we may take $q = p''$. So suppose that $p'' = 0$, so $p \cdot c \lesssim p \cdot e$. Since $p \cdot c$ is purely infinite, there exist, by Lemmas 4-3.3 and 4-3.4, $q \in \mathrm{Proj}\, S$ and an infinite cardinal β such that $\beta \cdot (qp \cdot c) \sim qp \cdot e$ and $qp \cdot c > 0$. After replacing q by qp, we have $q \in (0, p]$, with

$$\beta \cdot (q \cdot c) \sim q \cdot e \tag{4-3.1}$$

and $q \cdot c > 0$. Since $q \cdot a \lesssim_{\mathrm{rem}} q \cdot c$, we must have $q \cdot c \not\lesssim q \cdot a$, whence $q \cdot c \not\lesssim a$, and so $q \cdot c \in F$. Hence $\beta < \nu(e) = \alpha$. Hence, by Claim 1, there exists $r \in (0, q]$ such that $\beta \cdot (r \cdot c) \sim r \cdot c$. Therefore, by (4-3.1), $r \cdot e \sim \beta \cdot (r \cdot c) \sim r \cdot c$. □ Claim 2.

By Claim 2 and by Proposition 4-1.31, $\mathrm{cc}(c) \cdot e \lesssim c$. However, by assumption, $\mathrm{cc}(e) \leq \mathrm{cc}(c)$, thus $\mathrm{cc}(c) \cdot e = e$ (see Lemma 2-5.3), so $e \lesssim c$. □

And now, Axiom (M6) (recall that $\Delta(c)^{\perp} = \Delta(b)^{\perp}$ if and only if $\mathrm{cc}(c) = \mathrm{cc}(b)$; see Definition 2-5.14).

PROPOSITION 4-3.8. *Let a, $b \in L$ be purely infinite such that $a \lesssim_{\mathrm{rem}} b$. Then there exists a purely infinite $e \leq b$ such that*

(i) *$a \lesssim_{\mathrm{rem}} e$ and $\mathrm{cc}(e) = \mathrm{cc}(b)$.*
(ii) *$e \lesssim c$, for all purely infinite $c \in L$ such that $a \lesssim_{\mathrm{rem}} c$ and $\mathrm{cc}(c) = \mathrm{cc}(b)$.*

PROOF. We first claim that it will suffice to find a purely infinite element $e \leq b$ satisfying (i) and the the statement

(i′) $e \lesssim c$, for all purely infinite $c \leq b$ such that $a \lesssim_{\mathrm{rem}} c$ and $\mathrm{cc}(c) = \mathrm{cc}(b)$.

Indeed, suppose that (i) and (ii′) are satisfied. Let $c \in L$ such that $a \lesssim_{\mathrm{rem}} c$ and $\mathrm{cc}(c) = \mathrm{cc}(b)$. Then $\Delta(a) \lesssim_{\mathrm{rem}} \Delta(b), \Delta(c)$, and so $\Delta(a) \lesssim_{\mathrm{rem}} \Delta(b) \wedge \Delta(c)$ by Corollary 2-5.9. There exists $d \leq b$ such that $\Delta(d) = \Delta(b) \wedge \Delta(c)$, whence $a \lesssim_{\mathrm{rem}} d \lesssim c$. Moreover, d is purely infinite by Lemma 2-5.11, and $\mathrm{cc}(d) = \mathrm{cc}(b) \wedge \mathrm{cc}(c) = \mathrm{cc}(b)$ by Lemma 2-5.16(ii). Since any element $e \lesssim d$ would then satisfy $e \lesssim c$, the claim is proved.

Let P be the set of all pairs $(p, x) \in (\operatorname{Proj} S) \times L$ such that x is purely infinite and the following conditions hold:

(a) $p \leq \mathrm{cc}(b)$ and $x \leq p \cdot b$.
(b) $p \cdot a \lesssim_{\mathrm{rem}} x$ and $\mathrm{cc}(x) = p$.
(c) For all purely infinite $y \leq p \cdot b$, the conditions $p \cdot a \lesssim_{\mathrm{rem}} y$ and $\mathrm{cc}(y) = p$ imply that $x \lesssim y$.

Let $\{(p_i, x_i) \mid i \in I\}$ be a subset of P, maximal with the property that the p_i are nonzero and pairwise orthogonal. We observe that since $\{x_i \mid i \in I\}$ is majorized (by b) and since the p_i are pairwise orthogonal, it follows from Corollary 4-1.20 that $(x_i)_{i \in I}$ is an orthogonal family of L. We put

$$p = \bigvee_{i \in I} p_i \quad \text{and} \quad x = \oplus_{i \in I} x_i.$$

CLAIM 1. *The pair (p, x) belongs to P.*

PROOF OF CLAIM. Observe that $\mathrm{cc}(x) = p \leq \mathrm{cc}(b)$ and $x \leq p \cdot b$. Since all the x_i are purely infinite, x is purely infinite. Furthermore, $p_i \cdot a \lesssim_{\mathrm{rem}} x_i \leq x$ for all i, thus, by Lemma 2-5.6, $p_i \cdot a \lesssim_{\mathrm{rem}} x$. This holds for all i, thus, by Lemma 2-5.8(ii) and Proposition 3-4.2, $p \cdot a \lesssim_{\mathrm{rem}} x$.

Let $y \leq p \cdot b$ be a purely infinite element of L such that $p \cdot a \lesssim_{\mathrm{rem}} y$ and $\mathrm{cc}(y) = p$. For all $i \in I$, $p_i \cdot a \lesssim_{\mathrm{rem}} p_i \cdot y$ and, by Lemma 2-5.16(ii), $\mathrm{cc}(p_i \cdot y) = p_i$, so $p_i \cdot x = x_i \lesssim p_i \cdot y$. This holds for all i, thus $x = p \cdot x \lesssim y$. □ Claim 1.

So, it suffices to prove that $p = \mathrm{cc}(b)$. Until the end of the proof, we suppose otherwise. Put $q = \mathrm{cc}(b)p^{\perp} > 0$. Since $0 < q \leq \mathrm{cc}(b)$ and $a \lesssim_{\mathrm{rem}} b$, the relation $q \cdot b \not\lesssim q \cdot a$ holds. Since $q \cdot a \lesssim_{\mathrm{rem}} q \cdot b$, there exists, by Lemma 4-3.7, a purely infinite $x^* \leq q \cdot b$ such that

(α) $x^* \not\lesssim q \cdot a$;
(β) $x^* \lesssim y^*$, for all purely infinite $y^* \leq q \cdot b$ such that $q \cdot a \lesssim_{\mathrm{rem}} y^*$ and $\mathrm{cc}(x^*) \leq \mathrm{cc}(y^*)$.

Now put $r = \|x^* \lesssim a\|$. So, by definition, $r \cdot x^* \lesssim r \cdot a$. If $r \cdot x^* = x^*$, then $x^* \lesssim r \cdot a$, but $x^* \lesssim q \cdot b$, so $x^* = q \cdot x^* \lesssim qr \cdot a \lesssim q \cdot a$, which contradicts ($\alpha$) above. So, $r \cdot x^* \neq x^*$, thus, since $x^* \in \mathrm{cc}(b)S$, the projection $p' = \mathrm{cc}(b)r^{\perp}$ is nonzero.

Now, from $pq = 0$ it follows that $p \cdot x^* = 0$, so $p \leq r$, that is, $p \perp p'$. Hence, $p' = qr^\perp$. Moreover, $\mathrm{cc}(b)\,\mathrm{cc}(x^*)^\perp \leq \mathrm{cc}(x^*)^\perp \leq r$, whence, taking complements in $\mathrm{cc}(b)$, $p' \leq \mathrm{cc}(x^*)$. In particular, $\mathrm{cc}(p' \cdot x^*) = p'$.

By general comparability, there exists $g \leq \mathrm{cc}(b)$ in $\operatorname{Proj} S$ such that $g \cdot a \lesssim g \cdot x^*$ and $g^\perp \cdot x^* \lesssim g^\perp \cdot a$. Then $\mathrm{cc}(b)g^\perp \leq r$, whence $p' \leq g$, and so $p' \cdot a \lesssim p' \cdot x^*$. By the definition of r and of p', $s \cdot x^* \not\lesssim s \cdot a$, for all $s \in (0, p']$, thus, by Lemma 2-5.13, $p' \cdot a \lesssim_{\mathrm{rem}} p' \cdot x^*$.

Consider a purely infinite $c' \leq p' \cdot b$ such that $p' \cdot a \lesssim_{\mathrm{rem}} c'$ and $\mathrm{cc}(c') = p'$. Since $c' \leq p' \cdot b$ and $qr \perp p'$, the element $c^* = c' \oplus qr \cdot b$ is defined and $c^* \leq (p' \vee qr) \cdot b = q \cdot b$. Furthermore, since $q \leq \mathrm{cc}(b)$, $\mathrm{cc}(c^*) = p' \vee qr = q$. Since $p' \cdot a \lesssim_{\mathrm{rem}} c'$ and $a \lesssim_{\mathrm{rem}} b$, it follows from Lemma 2-5.8 that $q \cdot a \lesssim_{\mathrm{rem}} c^*$.

Therefore, by part (β) of the definition of x^*, $x^* \lesssim c^*$, thus $p' \cdot x^* \lesssim p' \cdot c^* = c'$. So we have proved that $(p', p' \cdot x^*) \in P$, with p' nonzero and orthogonal to all the p_i for $i \in I$, which contradicts the maximality of $\{(p_i, x_i) \mid i \in I\}$. So, $p = \mathrm{cc}(b)$. □

Proposition 4-3.8 concludes the proof of Theorem A. A more complete form of Theorem A is the following.

Theorem 4-3.9. *Let $(L, \leq, \perp, \sim)$ be an espalier. Then the quotient* $\operatorname{Drng} L = L/{\sim}$ *can be endowed with a partial addition $+$, defined by the rule*

$$\Delta(c) = \Delta(a) + \Delta(b), \quad \textit{for all } a,\ b,\ c \in L \textit{ such that } c = a \oplus b,$$

that makes it a continuous dimension scale, with zero element $\Delta(0)$.

4-4. D-universal classes of espaliers

One of the questions that we shall regularly encounter throughout the study of various classes of espaliers, in Chapter 5, will be what are the possible dimension ranges of members of a given class of espaliers.

Definition 4-4.1. A class $\mathcal{E}$ of espaliers is *D-universal*, if every continuous dimension scale admits a lower embedding into the dimension range of some member of $\mathcal{E}$.

We recall that the class of espaliers is closed under so-called *lower subespaliers*, and also under *direct products* of espaliers, see Proposition 4-1.2. The following lemma records some elementary facts about these notions. We leave its easy proof to the reader.

Lemma 4-4.2.

(i) *Let K and L be espaliers, let $\varphi\colon K \to L$ be a lower embedding (see Lemma* 4-1.3*). Then the rule $\Delta_K(x) \mapsto \Delta_L(\varphi(x))$ defines a lower embedding from* $\operatorname{Drng} K$ *into* $\operatorname{Drng} L$.

(ii) *Let $(L, \leq, \perp, \sim)$ be an espalier, let S be a lower subset of* $\operatorname{Drng} L$, *put*

$$K = \{x \in L \mid \Delta_L(x) \in S\}.$$

Then K is a lower subespalier of L, and the rule $\Delta_K(x) \mapsto \Delta_L(x)$ defines an isomorphism from $\operatorname{Drng} K$ *onto S.*

(iii) *Let $(L_i)_{i \in I}$ be a family of espaliers, let $L = \prod_{i \in I} L_i$ be its direct product. Then the rule $(\Delta_{L_i}(x_i))_{i \in I} \mapsto \Delta_L((x_i)_{i \in I})$ defines an isomorphism from* $\prod_{i \in I} \operatorname{Drng} L_i$ *onto* $\operatorname{Drng} L$.

In the context of Lemma 4-4.2(i), we shall of course write $\operatorname{Drng}\varphi\colon \operatorname{Drng}K \to \operatorname{Drng}L$ to denote the map that sends $\Delta_K(x)$ to $\Delta_L(\varphi(x))$, for every $x \in K$.

As a consequence of Lemma 4-4.2, the dimension ranges of members of D-universal classes of espaliers can be nearly anything reasonable.

Proposition 4-4.3. *Let $\mathcal{E}$ be a D-universal class of espaliers.*

(i) *If every bounded lower subespalier of every member of $\mathcal{E}$ belongs to $\mathcal{E}$, then every bounded continuous dimension scale is isomorphic to the dimension range of some bounded member of $\mathcal{E}$.*
(ii) *If every lower subespalier of every member of $\mathcal{E}$ belongs to $\mathcal{E}$, then every continuous dimension scale is isomorphic to the dimension range of some member of $\mathcal{E}$.*

Proof. (i) Let S be a bounded continuous dimension scale, denote by $\boldsymbol{a}$ the largest element of S. Since $\mathcal{E}$ is D-universal, there exists $L \in \mathcal{E}$ such that S is (isomorphic to) a lower subset of $\operatorname{Drng}L$. Let $a \in L$ such that $\Delta_L(a) = \boldsymbol{a}$, put $K = (a]$, a lower subespalier of L. It follows from the assumption and Lemma 4-4.2(i) that K belongs to $\mathcal{E}$ and $\operatorname{Drng}K$ is isomorphic to S. Observe that K is bounded.

(ii) Let S be a continuous dimension scale. Since $\mathcal{E}$ is D-universal, there exists $L \in \mathcal{E}$ such that S is (isomorphic to) a lower subset of $\operatorname{Drng}L$. Put $K = \{x \in L \mid \Delta_L(x) \in S\}$. It follows from the assumption and Lemma 4-4.2(ii) that K belongs to $\mathcal{E}$ and $\operatorname{Drng}K$ is isomorphic to S. □

The following result gives us a sufficient condition for D-universality.

Lemma 4-4.4. *Let $\mathcal{E}$ be a class of espaliers satisfying the following conditions:*

(i) *$\mathcal{E}$ is closed under finite direct products.*
(ii) *For every ordinal γ and every complete Boolean space Ω, there are L_{I}, L_{II}, $L_{\mathrm{III}} \in \mathcal{E}$ such that $\mathbf{C}(\Omega, \mathbb{Z}_\gamma)$ has a lower embedding into $\operatorname{Drng}L_{\mathrm{I}}$, $\mathbf{C}(\Omega, \mathbb{R}_\gamma)$ has a lower embedding into $\operatorname{Drng}L_{\mathrm{II}}$, and $\mathbf{C}(\Omega, \mathbf{2}_\gamma)$ has a lower embedding into $\operatorname{Drng}L_{\mathrm{III}}$.*

Then $\mathcal{E}$ is D-universal.

Proof. It follows from Lemma 4-4.2 and assumptions (i), (ii) above that for every ordinal γ and any complete Boolean spaces Ω_{I}, Ω_{II}, and Ω_{III}, there exists $L \in \mathcal{E}$ such that the continuous dimension scale

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma)$$

embeds into $\operatorname{Drng}L$. The conclusion follows from Theorem 3-8.9. □

The results of Section 4-4 will make it possible to prove further results of D-universality.

4-5. Existence of large constants

Taking account of the various examples of espaliers discussed in the Introduction, one is led to the conjecture that the appearance of large cardinal values in the functional representation of the dimension range of an espalier should be closely related to the existence of certain large orthogonal sums within the espalier. (See also the proof of Lemma 4-3.7.) Moreover, in the construction of espaliers of different types, we will need to know what ingredients will ensure that the dimension range of an example will be as large as desired. In the present section, we provide some answers to the above questions.

> *Standing hypotheses: L is an espalier, S is the dimension range of L, and $\Delta\colon L \twoheadrightarrow S$ is the canonical map. Moreover, Ω, Ω_{I}, Ω_{II}, Ω_{III} are as in Section* 3-7.
>
> *Let γ be the ordinal and $\mu\colon S \to \mathbf{C}(\Omega, \mathbf{2}_\gamma)$ the dimension function defined in Section* 3-6.
>
> *We put $\overline{S} = \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma; \Omega_{\mathrm{II}}, \mathbb{R}_\gamma; \Omega_{\mathrm{III}}, \mathbf{2}_\gamma)$. We pick a lower embedding $\delta\colon S_{\mathrm{fin}} \hookrightarrow \overline{S}$ as in Proposition* 3-7.9. *Let $\varepsilon\colon S \hookrightarrow \overline{S}$ be the corresponding lower embedding defined in Section* 3-8.

Lemma 4-5.1. *Let $a, b \in L$ be purely infinite elements with $a \lesssim b$. Suppose that there is an infinite cardinal κ such that $\kappa \cdot b \lesssim b$ but $\kappa \cdot c \not\lesssim a$ for all nonzero $c \in L$. Then $a \lesssim_{\mathrm{rem}} b$.*

Proof. Suppose $x \in L$ and $b \lesssim a \oplus x$. Because of general comparability in L (Proposition 4-1.28 and comment following), there is some $p \in \operatorname{Proj} S$ such that $p \cdot a \lesssim p \cdot x$ and $p^\perp \cdot x \lesssim p^\perp \cdot a$. Now

$$p^\perp \cdot b \lesssim (p^\perp \cdot a) \oplus (p^\perp \cdot x) \lesssim p^\perp \cdot (2 \cdot a) \sim p^\perp \cdot a$$

by Lemmas 4-1.25 and 4-1.29. Moreover, these lemmas imply $\kappa \cdot (p^\perp \cdot b) \lesssim p^\perp \cdot b$, and so we have $\kappa \cdot (p^\perp \cdot b) \lesssim a$. Our assumptions on a now imply that $p^\perp \cdot b = 0$, and hence $b = p \cdot b$ by Lemma 4-1.27.

Since $p \cdot a$ is purely infinite and $p \cdot a \lesssim p \cdot x$, we have $(p \cdot a) \oplus (p \cdot x) \sim p \cdot x$, and so

$$b = p \cdot b \lesssim (p \cdot a) \oplus (p \cdot x) \sim p \cdot x \leq x.$$

Therefore $a \lesssim_{\mathrm{rem}} b$. □

In many examples of espaliers, the orthogonality relation coincides with disjointness in the (partial) lattice: $a \perp b \iff a \wedge b = 0$. Let us abbreviate this condition by the symbol $(\perp = \wedge 0)$.

Lemma 4-5.2. *Assume $(\perp = \wedge 0)$. Let $x \in L$ be purely infinite, and let η be an infinite cardinal such that x is not equal to any orthogonal sum of more than η nonzero elements. Let $y \in L$ and let $\beta \geq \eta$ be a cardinal number such that $\beta \cdot x \sim y$.*

Then y does not majorize any orthogonal sum of more than β nonzero elements. In particular, $\alpha \cdot u \not\lesssim y$ for all $\alpha > \beta$ and all nonzero $u \in L$.

Proof. By assumption, $y = \oplus_{i \in I} y_i$ with $|I| = \beta$ and each $y_i \sim x$. Suppose that $y \geq \oplus_{j \in J} z_j$ where $|J| > \beta$ and all $z_j \neq 0$. Since $z_j \wedge y \neq 0$, we have $z_j \not\perp y$. Axiom (L4) then yields a finite subset $I_j \subset I$ such that $z_j \not\perp \oplus_{i \in I_j} y_i$. Since the set of finite subsets of I has cardinality β, the fibres of the map $j \mapsto I_j$ cannot all have cardinality at most β. Hence, there exist a subset $J' \subseteq J$ with $|J'| > \beta$ and a finite subset $I' \subset I$ such that $I_j = I'$ for all $j \in J'$. Thus, the element $y^* = \oplus_{i \in I'} y_i$ satisfies $z_j \not\perp y^*$, and so $z_j \wedge y^* \neq 0$, for all $j \in J'$, because of $(\perp = \wedge 0)$. Consequently, y^* majorizes an orthogonal sum of more than β nonzero elements, and after adjoining an additional element if necessary, we may assume that y^* equals such an orthogonal sum. However, $y^* \sim x$ because x is purely infinite, and so Axiom (L6) implies that x is an orthogonal sum of more than β nonzero elements. This contradicts our hypotheses. □

Lemma 4-5.3. *Assume* $(\perp = \wedge 0)$. *Let* $x \in L$ *be purely infinite, put* $p = \mathrm{cc}(x)$, *and let* σ *be an ordinal such that* x *is not equal to any orthogonal sum of more than* $\aleph_\sigma$ *nonzero elements. Let* $y \in L$ *and let* τ *be an ordinal with* $\aleph_{\sigma+\tau} \cdot x \sim y$. *Then* $\langle p \mid \aleph_\tau\rangle$ *is defined, and* $\langle p \mid \aleph_\tau\rangle \leq \Delta(y)$.

Proof. We proceed by induction on τ.

Assume first that $\tau = 0$. The set $X = \{a \in S|_\infty \mid \mathrm{cc}(a) = p\}$ is nonempty, as it contains $\Delta(x)$. Since the element $\langle p \mid 0\rangle = 0$ is removable from any element of L, the element $\langle p \mid \aleph_0\rangle$ is defined as the least element of X. Thus, $\langle p \mid \aleph_0\rangle \leq \Delta(x) \leq \Delta(y)$.

Next, suppose that $\tau = \rho + 1$ for some ordinal ρ. There is some $z \leq y$ such that $\aleph_{\sigma+\rho} \cdot x \sim z$. By induction, $\langle p \mid \aleph_\rho\rangle$ is defined, and $\langle p \mid \aleph_\rho\rangle \leq \Delta(z)$. Now $\langle p \mid \aleph_\rho\rangle = \Delta(a)$ for some purely infinite $a \leq z$, and Lemma 4-5.2 shows that $\aleph_{\sigma+\tau} \cdot u \not\lesssim z$ for all nonzero $u \in L$. On the other hand, since $\aleph_{\sigma+\tau} \cdot x \sim y$, we have $\aleph_{\sigma+\tau} \cdot y \sim y$. Hence, $z \lnsim_{\mathrm{rem}} y$ by Lemma 4-5.1, and so $a \lnsim_{\mathrm{rem}} y$. Therefore $\langle p \mid \aleph_\rho\rangle \ll_{\mathrm{rem}} \Delta(y)$. Since $\Delta(y)$ is a purely infinite element with central cover p, it follows that $\langle p \mid \aleph_\tau\rangle$ is defined and majorized by $\Delta(y)$.

Finally, suppose that τ is a limit ordinal. For each ordinal $\rho < \tau$, there exists $y_\rho \leq y$ such that $\aleph_{\sigma+\rho} \cdot x \sim y_\rho$. By induction, $\langle p \mid \aleph_\rho\rangle$ is defined and $\langle p \mid \aleph_\rho\rangle \leq \Delta(y_\rho) \leq \Delta(y)$. Therefore $\langle p \mid \aleph_\tau\rangle$ is defined, and $\langle p \mid \aleph_\tau\rangle = \bigvee_{\rho<\tau} \langle p \mid \aleph_\rho\rangle \leq \Delta(y)$. ☐

Proposition 4-5.4. *Assume* $(\perp = \wedge 0)$. *Let* x, $y \in L$ *be purely infinite elements such that* $\mathrm{cc}(x) = \mathrm{cc}(y) = 1$, *and let* σ, τ *be ordinals, such that* x *is not equal to any orthogonal sum of more than* $\aleph_\sigma$ *nonzero elements, and* $\aleph_{\sigma+\tau} \cdot x \sim y$. *Then the following statements hold:*

(i) $\mu(\Delta(y))(\mathfrak{a}) \geq \aleph_\tau$ *for all* $\mathfrak{a} \in \Omega$.

(ii) *There exists a purely infinite element* $u_\tau \in L$ *such that* $\mu(\Delta(u_\tau))$ *equals the constant function with value* $\aleph_\tau$.

(iii) *Set* $L_\tau = [0, u_\tau] \subseteq L$, *and restrict* $\leq$, $\perp$, $\sim$ *from* L *to* L_τ. *Then* L_τ *is an espalier, and* $\mathrm{Drng}\, L_\tau \cong \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\tau; \Omega_{\mathrm{II}}, \mathbb{R}_\tau; \Omega_{\mathrm{III}}, \mathbf{2}_\tau)$.

Proof. (i) In view of Lemma 4-5.3, $\langle 1 \mid \aleph_\tau\rangle$ is defined and majorized by $\Delta(y)$. Since $1 \in \mathfrak{a}$ for all $\mathfrak{a} \in \Omega$, we get $\mu(\Delta(y))(\mathfrak{a}) \geq \aleph_\tau$ for all $\mathfrak{a}$.

(ii) Because of (i), the lower embedding $\varepsilon : S \hookrightarrow \overline{S}$ sends $\Delta(y)$ to a function $f \in \overline{S}$ with $f(\mathfrak{a}) \geq \aleph_\tau$ for all $\mathfrak{a} \in \Omega$. In particular, $\tau \leq \gamma$, and $\overline{S}$ contains the constant function t_τ with $t_\tau(\mathfrak{a}) = \aleph_\tau$ for all $\mathfrak{a} \in \Omega$. Since ε is a lower embedding, there is some $w_\tau \in S$ such that $\varepsilon(w_\tau) = t_\tau$. Note that w_τ is purely infinite, because t_τ is. Hence, $\mu(w_\tau) = \varepsilon(w_\tau) = t_\tau$. It just remains to note that $w_\tau = \Delta(u_\tau)$ for some purely infinite element $u_\tau \in L$.

(iii) That L_τ is an espalier follows from Proposition 4-1.2(i). It is clear that $\mathrm{Drng}\, L_\tau$ is isomorphic to the submonoid $S_\tau = [0, \Delta(u_\tau)] \subseteq S$. Since ε is a lower embedding, it maps S_τ isomorphically onto $\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\tau; \Omega_{\mathrm{II}}, \mathbb{R}_\tau; \Omega_{\mathrm{III}}, \mathbf{2}_\tau)$. ☐

In case $(\perp = \wedge 0)$ does not hold, it is not clear whether large orthogonal sums are sufficient to imply large constants. For use in that situation, we record the following more elementary approach.

Lemma 4-5.5. *Let* κ *be an infinite cardinal, and* $(b_\xi)_{\aleph_0 \leq \xi \leq \kappa}$ *a family of purely infinite elements of* S *(indexed by infinite cardinals). Set* $p = \mathrm{cc}(b_{\aleph_0})$. *For all infinite cardinals* $\xi < \eta \leq \kappa$, *assume that* $b_\xi \leq b_\eta$ *but* $q(b_\eta) \nleq q(b_\xi)$ *for all nonzero projections* $q \leq p$. *Then* $\langle p \mid \kappa\rangle$ *is defined, and* $\langle p \mid \kappa\rangle \leq b_\kappa$.

PROOF. We show, by induction on ξ, that $\langle p \mid \xi\rangle$ is defined and majorized by b_ξ, for all infinite cardinals $\xi \le \kappa$. Since $b_{\aleph_0}$ is a purely infinite element with central cover p, it is clear from the definition that $\langle p \mid \aleph_0\rangle$ is defined and $\langle p \mid \aleph_0\rangle \le b_{\aleph_0}$.

Next, suppose that ξ is an infinite cardinal less than κ, such that the element $a = \langle p \mid \xi\rangle$ is defined and $a \le b_\xi$. Note that the elements a and $p(b_{\xi^+})$ both have central cover p. By assumption, $q(b_{\xi^+}) \not\lesssim q(a)$ for all nonzero projections $q \le p$, whence Corollary 2-5.15 implies that $\langle p \mid \xi\rangle = a \ll_{\mathrm{rem}} p(b_{\xi^+})$. Thus, $\langle p \mid \xi^+\rangle$ is defined and $\langle p \mid \xi^+\rangle \le p(b_{\xi^+}) \le b_{\xi^+}$.

Finally, if ξ is a limit cardinal less than or equal to κ, such that $\langle p \mid \eta\rangle$ is defined and majorized by b_η for all infinite cardinals $\eta < \xi$, then $\langle p \mid \eta\rangle \le b_\xi$ for all η, whence $\langle p \mid \xi\rangle = \bigvee_{\aleph_0 \le \eta < \xi} \langle p \mid \eta\rangle$ is defined and $\langle p \mid \xi\rangle \le b_\xi$. □

PROPOSITION 4-5.6. *Let τ be an ordinal and $(x_\alpha)_{\alpha\le\tau}$ a family of purely infinite elements of L with central cover 1. For all ordinals $\alpha < \beta \le \tau$, assume that $x_\alpha \lesssim x_\beta$ but $q \cdot x_\beta \not\lesssim q \cdot x_\alpha$ for all nonzero projections $q \in \operatorname{Proj} S$. Then the following statements hold:*

(i) *$\mu(\Delta(x_\tau))(\mathfrak{a}) \ge \aleph_\tau$ for all $\mathfrak{a} \in \Omega$.*
(ii) *There exists a purely infinite element $u_\tau \in L$ such that $\mu(\Delta(u_\tau))$ equals the constant function with value $\aleph_\tau$.*
(iii) *Set $L_\tau = [0, u_\tau] \subseteq L$, and restrict $\le$, $\perp$, $\sim$ from L to L_τ. Then L_τ is an espalier, and $\operatorname{Drng} L_\tau \cong \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\tau; \Omega_{\mathrm{II}}, \mathbb{R}_\tau; \Omega_{\mathrm{III}}, \mathbf{2}_\tau)$.*

PROOF. (i) Set $b_{\aleph_\alpha} = \Delta(x_\alpha)$ for all ordinals $\alpha \le \tau$. Then $(b_\xi)_{\aleph_0\le\xi\le\aleph_\tau}$ is a family of purely infinite elements of S with central cover 1, such that for all infinite cardinals $\xi < \eta \le \aleph_\tau$, we have $b_\xi \le b_\eta$ but $q(b_\eta) \not\lesssim q(b_\xi)$ for all nonzero $q \in \operatorname{Proj} S$. Thus, by Lemma 4-5.5, $\langle 1 \mid \aleph_\tau\rangle$ is defined and $\langle 1 \mid \aleph_\tau\rangle \le \Delta(x_\tau)$. Since $1 \in \mathfrak{a}$ for all $\mathfrak{a} \in \Omega$, we get $\mu(\Delta(x_\tau))(\mathfrak{a}) \ge \aleph_\tau$ for all $\mathfrak{a}$.

(ii) and (iii) follow from (i) just as in Proposition 4-5.4. □

CHAPTER 5

Classes of espaliers

5-1. Abstract measure theory; Boolean espaliers

Definition 5-1.1. An espalier $(L, \leq, \perp, \sim)$ is *Boolean*, if L is a Boolean lattice and $x \perp y$ if and only if $x \wedge y = 0$, for all x, $y \in L$.

Of course, the underlying Boolean algebra of a Boolean espalier is complete. For a Boolean algebra B, we will denote by $\perp_B$ the canonical orthogonality relation of B, that is, $x \perp_B y$ if and only if $x \wedge y = 0$, for all x, $y \in B$. We say that a family $(a_i)_{i\in I}$ of elements of B is *disjoint*, if $a_i \wedge a_j = 0$ for all $i \neq j$ in I, and then we let $\oplus_{i\in I} a_i$ denote its join.

Many of the axioms defining the class of espaliers do not need checking in the Boolean case.

Proposition 5-1.2. *Let B be a complete Boolean algebra, let $\sim$ be a binary relation on B. Then $(B, \leq_B, \perp_B, \sim)$ is a Boolean espalier if and only if the following conditions hold:*

(B0) $x \sim 0$ *implies that* $x = 0$, *for all* $x \in B$.

(B1) *The binary relation $\sim$ is unrestrictedly refining, that is, for every $a \in B$ and every disjoint family $(b_i)_{i\in I}$ of elements of B, if $a \sim \oplus_{i\in I} b_i$, then there exists a decomposition $a = \oplus_{i\in I} a_i$ such that $a_i \sim b_i$ for all $i \in I$.*

(B2) *The binary relation $\sim$ is unrestrictedly additive, that is, for all disjoint families $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ of elements of B, if $a_i \sim b_i$ for all $i \in I$, then $\oplus_{i\in I} a_i \sim \oplus_{i\in I} b_i$.*

We leave to the reader the straightforward proof of Proposition 5-1.2.

Boolean espaliers can often be constructed from the following objects.

Definition 5-1.3. A *Boolean pre-espalier* is a pair $(B, \sim)$, where B is a Boolean algebra and $\sim$ is an equivalence relation on B satisfying Axioms (B0) and (B1).

We observe that the underlying Boolean algebra of a Boolean pre-espalier need not be complete. We recall (see, for example, T. Jech [**27**]) that for every Boolean algebra B, there exists a unique (up to isomorphism) complete Boolean algebra, that we shall denote by $\overline{B}$ and call the *completion* of B, such that B is dense in $\overline{B}$. The following result makes it possible to extend to $\overline{B}$ any Boolean pre-espalier structure on B.

Lemma 5-1.4. *Let $(B, \sim)$ be a Boolean pre-espalier. Define a binary relation $\sim^*$ on $\overline{B}$ by the rule*

$$x \sim^* y \Longleftrightarrow \text{there are are decompositions } x = \oplus_{i\in I} x_i,\ y = \oplus_{i\in I} y_i \\ \text{such that } x_i,\ y_i \in B \text{ and } x_i \sim y_i,\ \text{for all } i \in I, \quad (5\text{-}1.1)$$

for all x, $y \in \overline{B}$. *Then* $(\overline{B}, \sim^*)$ *is a Boolean espalier. Furthermore,* $\sim^*$ *is the smallest equivalence relation* $\sim'$ *on* $\overline{B}$ *containing* $\sim$ *such that* $(\overline{B}, \sim')$ *is an espalier.*

We shall call $\sim^*$ the *espalier closure* of $\sim$.

Proof. It is clear that every equivalence relation $\sim'$ on $\overline{B}$ containing $\sim$, such that $(\overline{B}, \sim')$ is an espalier, also contains $\sim^*$, hence it suffices to prove that $(\overline{B}, \sim^*)$ is an espalier.

Since every element of $\overline{B}$ can be written $\oplus_{i \in I} x_i$, where all the x_i-s belong to B, the binary relation $\sim^*$ is reflexive. It is obviously symmetric. Now let a, b, $c \in \overline{B}$ such that $a \sim^* b$ and $b \sim^* c$. There are decompositions of the form

$$a = \oplus_{i \in I} a_i,\ b = \oplus_{i \in I} b'_i = \oplus_{j \in J} b''_j,\ c = \oplus_{j \in J} c_j,$$

with $a_i \sim b'_i$ in B, for all $i \in I$, and $b''_j \sim c_j$, for all $j \in J$. For any $i \in I$, $a_i \sim b'_i = \oplus_{j \in J}(b'_i \wedge b''_j)$, thus, since $\sim$ satisfies (B1), there exists a decomposition $a_i = \oplus_{j \in J} a_{i,j}$ with $a_{i,j} \sim b'_i \wedge b''_j$, for all $j \in J$. For $j \in J$, since $c_j \sim b''_j = \oplus_{i \in I}(b'_i \wedge b''_j)$ and by (B1), there exists a decomposition $c_j = \oplus_{i \in I} c_{i,j}$ such that $b'_i \wedge b''_j \sim c_{i,j}$, for all $i \in I$. Therefore, $a_{i,j} \sim c_{i,j}$, for all $(i,j) \in I \times J$, and $a = \oplus_{(i,j) \in I \times J} a_{i,j}$ and $c = \oplus_{(i,j) \in I \times J} c_{i,j}$; whence $a \sim^* c$. Therefore, $\sim^*$ is an equivalence relation on $\overline{B}$. It is obvious that $\sim^*$ satisfies (B0).

Now let $a \sim^* \oplus_{i \in I} b_i$ in $\overline{B}$. By definition, there are decompositions $a = \oplus_{j \in J} a'_j$ and $\oplus_{i \in I} b_i = \oplus_{j \in J} b'_j$ such that $a'_j \sim b'_j$, for all $j \in J$. For $j \in J$, since $a'_j \sim b'_j = \oplus_{i \in I}(b_i \wedge b'_j)$, there exists a decomposition $a'_j = \oplus_{i \in I} a_{i,j}$ such that $a_{i,j} \sim b_i \wedge b'_j$, for all $i \in I$. Observe that $a = \oplus_{(i,j) \in I \times J} a_{i,j}$; put $a_i = \oplus_{j \in J} a_{i,j}$, for all $i \in I$. Thus $a = \oplus_{i \in I} a_i$, and, by the definition of $\sim^*$, $a_i \sim^* \oplus_{j \in J}(b_i \wedge b'_j) = b_i$, for all $i \in I$. Therefore, $\sim^*$ satisfies (B1).

Finally let $a = \oplus_{i \in I} a_i$ and $b = \oplus_{i \in I} b_i$ with $a_i \sim^* b_i$, for all $i \in I$. By definition, for all $i \in I$, there are decompositions $a_i = \oplus_{j \in J_i} a_{i,j}$ and $b_i = \oplus_{j \in J_i} b_{i,j}$ such that $a_{i,j} \sim b_{i,j}$, for all $i \in I$ and all $j \in J_i$. Put $J = \bigcup_{i \in I}(\{i\} \times J_i)$, then $a = \oplus_{(i,j) \in J} a_{i,j}$ and $b = \oplus_{(i,j) \in J} b_{i,j}$, whence $a \sim^* b$. Therefore, $\sim^*$ satisfies (B2). ☐

For a Boolean espalier $(B, \sim)$ and a set I, we let the permutation group $\mathfrak{S}_I$ of I act on the Boolean algebra B^I by translation: namely,

$$(\sigma x)(i) = x(\sigma^{-1}(i)),\ \text{for all } x \in B^I \text{ and all } \sigma \in \mathfrak{S}_I.$$

Next, let $\sim_I$ be the equivalence relation on B^I associated with this action and $\sim$, that is,

$$x \sim_I y \Leftrightarrow \exists \sigma \in \mathfrak{S}_I \text{ such that } y(i) \sim x(\sigma(i)),\ \text{for all } x,\ y \in B^I.$$

Since $\mathfrak{S}_I$ acts on B^I by automorphisms (of the Boolean algebra B^I) and $(B, \sim)$ is an espalier, it is easy to see that $(B^I, \sim_I)$ is a Boolean pre-espalier. Since B^I is already complete, the espalier closure of $(B^I, \sim_I)$ is an equivalence relation on B^I, that we shall denote by $\sim^I$. For $i \in I$, we denote by $\varphi_i \colon B \hookrightarrow B^I$ the canonical map, that is, $\varphi_i(x)(j)$ is equal to x if $i = j$, to 0 otherwise, for all $x \in B$ and all $j \in J$.

Lemma 5-1.5. *The following statements hold, for any* $i \in I$*:*

(i) *The map* φ_i *is a lower embedding of espaliers.*
(ii) *The map* $\operatorname{Drng} \varphi_i$ *is a lower embedding with dense image from* $B/{\sim}$ *into* $B^I/{\sim^I}$.

PROOF. (i) It is obvious that φ_i is a $(\leq, \perp)$-isomorphism from B onto a lower subset of B^I, and that $x \sim y$ implies that $\varphi_i(x) \sim^I \varphi_i(y)$, for all x, $y \in B$. Now suppose that $\varphi_i(x) \sim^I \varphi_i(y)$. There are decompositions of the form $x = \oplus_{j\in J} x_j$ and $y = \oplus_{j\in J} y_j$ in B such that $\varphi_i(x_j) \sim_I \varphi_i(y_j)$, for all $j \in J$. Hence $x_j \sim y_j$, for all $j \in J$, whence, since $(B, \sim)$ satisfies (B2), $x \sim y$. This completes the proof of (i).

Let $\boldsymbol{a} \in B^I/{\sim^I}$ be nonzero; so $\boldsymbol{a} = \Delta(a)$, for some $a \in B^I \setminus \{0\}$. Since a has a nonzero component, there are $b \in B \setminus \{0\}$ and $\sigma \in \mathfrak{S}_I$ such that $\sigma\varphi_i(b) \leq a$. Hence,

$$0 < (\operatorname{Drng} \varphi_i)(\Delta(b)) = \Delta(\varphi_i(b)) = \Delta(\sigma\varphi_i(b)) \leq \Delta(a) = \boldsymbol{a},$$

which completes the proof of (ii). $\square$

For a Boolean algebra B, we define a cardinal number $\operatorname{wd}(B)$ by

$$\operatorname{wd}(B) = \sup\{|X| \mid X \text{ is an antichain of } B\}.$$

Furthermore, for a set I, we put

$$\operatorname{supp}(x) = \{i \in I \mid x(i) \neq 0\}, \text{for all } x \in B^I.$$

The coming set of lemmas, from 5-1.6 to 5-1.8, is aimed at constructing Boolean espaliers whose dimension ranges have large constants. Instead of accomodating the results of Section 4-5 to the present context, we propose direct proofs, probably of more interest to the Boolean algebra-oriented reader.

The following lemma expresses, essentially, the well-known fact that large enough cardinals are preserved from the ground universe V to the Boolean-valued universe V^B.

LEMMA 5-1.6. *Let $(B, \sim)$ be a Boolean espalier, let I be a set. If $x \sim^I y$, then $|\operatorname{supp}(y)| \leq |\operatorname{supp}(x)| \cdot \operatorname{wd}(B)$, for all x, $y \in B^I$.*

PROOF. Put $\kappa = |\operatorname{supp}(x)| \cdot \operatorname{wd}(B)$. Since $x \sim^I y$, there are decompositions of the form $x = \oplus_{j\in J} x_j$ and $y = \oplus_{j\in J} y_j$ such that $x_j \sim_I y_j$ for all $j \in J$. By decomposing further the x_j-s and the y_j-s as disjoint sums of elements of $D = \bigcup_{i\in I} \varphi_i[B \setminus \{0\}]$, we may assume, without loss of generality, that both x_j and y_j belong to D, for all $j \in J$. Hence, for all $j \in J$, there are $a(j)$, $b(j) \in I$ and $\bar{x}_j$, $\overline{y_j} \in B \setminus \{0\}$ such that $x_j = \varphi_{a(j)}(\bar{x}_j)$ and $y_j = \varphi_{b(j)}(\bar{y}_j)$. We observe that

$$\operatorname{supp}(x) = \{a(j) \mid j \in J\} \text{ and } \operatorname{supp}(y) = \{b(j) \mid j \in J\}.$$

Put $J_i = \{j \in J \mid a(j) = i\}$, for all $i \in \operatorname{supp}(x)$. Since the family $(x_j)_{j\in J_i}$ is the image under φ_i of the antichain $(\bar{x}_j)_{j\in J_i}$ of B, the inequality $|J_i| \leq \operatorname{wd}(B)$ holds. Therefore,

$$|\operatorname{supp}(y)| \leq |J| = \sum_{i\in\operatorname{supp}(x)} |J_i| \leq \kappa. \qquad \square$$

For a Boolean algebra B, a set I, and a subset X of I, we put $X \cdot 1 = (x_i)_{i\in I}$, where $x_i = 1_B$ if $i \in X$ and $x_i = 0_B$ if $i \notin X$; hence $X \cdot 1 \in B^I$.

LEMMA 5-1.7. *Let $(B, \sim)$ be a Boolean espalier, let I be a set. Then $|X| = |Y|$ implies that $X \cdot 1 \sim^I Y \cdot 1$, for all X, $Y \subseteq I$.*

PROOF. Let $\sigma \colon X \twoheadrightarrow Y$ be a bijection. If X is finite, then σ can be extended to a permutation τ of I, thus $X \cdot 1 \sim^I \tau(X \cdot 1) = Y \cdot 1$. Suppose now that X is infinite. There exists a partition $X = X_0 \sqcup X_1$ of X such that $|X_0| = |X_1| = |X|$. Put

$Y_i = \sigma[X_i]$, for $i < 2$. Then the restriction of σ from X_i onto Y_i can be extended to a permutation τ_i of I, for all $i < 2$. Therefore,

$$X \cdot 1 = (X_0 \cdot 1) \oplus (X_1 \cdot 1) \sim^I \tau_0(X_0 \cdot 1) \oplus \tau_1(X_1 \cdot 1) = (Y_0 \cdot 1) \oplus (Y_1 \cdot 1) = Y \cdot 1.$$

□

The following lemma makes it possible to find Boolean espaliers with long $\lesssim_{\mathrm{rem}}$-chains.

LEMMA 5-1.8. *Let $(B, \sim)$ be a Boolean espalier, let α, β, $\mathfrak{m}$ be infinite cardinals such that* $\mathrm{wd}(B) \leq \alpha < \beta \leq \mathfrak{m}$. *Then the relation $\alpha \cdot 1 \lesssim_{\mathrm{rem}} \beta \cdot 1$ holds in the espalier $(B^{\mathfrak{m}}, \sim^{\mathfrak{m}})$.*

PROOF. Let x, $y \in B^{\mathfrak{m}}$ such that $\beta \cdot 1 = x \oplus y$ and $x \sim^{\mathfrak{m}} \alpha \cdot 1$, we prove that $y \sim^{\mathfrak{m}} \beta \cdot 1$. It follows from Lemma 5-1.6 that $|\operatorname{supp}(x)] \leq \alpha$, thus, putting $Y = \beta \setminus \operatorname{supp}(x)$, we obtain that $|Y| = \beta$ and $Y \cdot 1 \leq y$. From Lemma 5-1.7 it follows that $Y \cdot 1 \sim \beta \cdot 1$, whence $\beta \cdot 1 \lesssim y$. Since $y \leq \beta \cdot 1$ and by Lemma 4-1.12, it follows that $y \sim^{\mathfrak{m}} \beta \cdot 1$. □

LEMMA 5-1.9. *Let Ω be a complete Boolean space, let γ be an ordinal. Then there exists a Boolean espalier $(B, \sim)$ such that $B/{\sim} \cong \mathbf{C}(\Omega, \mathbb{Z}_\gamma)$.*

PROOF. Denote by D the complete Boolean algebra of clopen subsets of Ω. Let θ be an ordinal such that $\mathrm{wd}(D) \leq \aleph_\theta$, put $\mathfrak{m} = \omega_{\theta+\gamma}$, endow the direct power $D^{\mathfrak{m}}$ with the previously introduced $\sim^{\mathfrak{m}}$ defined from the espalier $(D, =)$. We consider the map $\varphi_0 \colon D \hookrightarrow D^{\mathfrak{m}}$ introduced earlier. It follows from Lemma 5-1.5 that φ_0 is a lower embedding of espaliers and $\mathrm{Drng}\,\varphi_0$ is a lower embedding of continuous dimension scales with dense image. In particular, $\mathrm{Proj}\, D^{\mathfrak{m}} \cong \mathrm{Proj}\, D \cong D$. Since Ω is isomorphic to the ultrafilter space of D, it follows from Theorems 3-8.9 and 4-3.9 that $D^{\mathfrak{m}}/{\sim^{\mathfrak{m}}}$ has a lower embedding into an espalier of the form

$$\overline{S} = \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\alpha; \Omega_{\mathrm{II}}, \mathbb{R}_\alpha; \Omega_{\mathrm{III}}, \mathbf{2}_\alpha),$$

for some ordinal α and a partition $\Omega = \Omega_{\mathrm{I}} \sqcup \Omega_{\mathrm{II}} \sqcup \Omega_{\mathrm{III}}$ of Ω into clopen sets. However, the continuous dimension scale $D \cong \mathbf{C}(\Omega, \{0, 1\})$ has a lower embedding into $D^{\mathfrak{m}}/{\sim^{\mathfrak{m}}}$, thus $\Omega_{\mathrm{II}} = \Omega_{\mathrm{III}} = \varnothing$ and $\Omega_{\mathrm{I}} = \Omega$. Finally, it follows from Lemma 5-1.8 that the $(\gamma+1)$-sequence $(\Delta(\omega_{\theta+\xi} \cdot 1))_{\xi \leq \gamma}$ is $\ll_{\mathrm{rem}}$-increasing in $D^{\mathfrak{m}}/{\sim^{\mathfrak{m}}}$, but all the members of this sequence have central cover 1, thus the image of $D^{\mathfrak{m}}/{\sim^{\mathfrak{m}}}$ in $\overline{S}$ contains a function whose values are all above $\aleph_\gamma$, in particular, $\mathbf{C}(\Omega, \mathbb{Z}_\gamma)$ has a lower embedding into $D^{\mathfrak{m}}/{\sim^{\mathfrak{m}}}$. The argument of Proposition 4-4.3 shows then that there exists a bounded lower subespalier $(B, \sim)$ of $(D^{\mathfrak{m}}, \sim^{\mathfrak{m}})$ such that $B/{\sim}$ is isomorphic to $\mathbf{C}(\Omega, \mathbb{Z}_\gamma)$. □

LEMMA 5-1.10. *Let Ω be a complete Boolean space, let γ be an ordinal. Then there exists a Boolean espalier $(B, \sim)$ such that $B/{\sim} \cong \mathbf{C}(\Omega, \mathbb{R}_\gamma)$.*

PROOF. The proof of Lemma 5-1.10 requires some familiarity with forcing and complete Boolean algebras, in particular, the random real extension and two-step iterated forcing, see [**27**]. We denote by B_ω the Boolean algebra of all Borel subsets of the real unit interval $[0, 1]$ modulo null sets, the *random algebra*, and by $\mathsf{m} \colon B_\omega \to [0, 1]$ the Lebesgue measure on B_ω. Furthermore, let $x \sim y$ hold, if $\mathsf{m}(x) = \mathsf{m}(y)$, for all x, $y \in B_\omega$. Let $\alpha = \sum_{i \in I} \alpha_i$, for a family $(\alpha_i)_{i \in I}$ of elements of $[0, 1]$, mean that α is the supremum over all finite subsets J of I of $\sum_{i \in J} \alpha_i$. We need a couple of standard facts on the measure m, summed up in the following claims.

CLAIM 1.

(i) $\mathsf{m}(\oplus_{i\in I}x_i) = \sum_{i\in I}\mathsf{m}(x_i)$, *for any disjoint family* $(x_i)_{i\in I}$ *of elements of* B_ω.

(ii) *Let* $x \in B_\omega$ *and let* $(\alpha_i)_{i\in I}$ *be a family of elements of* $[0,1]$. *If* $\mathsf{m}(x) = \sum_{i\in I}\alpha_i$, *then there exists a decomposition* $x = \oplus_{i\in I}x_i$ *in* B_ω *such that* $\mathsf{m}(x_i) = \alpha_i$, *for all* $i \in I$.

PROOF OF CLAIM. (i) Observe that the assumptions imply that the set $\{i \in I \mid x_i > 0\}$ is countable; the conclusion follows from countable additivity of Lebesgue measure.

(ii) Again, $\{i \in I \mid \alpha_i > 0\}$ is countable, so we may assume without loss of generality that $I = \omega$. It is then easy to construct inductively a nondecreasing sequence $(a_i)_{i<\omega}$ of elements of $[0,1]$ satisfying the conditions $\mathsf{m}(x\cap[0,a_0]) = \alpha_0$ and $\mathsf{m}(x\cap[a_n,a_{n+1}]) = \alpha_{n+1}$, for all $n<\omega$. Put $a = \sup_{n<\omega}a_n$, then $x_0 = x\setminus[a_0,a]$ and $x_n = x\cap[a_n,a_{n+1}]$, for all $n<\omega$, satisfy the desired conclusion. □ Claim 1.

As an immediate corollary, we obtain the following claim.

CLAIM 2. *The structure* $(B_\omega,\sim)$ *is a Boolean espalier, with dimension range isomorphic to* $[0,1]$.

Now we work under the assumptions of Lemma 5-1.10. Denote by C the complete Boolean algebra of clopen subsets of Ω. We consider the quotient of the Scott-Solovay C-valued universe V^C of set theory under the equivalence relation that identifies names $\dot{x}$ and $\dot{y}$ if and only if $\|\dot{x}=\dot{y}\| = 1$. We still denote by V^C the quotient, endowed with its natural Boolean value, so, now, $\|\boldsymbol{x}=\boldsymbol{y}\| = 1$ if and only if $\boldsymbol{x}=\boldsymbol{y}$, for all $\boldsymbol{x}$, $\boldsymbol{y}\in V^C$. Furthermore, let $\boldsymbol{B_\omega}$ denote the (equivalence class of the) C-valued name for B_ω, and put $D = C * \boldsymbol{B_\omega}$, the two-step iterated forcing of C by the random algebra of V^C. Hence D is the complete Boolean algebra of all $\boldsymbol{x}\in V^C$ such that $\|\boldsymbol{x}\in\boldsymbol{B_\omega}\| = 1$, the partial ordering $\leq$ being defined by $\boldsymbol{x}\leq\boldsymbol{y}$ if and only if $\|\boldsymbol{x}\leq\boldsymbol{y}\| = 1$. Hence, orthogonality in D is defined by $x\perp y$ if and only if $\|\boldsymbol{x}\wedge\boldsymbol{y}=0\| = 1$. Let $\sim$ be the binary relation defined on D by $\boldsymbol{x}\sim\boldsymbol{y}$ if and only if $\|\boldsymbol{x}\sim\boldsymbol{y}\| = 1$, for all $\boldsymbol{x}$, $\boldsymbol{y}\in D$.

CLAIM 3. *The structure* $(D,\sim)$ *is a Boolean espalier.*

PROOF OF CLAIM. It is obvious that $\sim$ satisfies (B0). Now let $\boldsymbol{a}\sim\boldsymbol{b}$ in D, with $\boldsymbol{b}$ decomposed as $\boldsymbol{b} = \oplus_{i\in I}\boldsymbol{b}_i$. If $p\mapsto p^*$ denotes the canonical embedding from C into $C*\boldsymbol{B_\omega}$, the relation $p \leq \|\boldsymbol{x}\leq\boldsymbol{y}\|$ is equivalent to $p^*\wedge\boldsymbol{x}\leq\boldsymbol{y}$, for all $\boldsymbol{x}$, $\boldsymbol{y}\in D$. It is an easy exercise to deduce from this the relation $\left\|\boldsymbol{b} = \oplus_{i\in\check{I}}\boldsymbol{b}_i\right\| = 1$, where the symbol $\check{I}$ denotes the canonical name in V^C for I. Moreover,

$$\|\boldsymbol{a}\sim\boldsymbol{b}\| = \|(\boldsymbol{B_\omega},\sim)\text{ is a Boolean espalier}\| = 1$$

and V^C is a Boolean-valued model of set theory, in particular, V^C satisfies Claim 2. Therefore,

$$\left\|\exists(\boldsymbol{x}_i)_{i\in\check{I}}\text{ such that }\boldsymbol{a} = \oplus_{i\in\check{I}}\boldsymbol{x}_i\text{ and }\boldsymbol{x}_i\sim\boldsymbol{b}_i,\ \text{ for all }i\in\check{I}\right\| = 1.$$

Since V^C is full and the notion of function is absolute, there exists a family $(\boldsymbol{a}_i)_{i\in I}$ of elements of D such that $\|\boldsymbol{a}_i\sim\boldsymbol{b}_i\| = 1$, for all $i\in I$, and

$$\left\|\boldsymbol{a} = \oplus_{i\in\check{I}}\boldsymbol{a}_i\right\| = 1.$$

Hence $\boldsymbol{a} = \oplus_{i\in I}\boldsymbol{a}_i$ and $\boldsymbol{a}_i\sim\boldsymbol{b}_i$, for all $i\in I$. □ Claim 3.

We shall now identify the dimension range of $(D, \sim)$. For every $\boldsymbol{a} \in D$, there exists a unique $\boldsymbol{\alpha} \in V^C$ such that $\|\boldsymbol{\alpha} = \mathsf{m}(\boldsymbol{a})\| = 1$. Observe, in particular, that $\|0 \leq \boldsymbol{\alpha} \leq 1\| = 1$. We put $\boldsymbol{\alpha} = \varepsilon(\boldsymbol{a})$.

We need the following standard fact.

CLAIM 4. *There exists an isomorphism of partial commutative monoids from* $\mathbf{C}(\Omega, [0,1])$ *onto the set of* $\boldsymbol{\alpha} \in V^C$ *such that* $\|0 \leq \boldsymbol{\alpha} \leq 1\| = 1$*, endowed with the addition defined by* $\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}$ *if and only if* $\|\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}\| = 1$.

PROOF OF CLAIM. This is a particular case of a much more general statement, see, for example, [**55**, Theorem 3.10]. □ Claim 4.

From now on we identify the elements of $\mathbf{C}(\Omega, [0,1])$ with the C-valued names of elements of $[0,1]$, *via* the isomorphism of Claim 4. We obtain the following result.

CLAIM 5. *The map* ε *can be factored through* $\sim$*, to an isomorphism from* $D/{\sim}$ *onto* $\mathbf{C}(\Omega, [0,1])$.

PROOF OF CLAIM. For $\boldsymbol{a}, \boldsymbol{b} \in D$, $\varepsilon(\boldsymbol{a}) = \varepsilon(\boldsymbol{b})$ if and only if $\|\mathsf{m}(\boldsymbol{a}) = \mathsf{m}(\boldsymbol{b})\| = 1$, if and only if $\|\boldsymbol{a} \sim \boldsymbol{b}\| = 1$, if and only if $\boldsymbol{a} \sim \boldsymbol{b}$. If $\boldsymbol{c} \in D$ and $\boldsymbol{c} = \boldsymbol{a} \oplus \boldsymbol{b}$, then $\|\boldsymbol{c} = \boldsymbol{a} \oplus \boldsymbol{b}\| = 1$, thus $\|\mathsf{m}(\boldsymbol{c}) = \mathsf{m}(\boldsymbol{a}) + \mathsf{m}(\boldsymbol{b})\| = 1$, thus $\varepsilon(\boldsymbol{c}) = \varepsilon(\boldsymbol{a}) + \varepsilon(\boldsymbol{b})$. Finally, let $\boldsymbol{\alpha} \in \mathbf{C}(\Omega, [0,1])$. There exists a unique $\boldsymbol{a} \in V^C$ such that $\|\boldsymbol{a} = [0, \boldsymbol{\alpha}]\| = 1$, thus $\boldsymbol{a} \in D$ and $\varepsilon(\boldsymbol{a}) = \boldsymbol{\alpha}$, so ε is surjective. □ Claim 5.

The rest of the proof proceeds like in the proof of Lemma 5-1.9, of which we shall keep the notation. It follows from Claim 5 that the Boolean algebra of projections of $(D, \sim)$ is isomorphic to C, see Claim 5 in the proof of Theorem 3-3.6. If θ is an ordinal such that $\mathrm{wd}(D) \leq \aleph_\theta$ and we put $\mathfrak{m} = \omega_{\theta+\gamma}$, then $D^{\mathfrak{m}}/{\sim^{\mathfrak{m}}}$ has a $\ll_{\mathrm{rem}}$-increasing chain of length $\gamma + 1$. Furthermore, observe that this time, since $\mathbf{C}(\Omega, [0,1])$ has a lower embedding into $D^{\mathfrak{m}}/{\sim^{\mathfrak{m}}}$, $\Omega_{\mathrm{I}} = \Omega_{\mathrm{III}} = \varnothing$ while $\Omega_{\mathrm{II}} = \Omega$, and the image of $D^{\mathfrak{m}}/{\sim^{\mathfrak{m}}}$ in $\overline{S}$ contains a function with values above $\aleph_\gamma$. Hence, there exists a bounded lower subespalier $(B, \sim)$ of $(D^{\mathfrak{m}}, \sim^{\mathfrak{m}})$ such that $B/{\sim} \cong \mathbf{C}(\Omega, \mathbb{R}_\gamma)$. □

LEMMA 5-1.11. *Let* Ω *be a complete Boolean space, let* γ *be an ordinal. Then there exists a Boolean espalier* $(B, \sim)$ *such that* $B/{\sim} \cong \mathbf{C}(\Omega, \mathbf{2}_\gamma)$.

PROOF. In order to give a proof of Lemma 5-1.11, it is also convenient to be familiar with forcing and complete Boolean algebras. We denote by C_ω the Boolean algebra of all Borel subsets of the Cantor space $\{0,1\}^\omega$ modulo meager sets, the *Cohen algebra*. Furthermore, for $x \in C_\omega$, we define $\mathsf{n}(x) = \aleph_0$ if $x > 0$ while $\mathsf{n}(0) = 0$. Let $x \sim y$ hold, if $\mathsf{n}(x) = \mathsf{n}(y)$, for all $x, y \in C_\omega$.

CLAIM 1. *The structure* $(C_\omega, \sim)$ *is a Boolean espalier, with dimension range isomorphic to* $\{0, \aleph_0\}$.

PROOF OF CLAIM. Every nonzero element of C_ω can be decomposed as a disjoint union of two (resp., ω) nonzero elements of C_ω. Furthermore, if $u = \oplus_{i \in I} u_i$ in C_ω, then $\{i \in I \mid u_i > 0\}$ is countable. It follows easily that $\sim$ satisfies (B1). It obviously satisfies (B0) and (B2). Since every nonzero element of C_ω can be decomposed as a disjoint union of two nonzero elements of C_ω, every element of C_ω is purely infinite. It follows that $C_\omega/{\sim} \cong \{0, \aleph_0\}$. □ Claim 1.

Now we work under the assumptions of Lemma 5-1.11. Denote by C the complete Boolean algebra of clopen subsets of Ω. We define the (quotiented) Scott-Solovay universe V^C of set theory as in the proof of Lemma 5-1.10. Furthermore, let $\boldsymbol{C_\omega}$ denote the (equivalence class of the) C-valued name for C_ω, and put $D = C{*}\boldsymbol{C_\omega}$, the two-step iterated forcing of C by the Cohen algebra of V^C. Hence D is the complete Boolean algebra of all $\boldsymbol{x} \in V^C$ such that $\|\boldsymbol{x} \in \boldsymbol{C_\omega}\| = 1$, the partial ordering $\leq$ being defined by $\boldsymbol{x} \leq \boldsymbol{y}$ if and only if $\|\boldsymbol{x} \leq \boldsymbol{y}\| = 1$. Hence, orthogonality in D is defined by $x \perp y$ if and only if $\|\boldsymbol{x} \wedge \boldsymbol{y} = 0\| = 1$. Let $\sim$ be the binary relation defined on D by $\boldsymbol{x} \sim \boldsymbol{y}$ if and only if $\|\boldsymbol{x} \sim \boldsymbol{y}\| = 1$, for all $\boldsymbol{x}$, $\boldsymbol{y} \in D$.

The proof of the following Claim 2 is, *mutatis mutandis*, the same as the one for Claim 3 in the proof of Lemma 5-1.10.

Claim 2. *The structure $(D, \sim)$ is a Boolean espalier.*

We shall now identify the dimension range of $(D, \sim)$. For every $\boldsymbol{a} \in D$, there exists a unique $\boldsymbol{\alpha} \in V^C$ such that $\|\boldsymbol{\alpha} = \mathsf{n}(\boldsymbol{a})\| = 1$. Observe, in particular, that $\|\boldsymbol{\alpha} \in \{0, \aleph_0\}\| = 1$. We put $\boldsymbol{\alpha} = \varepsilon(\boldsymbol{a})$.

The analogue of Claim 4 of Lemma 5-1.10 takes the following simple form, with a much more direct proof.

Claim 3. *There exists an isomorphism of partial commutative monoids from $\mathbf{C}(\Omega, \{0, \aleph_0\})$ (isomorphic to C) onto the set of $\boldsymbol{\alpha} \in V^C$ such that $\|\boldsymbol{\alpha} \in \{0, \aleph_0\}\| = 1$, endowed with the addition defined by $\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}$ if and only if $\|\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}\| = 1$.*

From now on we identify the elements of $\mathbf{C}(\Omega, \{0, \aleph_0\})$ with the C-valued names of elements of $\{0, \aleph_0\}$, *via* the isomorphism of Claim 3. We obtain the following result.

Claim 4. *The map ε can be factored through $\sim$, to an isomorphism from $D/{\sim}$ onto $\mathbf{C}(\Omega, \{0, \aleph_0\})$.*

Proof of Claim. The proof that ε is an embedding for $\leq$ and for $+$ is the same as in the proof of Claim 4 of Lemma 5-1.10. Let $\boldsymbol{\alpha} \in \mathbf{C}(\Omega, \{0, \aleph_0\})$. There exists $\boldsymbol{a} \in V^C$ such that $\|\boldsymbol{a} = 0\| = \|\boldsymbol{\alpha} = 0\|$, thus $\boldsymbol{a} \in D$ and $\varepsilon(\boldsymbol{a}) = \boldsymbol{\alpha}$, so ε is surjective. □ Claim 4.

The rest of the proof proceeds like in the proof of Lemma 5-1.10. It follows from Claim 4 that the Boolean algebra of projections of $(D, \sim)$ is isomorphic to C. If θ is an ordinal such that $\mathrm{wd}(D) \leq \aleph_\theta$ and we put $\mathfrak{m} = \omega_{\theta+\gamma}$, then $D^{\mathfrak{m}}/{\sim}^{\mathfrak{m}}$ has a $\ll_{\mathrm{rem}}$-increasing chain of length $\gamma + 1$. Since $\mathbf{C}(\Omega, \{0, \aleph_0\})$ has a lower embedding into $D^{\mathfrak{m}}/{\sim}^{\mathfrak{m}}$, $\Omega_{\mathrm{I}} = \Omega_{\mathrm{II}} = \varnothing$ while $\Omega_{\mathrm{III}} = \Omega$, and the image of $D^{\mathfrak{m}}/{\sim}^{\mathfrak{m}}$ in $\overline{S}$ contains a function with values above $\aleph_\gamma$. Hence, there exists a bounded lower subespalier $(B, \sim)$ of $(D^{\mathfrak{m}}, \sim^{\mathfrak{m}})$ such that $B/{\sim} \cong \mathbf{C}(\Omega, \mathbf{2}_\gamma)$. □

Remark 5-1.12. Since the Boolean algebra C_ω has an absolute (in set-theoretical sense) dense subalgebra, namely, the Boolean algebra F_ω of clopen subsets of $\{0,1\}^\omega$, the forcing could, in principle, have been eliminated from the proof of Lemma 5-1.11: for example, one could have taken for D the completion of $C \otimes F_\omega$ (the tensor product for Boolean algebras is just the coproduct). Such an argument would not have worked for Lemma 5-1.10, because B_ω of the ground universe may not be dense in the B_ω of a generic extension.

By Lemma 4-4.4 and Proposition 4-4.3(i), we thus obtain the following result.

THEOREM 5-1.13. *The class of Boolean espaliers is D-universal. Moreover, every bounded continuous dimension scale is isomorphic to the dimension range of some Boolean espalier.*

5-2. Conditionally complete, meet-continuous, relatively complemented, modular lattices

We first recall some basic lattice-theoretical definitions, see [**19**]. A lattice $(L, \vee, \wedge)$ is *modular*, if $x \geq z$ implies that $x \wedge (y \vee z) = (x \wedge y) \vee z$, for all x, y, $z \in L$. We say that L is

— *complemented*, if it has a least element 0, a largest element 1, and every $x \in L$ has a complement, that is, $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$.
— *sectionally complemented*, if it has a least element 0 and every sublattice of the form $[0, a]$, for $a \in L$, is complemented;
— *relatively complemented*, if every sublattice of the form $[a, b]$, for $a \leq b$ in L, is complemented.

In general, these notions are unrelated. However, in the *modular* case, the following implications hold:

complemented $\Rightarrow$ sectionally complemented $\Rightarrow$ relatively complemented.

We say that the lattice L is *complete*, if every subset of L has an infimum—equivalently, every subset of L has a supremum. We say that L is *conditionally complete*, if every nonempty bounded subset of L has an infimum—equivalently, the interval $[a, b]$ is complete, for all $a \leq b$ in L. We say that L is *meet-continuous*, if for every $a \in L$ and every upward directed subset X of L admitting a supremum, the equality $a \wedge \bigvee X = \bigvee (a \wedge X)$ holds, where we put $a \wedge X = \{a \wedge x \mid x \in X\}$. If the dual condition holds, L is called *join continuous*, and if both conditions hold, L is *continuous*. (This definition of continuity is not equivalent to the one presented in G. Gierz *et al.* [**14**], which is nowadays often called "Scott continuity".) A *continuous geometry* is any complete, complemented, modular, continuous lattice. (This is the current terminology; von Neumann's original definition included hypotheses of irreducibility and lack of chain conditions. What we have called a continuous geometry was called a "reducible continuous geometry" or a "continuous geometry in the general sense" in some of the older literature.)

The *dimension monoid* of a lattice L, see [**56**], is the commutative monoid $\operatorname{Dim} L$ defined by generators $\Delta(a, b)$, for $a \leq b$ in L, and the following relations:

(D0) $\Delta(a, a) = 0$, for all $a \in L$.
(D1) $\Delta(a, c) = \Delta(a, b) + \Delta(b, c)$, for all $a \leq b \leq c$ in L.
(D2) $\Delta(a, a \vee b) = \Delta(a \wedge b, b)$, for all a, $b \in L$.

It is an open problem whether the dimension monoid of an arbitrary lattice is always a refinement monoid, however, this is solved in a few important particular cases: the case of *finite lattices*, of which the dimension monoids are so-called *primitive monoids*, and the case of *modular lattices*, for which an alternative presentation of the dimension monoid is given that implies refinement. We shall concentrate here on the latter.

In a modular lattice L with zero, let $x \perp y$ hold, if $x \wedge y = 0$, for any x, $y \in L$, and then we define $x \oplus y = x \vee y$. The following result is folklore, see, for example, [**56**, Proposition 8.1]; it says, essentially, that $\oplus$ is associative in modular lattices. We state it in a way that relates it to the axioms defining espaliers.

PROPOSITION 5-2.1. *Let L be a modular lattice with zero. Then the relation $\perp$ on L satisfies* (L2).

In a modular lattice L, we define the binary relations $\sim$ (*perspectivity*), $\sim_2$ (*bi-perspectivity*), $\approx$ (*projectivity*), and $\approxeq$ (*projectivity by decomposition*) as follows:

$$
\begin{aligned}
x \sim y &\iff \exists z \text{ such that } x \oplus z = y \oplus z;\\
x \sim_2 y &\iff \exists z \text{ such that } x \sim z \sim y;\\
x \approx y &\iff \exists n \in \mathbb{N},\ \exists z_0, \dots, z_n \text{ such that } x = z_0 \sim z_1 \sim \cdots \sim z_n = y;\\
x \approxeq y &\iff \exists n \in \mathbb{N},\ \exists x_0 \approx y_0, \dots, x_{n-1} \approx y_{n-1}\\
&\qquad\qquad \text{such that } x = \oplus_{i<n} x_i \text{ and } y = \oplus_{i<n} y_i.
\end{aligned}
$$

In case L is a relatively complemented lattice with zero, the dimension monoid $\operatorname{Dim} L$ of L is generated by the elements $\Delta(x) = \Delta(0, x)$, for $x \in L$ (see [**56**, Proposition 9.1]). We define the *dimension range* of L as $\operatorname{Drng} L = \{\Delta(x) \mid x \in L\}$. In case L is also modular, $\operatorname{Drng} L$ can be endowed with the partial addition defined by

$$\Delta(a) + \Delta(b) = \Delta(a \oplus b), \text{ for all } a,\, b \in L \text{ such that } a \perp b.$$

Furthermore, the partial commutative monoid $\operatorname{Drng} L$ determines the commutative monoid $\operatorname{Dim} L$, and any equality of the form $\Delta(a) = \Delta(b)$ can be tested in a very simple way, see Corollaries 9.4 and 9.5 in [**56**] and Proposition 2-1.13 of the present paper.

PROPOSITION 5-2.2. *Let L be a sectionally complemented modular lattice. Put $S = \operatorname{Drng} L$. Then the following statements hold:*

(i) $\Delta(a) = \Delta(b)$ *if and only if $a \approxeq b$, for all a, $b \in L$.*
(ii) $\operatorname{Dim} L = \widetilde{S}$, *the universal monoid of S.*

This is the point where the theory of espaliers and continuous dimension scales comes in. Our plan is to associate, with a sectionally complemented, modular lattice L, an espalier L^* such that the dimension range of L, as defined above, is the dimension range of L^*. The structure L^* is simply defined as $(L, \leq, \perp, \approxeq)$, for those $\perp$ and $\approxeq$ defined above, so all we need to do is find sufficient conditions for it to be an espalier. The following lemma sums up some of the hardest (in particular item (i)) and most useful results of [**56**].

LEMMA 5-2.3. *Let L be a conditionally complete, meet-continuous, sectionally complemented, modular lattice. Then the following statements hold:*

(i) *$x \approxeq y$ if and only if there are decompositions $x = x_0 \oplus x_1$ and $y = y_0 \oplus y_1$ in L such that $x_0 \sim y_0$ and $x_1 \sim y_1$, for all x, $y \in L$.*
(ii) *Let a, $b \in L$ and let $(b_i)_{i \in I}$ be a family of elements of L. If $a \sim \oplus_{i \in I} b_i$, then there exists a decomposition $a = \oplus_{i \in I} a_i$ such that $a_i \sim b_i$, for all $i \in I$.*
(iii) *Let $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ be families of elements of L such that $\oplus_{i \in I} a_i = \oplus_{j \in J} b_j$. Then there are families $(x_{i,j})_{(i,j) \in I \times J}$ and $(y_{i,j})_{(i,j) \in I \times J}$ of elements of L such that*

$$
\begin{aligned}
a_i &= \oplus_{j \in J} x_{i,j}, &&\textit{for all } i \in I,\\
b_j &= \oplus_{i \in I} y_{i,j}, &&\textit{for all } j \in J,
\end{aligned}
$$

and $x_{i,j} \sim_2 y_{i,j}$, for all $(i,j) \in I \times J$.

PROOF. (i) follows immediately from Lemma 10.4 and Theorem 13.2 in [**56**].

(ii) There exists $c \in L$ such that $a \oplus c = b \oplus c$. Let $\tau\colon [0,b] \twoheadrightarrow [0,a]$ be the perspective mapping with axis c, that is, $\tau(x) = (x \oplus c) \wedge a$, for all $x \in [0,b]$. Put $a_i = \tau(b_i)$, for all $i \in I$. Since τ is an isomorphism from $[0,b]$ onto $[0,a]$, the equality $a = \oplus_{i\in I} a_i$ holds. Furthermore, a_i is perspective to b_i with axis c, for all $i \in I$.

The proof of (iii) is virtually the same as the one of [**56**, Lemma 12.17], except that we replace countable families by transfinite ones. We give the proof here for the convenience of the reader. So, let λ and κ be ordinals, let $(a_\alpha)_{\alpha<\lambda}$ and $(b_\beta)_{\beta<\kappa}$ be orthogonal families of elements of L such that $\oplus_{\alpha<\lambda} a_\alpha = \oplus_{\beta<\kappa} b_\beta$. We put

$$\begin{aligned} \overline{a}_\alpha &= \oplus_{\xi<\alpha} a_\xi, && \text{for all } \alpha \leq \lambda,\\ \overline{b}_\beta &= \oplus_{\eta<\beta} b_\eta, && \text{for all } \beta \leq \kappa. \end{aligned}$$

Furthermore, we put $c_{\alpha,\beta} = (\overline{a}_{\alpha+1} \wedge \overline{b}_\beta) \vee (\overline{a}_\alpha \wedge \overline{b}_{\beta+1})$ and $d_{\alpha,\beta} = \overline{a}_{\alpha+1} \wedge \overline{b}_{\beta+1}$, for all $\alpha < \lambda$ and all $\beta < \kappa$. Since $c_{\alpha,\beta} \leq d_{\alpha,\beta}$ and L is sectionally complemented, there exists $z_{\alpha,\beta} \in L$ such that $c_{\alpha,\beta} \oplus z_{\alpha,\beta} = d_{\alpha,\beta}$.

CLAIM 1. *The statement $(\overline{a}_\alpha \wedge \overline{b}_\beta) \oplus (\oplus_{\eta<\beta} z_{\alpha,\eta}) = \overline{a}_{\alpha+1} \wedge \overline{b}_\beta$ holds, for all $\alpha < \lambda$ and all $\beta \leq \kappa$.*

PROOF OF CLAIM. We prove the conclusion by induction on β. It is trivial for $\beta = 0$. For β a limit ordinal, it follows easily from the meet-continuity of L and the induction hypothesis. Now suppose having proved the statement at step β, we prove it at step $\beta+1$. It follows from the induction hypothesis that $\oplus_{\eta<\beta} z_{\alpha,\eta} \leq \overline{a}_{\alpha+1} \wedge \overline{b}_\beta$, thus

$$\begin{aligned} \overline{a}_\alpha \wedge \overline{b}_{\beta+1} \wedge (\oplus_{\eta<\beta} z_{\alpha,\eta}) &= (\overline{a}_\alpha \wedge \overline{b}_{\beta+1}) \wedge (\overline{a}_{\alpha+1} \wedge \overline{b}_\beta) \wedge (\oplus_{\eta<\beta} z_{\alpha,\eta})\\ &= (\overline{a}_\alpha \wedge \overline{b}_\beta) \wedge \oplus_{\eta<\beta} z_{\alpha,\eta}\\ &= 0, \end{aligned}$$

that is, $\overline{a}_\alpha \wedge \overline{b}_{\beta+1} \perp \oplus_{\eta<\beta} z_{\alpha,\eta}$. Furthermore,

$$\begin{aligned} (\overline{a}_\alpha \wedge \overline{b}_{\beta+1}) \oplus (\oplus_{\eta<\beta} z_{\alpha,\eta}) &= (\overline{a}_\alpha \wedge \overline{b}_{\beta+1}) \vee (\overline{a}_\alpha \wedge \overline{b}_\beta) \vee (\oplus_{\eta<\beta} z_{\alpha,\eta})\\ &= (\overline{a}_\alpha \wedge \overline{b}_{\beta+1}) \vee (\overline{a}_{\alpha+1} \wedge \overline{b}_\beta)\\ &\qquad\qquad \text{(by the induction hypothesis)}\\ &= c_{\alpha,\beta}, \end{aligned}$$

whence

$$\begin{aligned} d_{\alpha,\beta} &= c_{\alpha,\beta} \oplus z_{\alpha,\beta}\\ &= \big((\overline{a}_\alpha \wedge \overline{b}_{\beta+1}) \oplus (\oplus_{\eta<\beta} z_{\alpha,\eta})\big) \oplus z_{\alpha,\beta}\\ &= (\overline{a}_\alpha \wedge \overline{b}_{\beta+1}) \oplus (\oplus_{\eta<\beta+1} z_{\alpha,\eta}). \qquad \square \text{ Claim 1.} \end{aligned}$$

Of course, by symmetry, the following claim is also valid.

CLAIM 2. *The statement $(\overline{a}_\alpha \wedge \overline{b}_\beta) \oplus (\oplus_{\xi<\alpha} z_{\xi,\beta}) = \overline{a}_\alpha \wedge \overline{b}_{\beta+1}$ holds, for all $\alpha \leq \lambda$ and all $\beta < \kappa$.*

In particular, using Claim 1 for $\beta = \kappa$ yields that $\overline{a}_\alpha \oplus (\oplus_{\eta<\kappa} z_{\alpha,\eta}) = \overline{a}_\alpha \oplus a_\alpha$, thus $\oplus_{\eta<\kappa} z_{\alpha,\eta} \sim a_\alpha$. Thus, by item (ii) above, there exists a decomposition of the form $a_\alpha = \oplus_{\eta<\kappa} x_{\alpha,\eta}$ such that $x_{\alpha,\eta} \sim z_{\alpha,\eta}$, for all $\eta < \kappa$. Similarly, for all $\beta < \kappa$, there exists a decomposition of the form $b_\beta = \oplus_{\xi<\lambda} y_{\xi,\beta}$ such that $z_{\xi,\beta} \sim y_{\xi,\beta}$, for all $\xi < \lambda$. It follows that $x_{\xi,\eta} \sim_2 y_{\xi,\eta}$, for all $\xi < \alpha$ and $\eta < \beta$, and the $x_{\xi,\eta}$-s and $y_{\xi,\eta}$-s are as desired. $\square$

PROPOSITION 5-2.4. *Let L be a conditionally complete, meet-continuous, sectionally complemented, modular lattice. Then $L^* = (L, \leq, \perp, \approxeq)$ is an espalier, and* $\operatorname{Drng} L = \operatorname{Drng} L^*$.

PROOF. The verifications in L^* of Axioms (L1) to (L5) and (L8) are either trivial or immediate consequences of the assumptions and Proposition 5-2.1.

Let a, b, b_i (for $i \in I$) be elements of L such that $a \approxeq b$ and $b = \oplus_{i\in I} b_i$. By Lemma 5-2.3(i), there are a', a'', b', $b'' \in L$ such that $a = a' \oplus a''$, $b = b' \oplus b''$, $a' \sim b'$, and $a'' \sim b''$. Applying Lemma 5-2.3(iii) to the equality $b' \oplus b'' = \oplus_{i\in I} b_i$, we obtain decompositions $b' = \oplus_{i\in I} x'_i$, $b'' = \oplus_{i\in I} x''_i$, $b_i = y'_i \oplus y''_i$, for all $i \in I$, such that $x'_i \sim_2 y'_i$ and $x''_i \sim_2 y''_i$, for all $i \in I$. Since $a' \sim b' = \oplus_{i\in I} x'_i$ and $a'' \sim b'' = \oplus_{i\in I} x''_i$, there are, by Lemma 5-2.3(ii), decompositions $a' = \oplus_{i\in I} a'_i$ and $a'' = \oplus_{i\in I} a''_i$ such that $a'_i \sim x'_i$ and $a''_i \sim x''_i$, for all $i \in I$. Observe that $a = a' \oplus a'' = \oplus_{i\in I} a_i$, where we put $a_i = a'_i \oplus a''_i$, for all $i \in I$. Furthermore, $a_i \approxeq b_i$, for all $i \in I$. Hence $\approxeq$ is unrestrictedly refining (Axiom (L6)).

The proof that $\approxeq$ is unrestrictedly additive (Axiom (L7)) is virtually the same as [**56**, Proposition 13.9], except that countable families are replaced by arbitrary families.

The statement that $\operatorname{Drng} L = \operatorname{Drng} L^*$ follows immediately from Proposition 5-2.2(i). $\square$

COROLLARY 5-2.5. *The dimension monoid of any conditionally complete, meet-continuous, sectionally complemented, modular lattice is a* (*total*) *continuous dimension scale.*

PROOF. By Proposition 5-2.4, L^* is an espalier and $\operatorname{Drng} L = \operatorname{Drng} L^*$, thus, by Theorem 4-3.9, $S = \operatorname{Drng} L$ is a continuous dimension scale. By Corollary 3-8.11, $\widetilde{S}$ is also a continuous dimension scale, but by Proposition 5-2.2, $\operatorname{Dim} L$ is isomorphic to $\widetilde{S}$, thus it is a continuous dimension scale. $\square$

Hence we can complete the program of determining the dimension theory of conditionally complete, meet-continuous, relatively complemented, modular lattices initiated by I. Halperin and J. von Neumann in [**22**].

THEOREM 5-2.6. *The dimension monoid of any conditionally complete, meet-continuous, relatively complemented modular lattice is a* (*total*) *continuous dimension scale.*

PROOF. Let L be a conditionally complete, meet-continuous, relatively complemented modular lattice. The interval $[a, b]$ is a conditionally complete, meet-continuous, complemented modular lattice, for all $a \leq b$ in L, thus, by Corollary 5-2.5, its dimension monoid $\operatorname{Dim}[a, b]$ is a continuous dimension scale. Furthermore, if $a' \leq a \leq b \leq b'$ in L, then the natural map from $\operatorname{Dim}[a, b]$ to $\operatorname{Dim}[a', b']$ is, by [**56**, Corollary 13.5], a lower embedding of commutative monoids. Express the lattice L as the direct limit of the direct system $\mathcal{I}$ of its closed intervals. Since the Dim functor preserves direct limits, it follows from Lemma 3-1.10 that $\operatorname{Dim} L = \varinjlim_{[a,b]\in\mathcal{I}} \operatorname{Dim}[a, b]$ is a continuous dimension scale. $\square$

We say that a (von Neumann) regular ring R is *right continuous*, if the lattice $\mathcal{L}(R)$ of all principal right ideals of R (cf. [**17**, Theorem 2.3]) is complete and meet-continuous. In particular, every right self-injective regular ring is right continuous, see [**17**, Corollary 13.5]. The connection between the present section and the upcoming Section 5-3 is made possible by the following immediate consequence of [**56**, Corollary 13.4].

Proposition 5-2.7. *Let R be a right continuous regular ring. Then the following statements hold:*

(i) *The monoid $V(R)$ of isomorphism classes of finitely generated projective right R-modules is isomorphic to the dimension monoid* $\operatorname{Dim}\mathcal{L}(R)$ *of $\mathcal{L}(R)$.*
(ii) *Two principal right ideals I and J of R are isomorphic if and only if there are decompositions $I = I_0 \oplus I_1$ and $J = J_0 \oplus J_1$ such that $I_0 \sim J_0$ and $I_1 \sim J_1$. In particular, $I \cong J$ if and only if $I \approxeq J$ in the lattice $\mathcal{L}(R)$.*

The results of Section 5-3 below imply that the espaliers of the form L^* appearing in Proposition 5-2.4 form a D-universal class. Here is a slightly sharper statement.

Theorem 5-2.8. *The class of espaliers of the form $(L, \leq, \perp, \approxeq)$, where L is a complete, meet-continuous, complemented, modular lattice, is D-universal.*

Proof. It follows from Theorem 5-3.14 below that every continuous dimension scale admits a lower embedding into the dimension range of an espalier of the form $(\mathcal{L}(R), \subseteq, \perp, \cong)$, for some regular, right self-injective ring R. It follows from Proposition 5-2.7(ii) that the relations $\cong$ and $\approxeq$ on $\mathcal{L}(R)$ coincide. Since $\mathcal{L}(R)$ is a complete, meet-continuous, complemented, modular lattice, the conclusion follows. $\square$

In particular, it follows from Propositions 5-2.4 and 5-2.7 and Theorem 5-3.14 that every continuous dimension scale admits a lower embedding into $\operatorname{Dim}\mathcal{L}(R)$, for some regular, right self-injective ring R. Observe again that $\mathcal{L}(R)$ is a complete, meet-continuous, complemented, modular lattice.

Corollary 5-2.9. *The dimension monoids of conditionally complete, meet-continuous, sectionally* (*resp., relatively*) *complemented, modular lattices are exactly the total continuous dimension scales.*

Hence, validating the possibility suggested in F. Wehrung [**57**], the dimension theory of conditionally complete, meet-continuous, relatively complemented, modular lattices is completely elucidated.

5-3. Self-injective regular rings and nonsingular injective modules

For notation, terminology, and standard results on the topics of this section, we refer to [**18, 15, 17**]. Throughout the section, let R denote a (von Neumann) regular (unital) ring; after some preliminary results, we shall assume that R is also right self-injective, that is, R is injective as a right module over itself.

Let $\mathcal{L}(R)$ denote the collection of principal right ideals of R. Regularity implies that $\mathcal{L}(R)$ is a complemented modular lattice, in which finite suprema and infima are given by sums and intersections, respectively (e.g., [**17**, Theorem 2.3]). Define orthogonality in $\mathcal{L}(R)$ to mean lattice disjointness: $A \perp B$ if and only if $A \cap B = 0$. For an equivalence relation on $\mathcal{L}(R)$, we shall use $\cong$, that is, isomorphism of right R-modules.

A small amount of category-theoretical notation will be helpful in dealing with R-modules. We write $\mathsf{Mod}\text{-}R$ for the category of all right R-modules, and $\mathsf{add}\text{-}A$ for the full subcategory of $\mathsf{Mod}\text{-}R$ whose objects are all direct summands of finite direct sums of copies of a given module A. In particular, the objects of $\mathsf{add}\text{-}R$ are precisely the finitely generated projective right R-modules. An expression such as "$A \in \mathsf{Mod}\text{-}R$" will abbreviate the assertion that A is an object in the category $\mathsf{Mod}\text{-}R$. We write $\mathrm{E}(A)$ to stand for an arbitrary injective hull of a module A, and if κ is a cardinal, $\kappa \cdot A$ stands for a direct sum of κ copies of A. For $A, B \in \mathsf{Mod}\text{-}R$, write $A \lesssim B$ to mean that A is isomorphic to a submodule of B. If $A, B \in \mathsf{add}\text{-}R$ and $A \lesssim B$, then regularity of R implies that A is isomorphic to a direct summand of B [**17**, Theorem 1.11].

We use the notation $V(R)$ for the monoid of isomorphism classes of objects from $\mathsf{add}\text{-}R$ (in which the addition operation is induced from the direct sum operation on modules). To match our notation for dimension ranges, we shall denote elements of $V(R)$ in the form $\Delta(A)$, rather than using a more common notation like $[A]$. This involves a slight but unproblematic abuse of notation in case $A \in \mathcal{L}(R)$, since the element $\Delta(A) \in V(R)$ stands for the isomorphism class of A within the class of all right R-modules, whereas once we have made $\mathcal{L}(R)$ into an espalier, the notation $\Delta(A)$ will also be used for the image of A in $\operatorname{Drng}\mathcal{L}(R)$, and in the latter case $\Delta(A)$ stands for the isomorphism class of A within $\mathcal{L}(R)$.

Lemma 5-3.1. *For any regular ring R, the monoid $V(R)$ is a refinement monoid, the interval $[0, \Delta(R)] \subseteq V(R)$ is a partial refinement monoid, and $V(R)$ is the universal monoid of $[0, \Delta(R)]$.*

Proof. Refinement in $V(R)$ is given by [**17**, Theorem 2.8], and it is clear that $S = [0, \Delta(R)]$ is a partial submonoid of $V(R)$, hence a partial refinement monoid in its own right. Since every object in $\mathsf{add}\text{-}R$ is isomorphic to a finite direct sum of principal right ideals of R [**17**, Proposition 2.6], every element of $V(R)$ is a sum of elements of S.

Let U denote the universal monoid of S, with canonical map $\phi\colon S \to U$. There exists a unique homomorphism $\psi\colon U \to V(R)$ such that $\psi\phi\colon S \to V(R)$ is the inclusion map. Given $x \in V(R)$, write $x = \sum_{i<n} x_i$ for some elements $x_i \in S$, and set $\theta(x) = \sum_{i<n} \phi(x_i) \in U$; this is well defined by refinement. Hence, we obtain a homomorphism $\theta\colon V(R) \to U$. Obviously $\psi\theta$ is the identity map on $V(R)$, and $\theta\psi\phi = \phi$, whence $\theta\psi$ is the identity map on U. Therefore ψ is an isomorphism. $\square$

We next determine the projections on $V(R)$ and on $[0, \Delta(R)]$. This requires working with orthogonality in $V(R)$ (as defined in Section 2-2), which is determined as follows [**17**, Proposition 2.21]: For any $A, B \in \mathsf{add}\text{-}R$, we have

$$\Delta(A) \perp \Delta(B) \iff \operatorname{Hom}_R(A, B) = 0 \iff \operatorname{Hom}_R(B, A) = 0.$$

Let $\mathrm{B}(R)$ denote the set of all central idempotents in R; this is a Boolean algebra whose operations are given by the rules

$$e \wedge f = ef \qquad e \vee f = e + f - ef \qquad e' = 1 - e$$

[**17**, p. 83]. If R is right self-injective, $\mathrm{B}(R)$ is complete [**17**, Proposition 9.9].

LEMMA 5-3.2. *Let R be a regular ring.*

(i) *For any $e \in \mathrm{B}(R)$, there is a projection p_e on $V(R)$ such that $p_e\Delta(A) = \Delta(Ae)$ for all $A \in \mathsf{add}\text{-}R$. Moreover, $p_e^{\perp} = p_{1-e}$.*
(ii) *The rule $e \mapsto p_e$ defines an isomorphism $\mathrm{B}(R) \xrightarrow{\cong} \operatorname{Proj} V(R)$.*
(iii) *The rule $e \mapsto p_e|_{[0,\Delta(R)]}$ defines an isomorphism $\mathrm{B}(R) \xrightarrow{\cong} \operatorname{Proj}[0, \Delta(R)]$.*

PROOF. (i) It is clear that there is an endomorphism p_e of $V(R)$ such that $p_e\Delta(A) = \Delta(Ae)$ for all $A \in \mathsf{add}\text{-}R$. Moreover, $\Delta(A) = \Delta(Ae) + \Delta(A(1-e))$, and $\Delta(A(1-e)) \perp \Delta(Be)$ for all $B \in \mathsf{add}\text{-}R$, so that $\Delta(A(1-e)) \in (p_eV(R))^{\perp}$. Therefore $p_e \in \operatorname{Proj} V(R)$. Similarly, $p_{1-e} \in \operatorname{Proj} V(R)$, and we observe that $V(R) = p_eV(R) \oplus p_{1-e}V(R)$. Therefore $p_{1-e} = p_e^{\perp}$.

(ii) Let $e, f \in \mathrm{B}(R)$. If $e \leq f$, then $e = fe$, whence $p_ep_f = p_e$ and so $p_e \leq p_f$. Conversely, if $p_e \leq p_f$, then $p_e(x) \leq p_f(x)$ for all $x \in V(R)$ (Lemma 2-3.8(i)), whence $Re \lesssim Rf$ (taking $x = \Delta(R)$). Consequently, $Re(1-f) = 0$, and so $e \leq f$. This shows that the map $e \mapsto p_e$ is an order-embedding of $\mathrm{B}(R)$ into $\operatorname{Proj} V(R)$.

Given $p \in \operatorname{Proj} V(R)$, we have $\Delta(R) = p\Delta(R) + p^{\perp}\Delta(R)$, and so $R = I \oplus J$ for some right ideals I, J such that $\Delta(I) = p\Delta(R)$ and $\Delta(J) = p^{\perp}\Delta(R)$. There is an idempotent $e \in R$ such that $eR = I$ and $(1-e)R = J$. Moreover, $\Delta(I) \perp \Delta(J)$, and thus $\operatorname{Hom}_R(I, J) = 0$. This homomorphism group being isomorphic to $(1-e)Re$, we conclude that $e \in \mathrm{B}(R)$ [**17**, Lemma 3.1]. In particular, we can now write $I = Re$ and $J = R(1-e)$. Any $A \in \mathsf{add}\text{-}R$ is isomorphic to a direct summand of $n \cdot R$ for some positive integer n, whence $p\Delta(A) \leq np\Delta(R) = n\Delta(Re) = p_e(n\Delta(R))$. On the other hand, Ae is isomorphic to a direct summand of $n \cdot (Re)$, and so $p_e\Delta(A) \leq n\Delta(Re) = p(n\Delta(R))$. Since $pV(R)$ and $p_eV(R)$ are ideals of $V(R)$, it follows that they are equal, and therefore $p = p_e$.

(iii) This is proved in the same manner as (ii). □

Recall that we have defined orthogonality in $\mathcal{L}(R)$ by the rule $A \perp B \iff A \cap B = 0$. When this occurs, the right ideal $A + B$ is both the orthogonal sum of A and B within $\mathcal{L}(R)$ and the module-theoretic direct sum of A and B, so that the two uses of the expression $A \oplus B$ coincide. However, infinite orthogonal sums in $\mathcal{L}(R)$ (when they exist) cannot be module-theoretic direct sums, since the direct sum of an infinite family of nonzero modules is not finitely generated. To distinguish these cases, let us write $\oplus^{\perp}_{i\in I} A_i$ for the orthogonal sum of a family $(A_i)_{i\in I}$ of elements of $\mathcal{L}(R)$ and $\bigoplus_{i\in I} A_i$ for the module-theoretic direct sum. (For either to exist, the family $(A_i)_{i\in I}$ must be independent.)

PROPOSITION 5-3.3. *Let $\mathcal{L}(R)$ be the lattice of principal right ideals of a regular, right self-injective ring R. Then $(\mathcal{L}(R), \subseteq, \perp, \cong)$ is an espalier, and its dimension range is isomorphic to the interval $[0, \Delta(R)] \subseteq V(R)$. Consequently, both $[0, \Delta(R)]$ and $V(R)$ are continuous dimension scales. In case R is purely infinite, $\operatorname{Drng} \mathcal{L}(R) \cong V(R)$.*

PROOF. By [**17**, Corollary 13.5], $\mathcal{L}(R)$ is complete and upper continuous (= meet-continuous). In particular, Axiom (L1) holds. As shown in the proof of [**17**, Proposition 13.3], arbitrary infima in $\mathcal{L}(R)$ are given by intersections, while the supremum of a family $(A_i)_{i\in I}$ of elements of $\mathcal{L}(R)$ is the unique principal right ideal of R which contains $\sum_{i\in I} A_i$ as an essential submodule. Since R is right self-injective, $\bigvee_{i\in I} A_i = \mathrm{E}(\sum_{i\in I} A_i)$. Hence, if $(A_i)_{i\in I}$ is an orthogonal family, $\oplus^{\perp}_{i\in I} A_i = \mathrm{E}(\bigoplus_{i\in I} A_i)$.

Axioms (L2), (L3), (L5), and (L8) are clear, (L2)(iv) and (L8) being standard properties of submodules of arbitrary modules. Axioms (L4), (L6), and (L7) are basic properties of injective hulls. Therefore $\mathcal{L}(R)$ is an espalier. It is clear that $\operatorname{Drng}\mathcal{L}(R) \cong [0, \Delta(R)]$. If R is purely infinite, then $n \cdot R \cong R$ for all positive integers n [**17**, Theorem 10.16], in which case $[0, \Delta(R)] = V(R)$.

Finally, $[0, \Delta(R)]$ is a continuous dimension scale by Theorem 4-3.9, and it follows from Lemma 5-3.1 and Corollary 3-8.11 that $V(R)$ is a continuous dimension scale. □

For the remainder of the section, assume that R is right self-injective. Before applying Theorem 3-8.9, we show that the type decomposition of R (see [**18**, Chapter VII] or [**17**, Chapter 10]) matches the type decomposition of $V(R)$ (Definition 3-7.8). Here it is natural to work with type decompositions of modules from $\mathsf{add}\text{-}R$, as in [**18**, Theorem 7.2] and [**17**, Theorem 10.31].

LEMMA 5-3.4. *Let $A \in \mathsf{add}\text{-}R$. The following statements hold:*

(i) *A is an abelian, directly finite, or purely infinite module, respectively, if and only if $\Delta(A)$ is a multiple-free, directly finite, or purely infinite element, respectively, of $V(R)$.*

(ii) *A is of Type* I, II, III, *respectively, if and only if $\Delta(A)$ lies in $V(R)_{\mathrm{I}}$, $V(R)_{\mathrm{II}}$, $V(R)_{\mathrm{III}}$, respectively.*

PROOF. (i) The equivalence for directly finite modules is clear from the definitions, and the other two equivalences follow from [**18**, Theorems 2.1, 6.2].

(ii) First, $\Delta(A) \in V(R)_{\mathrm{III}} = V(R)_{\mathrm{fin}}^{\perp}$ if and only if $\operatorname{Hom}_R(B, A) = 0$ for all directly finite $B \in \mathsf{add}\text{-}R$, if and only if A has no nonzero directly finite direct summands, if and only if A is of Type III [**18**, p. 37]. Similarly, $\Delta(A) \in V(R)_{\mathrm{mf}}^{\perp}$ if and only if $\operatorname{Hom}_R(B, A) = 0$ for all abelian $B \in \mathsf{add}\text{-}R$, if and only if A has no nonzero abelian direct summands. Consequently, $\Delta(A) \in V(R)_{\mathrm{I}}$ if and only if every nonzero direct summand of A contains a nonzero abelian direct summand, while $\Delta(A) \in V(R)_{\mathrm{III}}$ if and only if A has no nonzero abelian direct summands, but every nonzero direct summand of A contains a nonzero directly finite direct summand. Thus, the remainder of part (ii) follows from Theorems 5.1 and 5.5 of [**18**]. □

COROLLARY 5-3.5. *If R is of Type* I, II, III, *respectively, then $V(R)$ equals $V(R)_{\mathrm{I}}$, $V(R)_{\mathrm{II}}$, $V(R)_{\mathrm{III}}$, respectively.*

PROOF. [**18**, Theorem 5.11]. □

THEOREM 5-3.6. *Let R be a regular, right self-injective ring, and write $R = R_{\mathrm{I}} \times R_{\mathrm{II}} \times R_{\mathrm{III}}$ where R_{J} is of Type* J. *Let Ω_{J} be the ultrafilter space of* $\mathrm{B}(R_{\mathrm{J}})$. *Then there exists an ordinal γ such that $V(R)$ is isomorphic to a lower subset of $\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma)$.*

PROOF. Since $V(R) \cong V(R_{\mathrm{I}}) \times V(R_{\mathrm{II}}) \times V(R_{\mathrm{III}})$, it follows from Corollary 5-3.5 that each $V(R)_{\mathrm{J}} \cong V(R_{\mathrm{J}})$. By Proposition 5-3.3, $V(R)$ is a continuous dimension scale, and using Lemma 5-3.2(ii) we see that each Ω_{J} is homeomorphic to the ultrafilter space of $\operatorname{Proj} V(R)_{\mathrm{J}}$. Therefore the theorem follows from Theorem 3-8.9. □

We now turn our attention to nonsingular injective modules, which allows us to extend the above results to proper Continuous Dimension Scales, and which will allow us to show that the espaliers of the form $\mathcal{L}(R)$ form a D-universal class.

Let $\mathsf{NSI}\text{-}R$ denote the full subcategory of $\mathsf{Mod}\text{-}R$ whose objects are all nonsingular injective right R-modules. Note that if $A \in \mathsf{NSI}\text{-}R$, then $\mathsf{add}\text{-}A \subseteq \mathsf{NSI}\text{-}R$. Let $V(A)$ be the monoid of isomorphism classes of objects in $\mathsf{add}\text{-}A$, where—as above—we use $\Delta(B)$ to denote the isomorphism class of an object B. Note that for B, $C \in \mathsf{add}\text{-}A$, we have $\Delta(B) \leq \Delta(C)$ if and only if $B \lesssim C$.

For $A \in \mathsf{NSI}\text{-}R$, let $\mathcal{L}(A)$ denote the collection of those submodules of A which are direct summands. Then $\mathcal{L}(A)$ is a complete, complemented, modular lattice, with infima and suprema given just as in $\mathcal{L}(R)$ [**18**, Propositions 1.3, 1.6]. We define $\perp$ in $\mathcal{L}(A)$ as in $\mathcal{L}(R)$.

Lemma 5-3.7. *Let A be a nonsingular injective right R-module, and set $T = \operatorname{End}_R(A)$. Then T is a regular, right self-injective ring, and $V(T) \cong V(A)$. Consequently, $V(A)$ is a dimension interval.*

Moreover, $(\mathcal{L}(A), \subseteq, \perp, \cong)$ is an espalier, isomorphic to $(\mathcal{L}(T), \subseteq, \perp, \cong)$. Consequently, $\operatorname{Drng} \mathcal{L}(A) \cong [0, \Delta(A)] \subseteq V(A)$.

Proof. For the first statement, see, for example, [**17**, Corollary 1.23]. It is well known that $\mathsf{add}\text{-}A$ is equivalent to the category of finitely generated projective right T-modules (e.g., [**31**, Theorem 18.59]), and thus $V(A) \cong V(T)$. Therefore, Proposition 5-3.3 implies that $V(A)$ is a continuous dimension scale.

According to [**18**, Proposition 1.8], there is a lattice isomorphism $\phi\colon \mathcal{L}(T) \to \mathcal{L}(A)$ given by the rule $\phi(J) = JA$. Any pair of right ideals of T is given by eT, fT for some pair e, f of idempotents in T, and it is well known that $eT \cong fT$ if and only if $eA \cong fA$ (cf. the proof of [**17**, Proposition 2.4]). Hence, ϕ is an isomorphism of espaliers. It is clear that $\operatorname{Drng} \mathcal{L}(A) \cong [0, \Delta(A)]$. □

A major advantage of working with nonsingular injective modules is that any set of such modules can be combined to form a new one, by taking the injective hull of the direct sum. Consequently, we can pass from the category $\mathsf{NSI}\text{-}R$ to a proper Continuous Dimension Scale which contains "arbitrarily large" elements. Thus, let $V(\mathsf{NSI}\text{-}R)$ denote the (proper) Monoid consisting of all isomorphism classes $\Delta(A)$ of objects $A \in \mathsf{NSI}\text{-}R$, with addition induced by direct sum. (To help keep set-theoretic difficulties at bay, one might wish to pass from $\mathsf{NSI}\text{-}R$ to an equivalent skeletal category—a category in which isomorphic objects are equal—before forming this Monoid.) For any $A \in \mathsf{NSI}\text{-}R$, the ideal of $V(\mathsf{NSI}\text{-}R)$ generated by $\Delta(A)$ equals the monoid $V(A)$; in particular, this ideal is a set.

Lemma 5-3.8. *The Monoid $V(\mathsf{NSI}\text{-}R)$ is a Continuous Dimension Scale, and $V(R)$ is a generating lower subset of $V(\mathsf{NSI}\text{-}R)$.*

Proof. If S is a lower subset of $V(\mathsf{NSI}\text{-}R)$, then $S = \{\Delta(B_i) \mid i \in I\}$ for some set $\{B_i \mid i \in I\}$ of objects from $\mathsf{NSI}\text{-}R$. Form $B = \mathrm{E}(\bigoplus_{i\in I} B_i)$, and observe that $S \subseteq V(B)$. By Lemmas 5-3.7 and 3-1.9, $V(B)$ and S are continuous dimension scales. For any element $a = \Delta(A) \in V(\mathsf{NSI}\text{-}R)$, the class $(a]$ is contained in the set $V(A)$ and so it is a set. Thus, Axiom ($\mathrm{M_{ht}}$) is satisfied in $V(\mathsf{NSI}\text{-}R)$.

Since every object in $\mathsf{add}\text{-}R$ is injective (being a direct summand of some injective module $n \cdot R$), we have $\mathsf{add}\text{-}R \subseteq \mathsf{NSI}\text{-}R$ and $V(R) \subseteq V(\mathsf{NSI}\text{-}R)$. It is then clear that $V(R)$ is a lower subset of $V(\mathsf{NSI}\text{-}R)$. Given any nonzero object $A \in \mathsf{NSI}\text{-}R$, choose a nonzero element $x \in A$. By [**17**, Theorem 9.2], the cyclic module xR is both projective and injective. On the one hand, this means that $xR \in \mathsf{add}\text{-}R$ and $\Delta(xR) \in V(R)$, while on the other, $\Delta(xR) \leq \Delta(A)$. Thus, $V(R)$ is dense in $V(\mathsf{NSI}\text{-}R)$, and therefore $V(\mathsf{NSI}\text{-}R)$ satisfies Axiom ($\mathrm{M_{lh}}$). □

THEOREM 5-3.9. *Let R be a regular, right self-injective ring, and write $R = R_{\mathrm{I}} \times R_{\mathrm{II}} \times R_{\mathrm{III}}$ where R_{J} is of Type J. Let Ω_{J} be the ultrafilter space of* $\mathrm{B}(R_{\mathrm{J}})$. *Then*

$$V(\mathsf{NSI}\text{-}R) \cong \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\infty) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\infty) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\infty).$$

PROOF. As observed in the proof of Theorem 5-3.6, Ω_{J} is homeomorphic to the ultrafilter space of $\operatorname{Proj} V(R)_{\mathrm{J}}$ for J = I, II, III. Let E be a finitary unit of $V(R)_{\mathrm{fin}}$, and observe that since $V(R)$ is dense in $V(\mathsf{NSI}\text{-}R)$, the set E is dense in $V(\mathsf{NSI}\text{-}R)_{\mathrm{fin}}$. Thus, E is a finitary unit of $V(\mathsf{NSI}\text{-}R)$. By Theorem 3-10.5 and its proof, there is a lower embedding

$$\varepsilon\colon V(\mathsf{NSI}\text{-}R) \hookrightarrow \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\infty) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\infty) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\infty)$$

(unique with respect to our choice of E) such that whenever $A \in \mathsf{NSI}\text{-}R$ and $R \in \mathsf{add}\text{-}A$, the restriction of ε to $V(A)$ matches the embedding given in Theorem 3-8.9.

To see that every function in $\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\infty) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\infty) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\infty)$ lies in the image of ε, it suffices to show that for any infinite cardinal $\kappa = \aleph_\tau$, the constant function t_κ with $t_\kappa(x) = \kappa$ for all $x \in \Omega_{\mathrm{I}} \sqcup \Omega_{\mathrm{II}} \sqcup \Omega_{\mathrm{III}}$ lies in the image of ε.

Set $B = \mathrm{E}(\aleph_0 \cdot R)$, let $\aleph_\sigma$ be the cardinality of B, and set $A = \mathrm{E}(\aleph_{\sigma+\tau} \cdot B)$. In particular, B contains no direct sums of more than $\aleph_\sigma$ nonzero submodules, and $V(R) \subseteq V(B) \subseteq V(A)$. We may assume that B is an actual submodule of A. By Lemma 3-7.1, restriction from $V(A)$ to $V(R)$ provides an isomorphism $\operatorname{Proj} V(A) \xrightarrow{\cong} \operatorname{Proj} V(R)$.

According to Lemma 5-3.7, $\mathcal{L}(A)$ is an espalier whose dimension range is isomorphic to $V(A)$ (we have $[0, \Delta(A)] = V(A)$ because A is purely infinite). Now A and B are purely infinite elements of $\mathcal{L}(A)$ with central cover 1, and B is not equal to any orthogonal sum of more than $\aleph_\sigma$ nonzero elements. The module-theoretic statement $\mathrm{E}(\aleph_{\sigma+\tau} \cdot B) \cong A$, when written in the symbolism of espaliers, says that $\aleph_{\sigma+\tau} \cdot B \sim A$. Thus, Proposition 4-5.4 implies that there exists a purely infinite element $C \in \mathcal{L}(A)$ such that $\mu(\Delta(C))$ equals the constant function with value $\aleph_\tau$. Therefore we have $\Delta(C) \in V(\mathsf{NSI}\text{-}R)$ with $\varepsilon(\Delta(C)) = t_\kappa$, which completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 5-3.9, in view of the fact that $V(A)$ is a lower subset of $V(\mathsf{NSI}\text{-}R)$ for any $A \in \mathsf{NSI}\text{-}R$. If the reader wishes to avoid proper Continuous Dimension Scales, this result can be proved directly, using the same methods employed in the theorem.

COROLLARY 5-3.10. *Let R be a regular, right self-injective ring, and write $R = R_{\mathrm{I}} \times R_{\mathrm{II}} \times R_{\mathrm{III}}$, where R_{J} is of Type* J. *Let Ω_{J} be the ultrafilter space of* $\mathrm{B}(R_{\mathrm{J}})$. *Given any ordinal γ, there exists a nonsingular injective right R-module A such that*

$$V(A) \cong \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma). \qquad \square$$

To show that every continuous dimension scale appears as a lower subset of some $V(A)$, it only remains to construct regular, right self-injective rings of Types I, II, III having arbitrary complete Boolean algebras as their Boolean algebras of central idempotents. We shall make use of the concept of a *maximal quotient ring* (see, for example, [**15**, Chapter 2], [**31**, §13]) in part of the process.

PROPOSITION 5-3.11. *Given any complete Boolean algebra B, there exist regular, right self-injective rings R_{I} and R_{III} of Types I and III, respectively, such that* $\mathrm{B}(R_{\mathrm{I}}) \cong \mathrm{B}(R_{\mathrm{III}}) \cong B$.

PROOF. The quickest way to obtain a Type I example is to take R_{I} to be B itself, made into a ring in the canonical way. Then R_{I} is a commutative, regular, self-injective ring in which all elements are idempotent, and $\mathrm{B}(R_{\mathrm{I}}) \cong B$. For later use, we note that since R_{I} is commutative, $B \cong \mathrm{B}(R_{\mathrm{I}}) \cong \mathcal{L}(R_{\mathrm{I}})$. The self-injectivity of R_{I} implies that R_{I} is a continuous regular ring, thus yielding von Neumann's well-known result that B is continuous (see [**19**, Lemma II.4.10]). Since R_{I} is commutative, it is abelian, and hence is of Type I. As a ring, B has characteristic 2, while the reader may prefer examples having characteristic 0. We can construct examples which are algebras over any field F, as follows.

Let X be the ultrafilter space of B, so that B is isomorphic to the Boolean algebra of clopen subsets of X. Let S be the ring of all locally constant functions from X to F (that is, functions $f : X \to F$ such that each point of X has a neighborhood on which f is constant). Observe that S is a commutative regular ring, with $B \cong \mathrm{B}(S) \cong \mathcal{L}(S)$. As noted above, B is a continuous lattice; thus S is a continuous regular ring. Finally, let R_{I} be the maximal (right) quotient ring of S. Since S is regular, it is a nonsingular ring, and so R_{I} is regular and right self-injective ([**15**, Corollary 2.31], [**31**, Theorem 13.36]). Moreover, since S is commutative, so is R_{I} (see [**31**, Lemma 14.15]). Therefore R_{I} is of Type I. By [**17**, Theorem 13.13], all the idempotents of R_{I} lie in S (this is not hard to prove directly in the present case). Therefore $\mathrm{B}(R_{\mathrm{I}}) = \mathrm{B}(S) \cong B$.

Similar methods, worked out by Busqué [**8**], can be applied in the Type III case. First choose a commutative, regular, self-injective ring R_{I} with $\mathrm{B}(R_{\mathrm{I}}) \cong B$. By [**8**, Theorem 2.5], there exists a regular, right self-injective ring R_{III} of Type III whose center is isomorphic to R_{I}. Therefore $\mathrm{B}(R_{\mathrm{III}}) \cong B$. □

It appears that the constructions used in Proposition 5-3.11 do not always produce rings of Type II. We approach the Type II existence problem lattice-theoretically, *via* von Neumann's *Coordinatization Theorem* (e.g., [**51**, Theorem 14.1], [**38**, Chapter XI, Satz 3.2]).

PROPOSITION 5-3.12. *Given any complete Boolean algebra B, there exists a regular, right and left self-injective ring R of Type* $\mathrm{II_f}$ *with* $\mathrm{B}(R) \cong B$.

PROOF. Let L be an irreducible (i.e., indecomposable) continuous geometry such that the (unique) dimension function D on L is positive on all nonzero elements of L and the range of D is the unit interval $[0, 1]$. Such continuous geometries were constructed by von Neumann [**50**]. Alternatively, one could choose a simple, regular, right self-injective ring S of Type $\mathrm{II_f}$ (see [**18**, Corollary 11.10] and [**17**, Example 10.7, Theorem 10.27] for existence) and take $L = \mathcal{L}(S)$. Indecomposability of L then follows from indecomposability of S, and the properties of D follow from those of the unique rank function N on S (see [**17**, Corollary 16.15]), since D is given by the formula $D(xR) = N(x)$ for $x \in R$.

Next, let $L(B)$ be the (reducible) continuous geometry constructed from L and B by Halperin in [**21**, Theorem 1]. The center of $L(B)$ (i.e., the sublattice of neutral elements) is isomorphic to B by [**21**, Theorem 2], and $L(B)$ contains a sublattice (with the same largest element) isomorphic to L [**21**, Remark 2, p. 351]. For any positive integer n, the largest element $1 \in L$ can be written as the supremum of n independent pairwise perspective elements (e.g., because there exist elements $x \in L$ with $D(x) = 1/n$), and so the same occurs in $L(B)$. Consequently, $L(B)$ has order n (in von Neumann's sense) for all n.

In particular, since $L(B)$ has order 4, von Neumann's Coordinatization Theorem implies that there exists a regular ring R such that $\mathcal{L}(R) \cong L(B)$. Since $\mathcal{L}(R)$ is thus a continuous lattice, R is a continuous regular ring. Now R is unit-regular [**17**, Corollary 13.23], and hence perspectivity in $\mathcal{L}(R)$ is given by module isomorphism [**17**, Corollary 4.23]. Consequently, for each positive integer n, the module R is a direct sum of n pairwise isomorphic right ideals. In particular, there are no nonzero central idempotents $e \in R$ such that the ring eR is abelian, and therefore R is right and left self-injective [**17**, Corollary 13.18].

Since R is unit-regular, it is directly finite [**17**, Proposition 5.2]. Hence, [**17**, Theorems 10.13, 10.24] show that $R \cong \prod_{m=1}^{\infty} R_m \times R_{\mathrm{II}}$ where each R_m is of Type I_m and R_{II} is of Type II_{f}. Since the dimension theory of $\mathcal{L}(R_m)$ takes values in $\{0, 1/m, 2/m, \dots, 1\}$, the module R_m cannot be a direct sum of $m+1$ nonzero pairwise isomorphic right ideals (cf. [**18**, Theorem 10.10] or [**17**, Corollary 11.18]). Thus, all $R_m = 0$, and $R \cong R_{\mathrm{II}}$ is of Type II_{f}.

Finally, since the center of $\mathcal{L}(R)$ is isomorphic to $\mathrm{B}(R)$ [**38**, Chapter VI, Sätze 1.9, 3.5], we conclude that $\mathrm{B}(R) \cong B$. $\square$

Theorem 5-3.13. *Let Ω_{I}, Ω_{II}, Ω_{III} be arbitrary complete Boolean spaces (possibly empty). Then there exists a regular, right self-injective ring R such that*

$$V(\mathsf{NSI}\text{-}R) \cong \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\infty) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\infty) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\infty).$$

Proof. Propositions 5-3.11 and 5-3.12, together with Theorem 5-3.9. $\square$

Although the following result is a corollary of Theorem 5-3.13, we give an independent proof avoiding the use of proper Continuous Dimension Scales.

Theorem 5-3.14. *The class of espaliers of the form $(\mathcal{L}(R), \subseteq, \perp, \cong)$, for regular, right self-injective rings R, is D-universal.*

Proof. Let S be an arbitrary continuous dimension scale. By Propositions 5-3.11 and 5-3.12, together with Theorems 3-8.9 and 5-3.6, there exists a regular, right self-injective ring R' such that S is isomorphic to a lower subset S' of $V(R')$. Set $A = \mathrm{E}(\aleph_0 \cdot R')$ and $R = \mathrm{End}_{R'}(A)$. By Lemma 5-3.7, R is a regular, right self-injective ring and $\mathrm{Drng}\,\mathcal{L}(R) \cong \mathrm{Drng}\,\mathcal{L}(A) \cong [0, \Delta(A)]$. Since all finitely generated projective right R'-modules are isomorphic to direct summands of A, we see that $V(R')$ is a lower subset of $[0, \Delta(A)]$. Thus, S' is isomorphic to a lower subset of the dimension range of the espalier $\mathcal{L}(R)$. $\square$

The results above also allow us to determine the monoids $V(R)$ in the present context, as follows.

Corollary 5-3.15. *Let M be a commutative monoid. Then $M \cong V(R)$ for some regular, right self-injective ring R if and only if M is a continuous dimension scale containing an order-unit.*

Proof. If R is a regular, right self-injective ring, then $\Delta(R)$ is an order-unit in $V(R)$, and $V(R)$ is a continuous dimension scale by Proposition 5-3.3. Conversely, let M be a continuous dimension scale which contains an order-unit u. By Theorem 5-3.14 and Proposition 5-3.3, there exists a regular, right self-injective ring R'

such that M is isomorphic to a lower subset M' of $V(R')$. Let u' denote the image of u under this isomorphism; then M' equals the ideal of $V(R')$ generated by u'. Now $u' = \Delta(A)$ for some $A \in \mathsf{add}\text{-}R'$, and it is clear that $M' = V(A)$. By Lemma 5-3.7, $R = \mathrm{End}_{R'}(A)$ is a regular, right self-injective ring and $V(R) \cong V(A) \cong M$. □

5-4. Projection lattices of W*- and AW*-algebras

Our main references for W*-algebras will be the texts by J. Dixmier [**10**], R. V. Kadison and J. R. Ringrose [**29**], B.-R. Li [**32**], and S. Sakai [**46**]; for AW*-algebras, we rely on the text by S. K. Berberian [**3**] and the monograph by I. Kaplansky [**30**]. A *W*-algebra* (also called a *von Neumann algebra*) can be defined as any C*-algebra which is isomorphic (*qua* C*-algebra) to a *-subalgebra of $\mathcal{B}(H)$ (the algebra of all bounded linear operators) on some (complex) Hilbert space H which is closed in the strong operator topology (the topology of pointwise convergence). Kaplansky introduced the concept of an *AW*-algebra* (abbreviating "abstract W*-algebra") in order to obtain a more general class of C*-algebras defined (and analyzed) by purely algebraic properties. Before giving the definition, we recall a few basic concepts.

All W*- and AW*-algebras that we consider here will be assumed to be *unital*.

A *projection* in a C*-algebra A is any self-adjoint idempotent, that is, any element $p \in A$ with $p = p^2 = p^*$. The *right annihilator* of a subset S of A is the right ideal

$$\mathrm{ann_r}(S) = \{x \in A \mid sx = 0 \text{ for all } s \in S\}.$$

Finally, A is said to be an *AW*-algebra* if the right annihilator of any subset S of A is a principal right ideal generated by a projection, that is, $\mathrm{ann_r}(S) = pS$ for some (necessarily unique) projection $p \in A$. Every W*-algebra is AW* [**3**, §4, Proposition 9], but not conversely. For example, the (unital) commutative AW*-algebras are precisely (up to isomorphism) the algebras $\mathbf{C}(X, \mathbb{C})$ of continuous complex-valued functions on complete Boolean spaces X [**3**, §7, Theorem 1]; such an algebra is W* if and only if X is *hyperstonian* ([**9**, Théorème 2]; cf. [**32**, Theorems 5.3.3, 5.3.4]). By definition, X is hyperstonian (cf. [**9**, Définition 3]), if for any nonempty open subset U of X, there exists a Radon measure μ on X, vanishing on all nowhere dense subsets of X, such that $\mu(U) > 0$.

The set L of projections of an AW*-algebra A is equipped with the partial ordering $\leq$ defined by $p \leq q$ if and only if $pq = p$ (equivalently, $qp = p$), for p, $q \in L$. The poset $(L, \leq)$ is a complete lattice [**3**, §4, Proposition 1]. Furthermore, two projections p, $q \in L$ are *orthogonal*, in symbols $p \perp q$, if $pq = 0$ (equivalently, $qp = 0$). Then the sum $p + q$ is also a projection, and it is the join of $\{p, q\}$ in L: hence $p \oplus q = p + q$. Finally, two projections p and q of A are *Murray-von Neumann equivalent*, in symbols $p \sim q$, if there exists $x \in A$ such that $p = xx^*$ and $q = x^*x$. Equivalently, pA and qA are isomorphic as right A-modules, that is, there are x, $y \in A$ such that $p = xy$ and $q = yx$—this equivalence is nontrivial and contained in [**30**, Theorem 27].

A projection $p \in A$ is said to be *σ-finite* (or *countably decomposable*, or *orthoseparable*) if p does not majorize any uncountable orthogonal family of nonzero projections; if the projection $1 \in A$ has this property, then the algebra A itself is called σ-finite. This same terminology is also used for Boolean algebras and their elements. Let us say that a Boolean algebra B is *locally σ-finite* provided every element of B is a supremum of σ-finite elements. Furthermore, a Boolean space X is locally σ-finite, if its Boolean algebra of clopen subsets is locally σ-finite.

PROPOSITION 5-4.1. *Every hyperstonian Boolean space is locally σ-finite.*

PROOF. Let B denote the ultrafilter space of a hyperstonian Boolean space X. For a Radon measure μ on X, we say that a Borel subset A of X is *μ-self-supporting*, if $\mu(A \cap U) > 0$ whenever $U \subseteq X$ is open and $A \cap U \neq \varnothing$. Then every Borel subset A of X of positive measure contains a μ-self-supporting compact subset K of positive measure, see [**13**, §1.9]. If μ vanishes on all nowhere dense subsets, then $\mu(\overset{\circ}{K}) = \mu(K) > 0$, hence $\overset{\circ}{K}$ is a μ-self-supporting open subset of K with positive measure. As μ is a Radon measure, K contains a μ-self-supporting *clopen* subset with positive measure.

Let D denote the set of elements of B whose associated clopen set is μ-self-supporting with respect to some finite Radon measure μ on X vanishing on all nowhere dense subsets. It follows from the assumption that X is hyperstonian and the paragraph above that every element of B is a supremum of elements of D. But every element of D is clearly σ-finite. □

The converse of Proposition 5-4.1 does not hold as a rule.

EXAMPLE 5-4.2. As in Section 5-1, we denote by C_ω the Boolean algebra of all Borel subsets of the Cantor space $\{0,1\}^\omega$ modulo meager sets. Let X denote the ultrafilter space of C_ω. Then X is clearly σ-finite. However, there is no nontrivial Radon measure on X, as shown, for example, by the argument on pages 82–83 in [**44**, Chapter 21]. In particular, X is not hyperstonian.

On the other hand, the "measure" analogue of C_ω, that is, the random algebra B_ω (see Section 5-1) has, of course, hyperstonian ultrafilter space.

PROPOSITION 5-4.3. *Let L be the lattice of projections of an AW*-algebra A, endowed with the relations $\leq$, $\perp$, and $\sim$ defined above. Then L is an espalier.*

PROOF. Axiom (L1) follows from [**30**, Theorem 19] or [**3**, §4, Proposition 1]. Axioms (L2) and (L3) are easy exercises, Axiom (L5) is trivial.

Axiom (L4): for p, $r \in L$, the element $1 - p \in A$ is also a projection, and $p \perp r$ is equivalent to $r \leq 1 - p$. Now let $(q_i)_{i \in I}$ be an orthogonal family of elements of L such that $p \perp (\oplus_{i \in J} q_i)$, for all finite $J \subseteq I$. This means that $\oplus_{i \in J} q_i \leq 1 - p$, for all finite $J \subseteq I$, thus $\oplus_{i \in I} q_i \leq 1 - p$, that is, $p \perp (\oplus_{i \in I} q_i)$.

Axiom (L6) is Axiom (C) in [**30**, Chapter 4]; see [**30**, Theorem 24] or [**3**, §1, Proposition 9].

Axioms (L7) and (L8) are difficult results, proved in [**30**, Theorem 52, 62] and [**3**, §20, Theorem 1, §13, Theorem 1]. □

The "projections" of the espalier L are not the projections of A (which are the elements of L), but they correspond to the *central* projections of A. In fact, all central idempotents of A are projections [**3**, §3, Exercise 1], and so we may use without ambiguity the notation $\mathrm{B}(A)$ of Section 5-3 to stand for the Boolean algebra of central projections in A.

LEMMA 5-4.4. *Let L be the lattice of projections of an AW*-algebra A. For each $e \in \mathrm{B}(A)$, there is a projection $\pi_e \in \operatorname{Proj}\operatorname{Drng} L$ such that $\pi_e(\Delta(p)) = \Delta(ep)$ for all $p \in L$. The rule $e \mapsto \pi_e$ defines an isomorphism of $\mathrm{B}(A)$ onto $\operatorname{Proj}\operatorname{Drng} L$.*

PROOF. Set $S = \operatorname{Drng} L = L/{\sim}$. It is clear that for each $e \in \mathrm{B}(A)$, there is a projection $\pi_e \in \operatorname{Proj} S$ as described, and $\pi_{1-e} = \pi_e^{\perp}$. It is also clear that $e \leq f$ implies $\pi_e \leq \pi_f$, for $e, f \in \mathrm{B}(A)$. On the other hand, if $e \nleq f$, the projection $g = e(1-f)$ is nonzero. Note that $\pi_g(\Delta(g)) = \Delta(g) \neq 0$, whence $\pi_g \neq 0$. Since $\pi_g \leq \pi_e \wedge \pi_f^{\perp}$, it follows that $\pi_e \nleq \pi_f$. Thus, the map $\mathrm{B}(A) \to \operatorname{Proj} S$ given by $e \mapsto \pi_e$ is an order-embedding that respects complements. It only remains to show that this map is surjective.

Given $q \in \operatorname{Proj} S$, we have $\Delta(1) = q(\Delta(1)) + q^{\perp}(\Delta(1))$, and so there exist orthogonal projections $e, f \in L$ such that $1 = e \oplus f$ while $\Delta(e) = q(\Delta(1))$ and $\Delta(f) = q^{\perp}(\Delta(1))$. Further, $\Delta(e) \perp \Delta(f)$, and so there is no nonzero projection $p \in L$ such that $p \lesssim e, f$. We next show that e and f are central projections. Since $fAe = (eAf)^*$, it is enough to show that $eAf = 0$. Let x be an arbitrary element of eAf, and let p_r and p_l be the *right* and *left projections* of x, respectively ([**3**, §3, Definition 4], [**30**, p. 28]), that is, the unique projections such that $p_r^{\perp}$ and $p_l^{\perp}$ generate, respectively, the right and left annihilators of x. Since $xe = fx = 0$, we find that $e \leq p_r^{\perp}$ and $f \leq p_l^{\perp}$, that is, $p_r \leq e^{\perp} = f$ and $p_l \leq e$. However, $p_r \sim p_l$ (see [**3**, §20, Theorem 3] or [**30**, Theorem 63]), whence $p_r \lesssim e, f$ and so $p_r = 0$. Thus $x = xp_r = 0$, proving that $eAf = 0$, as desired. Consequently, e and f are central, as claimed. Now for any $r \in L$, we have $\Delta(r) = \Delta(er) + \Delta(fr)$ with $\Delta(er) \leq q(\Delta(1))$ and $\Delta(fr) \leq q^{\perp}(\Delta(1))$, whence $\Delta(er) \in q(S)$ and $\Delta(fr) \in q^{\perp}(S)$. Hence, we conclude that $\pi_e(\Delta(r)) = \Delta(er) = q(\Delta(r))$. Therefore $\pi_e = q$, completing the proof. □

In the context of Proposition 5-4.3, let us denote by $[p]$ the $\sim$-equivalence class of a projection p of A. The *addition* of these equivalence classes is defined by

$$[p] + [q] = [p \oplus q] = [p + q], \text{ for any orthogonal projections } p \text{ and } q.$$

The dimension range of L is, of course, $\operatorname{Drng} L = L/{\sim}$, equipped with the above partial addition.

Just as in Section 5-3, we can define the monoid $V(A)$ of isomorphism classes of finitely generated projective right A-modules. As noted above, projections p, $q \in A$ satisfy $p \sim q$ if and only if $pA \cong qA$, and so we obtain an embedding of partial monoids, $\operatorname{Drng} L \hookrightarrow V(A)$, where $[p] \mapsto \Delta(pA)$. Under this embedding, $[1] \mapsto \Delta(A)$. Any direct summand of the right module A has the form eA for an idempotent e, and since e is equivalent to a projection $p \in A$ [**30**, Theorem 26], we have $pA \cong eA$ and so $[p] \mapsto \Delta(eA)$. Similarly, any pair of orthogonal idempotents in A is equivalent to a pair of orthogonal projections, so that any pair of elements u, $v \in V(A)$ such that $u+v \leq \Delta(A)$ must be the image of a pair of elements of $\operatorname{Drng} L$ whose sum is defined. Thus, the embedding above maps $\operatorname{Drng} L$ isomorphically onto the interval $[0, \Delta(A)] \subseteq V(A)$, which we record in the theorem below.

THEOREM 5-4.5. *Let L be the lattice of projections of an AW*-algebra A. Then the dimension range* $\operatorname{Drng} L = L/{\sim}$ *is a (bounded) continuous dimension scale, and* $\operatorname{Drng} L \cong [0, \Delta(A)] \subseteq V(A)$. *If A is a W*-algebra, then the ultrafilter space of* $\operatorname{Proj} \operatorname{Drng} L$ *is hyperstonian.*

PROOF. That $\operatorname{Drng} L$ is a continuous dimension scale follows from Theorem 4-3.9. We have just seen above that $\operatorname{Drng} L \cong [0, \Delta(A)]$. Observe that $\operatorname{Proj} \operatorname{Drng} L \cong \mathrm{B}(A) \cong \mathrm{B}(Z)$, where Z is the center of A. If A is a W*-algebra, then so is Z, whence $Z \cong \mathbf{C}(X, \mathbb{C})$ for some hyperstonian complete Boolean space X. In particular, the ultrafilter space of $\operatorname{Proj}\operatorname{Drng} L$ is homeomorphic to X and thus it is hyperstonian. $\square$

In the context of Theorem 5-4.5, observe that by Proposition 5-4.1, $\operatorname{Proj}\operatorname{Drng} L$ is locally σ-finite in case A is a W*-algebra.

In particular, when L is the lattice of projections of an AW*-algebra A, it follows from Theorem 3-8.9 that the partial commutative monoid $L/{\sim}$ embeds as a lower subset into a commutative monoid of the form

$$\mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma),$$

for complete Boolean spaces Ω_{I}, Ω_{II}, Ω_{III}. Theorem 5-4.5 implies that these spaces must be hyperstonian in case A is a W*-algebra.

There exists a Type I, II, III decomposition for AW*-algebras (see [**30**]) which parallels that for regular, right self-injective rings; in fact, Kaplansky developed much of the Type I, II, III theory for *Baer rings* (rings in which the right or left annihilator of any element is generated by an idempotent), a class of rings which includes both AW*-algebras and regular, right self-injective rings. We shall use some of the terminology and results of this theory without explicit references. We point out that an AW*-algebra A is called a *factor* provided the center of A equals the complex field $\mathbb{C}$; equivalently, A is a factor if and only if A is nonzero and $\mathrm{B}(A) = \{0, 1\}$.

LEMMA 5-4.6. *Let γ be an ordinal and $\mathrm{J} \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. There exists a W*-factor A_{J} of Type J which contains a family $(p^{\mathrm{J}}_\alpha)_{\alpha\le\gamma}$ of nonzero purely infinite projections such that $p^{\mathrm{J}}_\alpha \lesssim p^{\mathrm{J}}_\beta$ but $p^{\mathrm{J}}_\beta \not\lesssim p^{\mathrm{J}}_\alpha$ for all ordinals $\alpha < \beta \le \gamma$.*

PROOF. Choose a Hilbert space H_γ with an orthonormal basis of cardinality $\aleph_\gamma$, and set $A_{\mathrm{I}} = \mathcal{B}(H_\gamma)$. For each ordinal $\alpha \le \gamma$, choose a projection $p^{\mathrm{I}}_\alpha \in A_{\mathrm{I}}$ such that the closed subspace $p^{\mathrm{I}}_\alpha H_\gamma$ of H_γ has an orthonormal basis of cardinality $\aleph_\alpha$. The desired properties of A_{I} and the p^{I}_α are clear.

Next, choose W*-factors B_{II} and B_{III} of Types II and III (e.g., [**10**, Part I, §9.4], [**29**, Chapters 6, 8], [**46**, Chapter 4]). These factors can be chosen as subalgebras of $\mathcal{B}(H_0)$ for a separable Hilbert space H_0 (e.g., [**10**, Remark, p. 155], [**32**, Theorem 7.3.16]), so that they are σ-finite[**32**, Proposition 1.14.3]. Now let A_{II} and A_{III} be the W*-tensor products $B_{\mathrm{II}} \mathbin{\overline{\otimes}} A_{\mathrm{I}}$ and $B_{\mathrm{III}} \mathbin{\overline{\otimes}} A_{\mathrm{I}}$. These algebras are of Types II and III, respectively (e.g., [**29**, Propositions 11.2.21, 11.2.26], [**46**, Proposition 2.6.3, Theorem 2.6.4]), and they are factors [**46**, Proposition 2.6.7].

Now let $\mathrm{J} = \mathrm{II}$ or III, and set $p^{\mathrm{J}}_\alpha = 1 \otimes p^{\mathrm{I}}_\alpha \in A_{\mathrm{J}}$ for all ordinals $\alpha \le \gamma$. It is clear that these p^{J}_α are purely infinite projections, and that $p^{\mathrm{J}}_\alpha \lesssim p^{\mathrm{J}}_\beta$ for all ordinals $\alpha < \beta \le \gamma$. Observe that the W*-algebra $p^{\mathrm{J}}_\alpha A_{\mathrm{J}} p^{\mathrm{J}}_\alpha$ is isomorphic to $B_{\mathrm{J}} \mathbin{\overline{\otimes}} p^{\mathrm{I}}_\alpha A_{\mathrm{I}} p^{\mathrm{I}}_\alpha$, which is in turn isomorphic to a W*-subalgebra of $\mathcal{B}(H_0 \otimes p^{\mathrm{I}}_\alpha H_\gamma)$. Since $H_0 \otimes p^{\mathrm{I}}_\alpha H_\gamma$ has an orthonormal basis of cardinality $\aleph_\alpha$, we thus see that p^{J}_α does not majorize any orthogonal family of more than $\aleph_\alpha$ nonzero projections. On the other hand, $\aleph_\beta \cdot p^{\mathrm{I}}_\beta \sim p^{\mathrm{I}}_\beta$, whence p^{J}_β majorizes an orthogonal family of $\aleph_\beta$ nonzero projections (equivalent to itself). Therefore $p^{\mathrm{J}}_\beta \not\lesssim p^{\mathrm{J}}_\alpha$. $\square$

We can now show that the class of projection lattices of W*-algebras, while not D-universal, is at least D-universal relative to continuous dimension scales for which the ultrafilter space of the Boolean algebra of projections is hyperstonian.

THEOREM 5-4.7. *Let Ω_{I}, Ω_{II}, Ω_{III} be arbitrary hyperstonian spaces (possibly empty), and let γ be an arbitrary ordinal. Then there exists a W*-algebra A such that*

$$V(A) \cong \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma).$$

PROOF. For J = I, II, III, choose W*-factors A_{J} and families $(p^{\mathrm{J}}_\alpha)_{\alpha\leq\gamma}$ of purely infinite projections as in Lemma 5-4.6. Let $C_{\mathrm{J}} = \mathbf{C}(\Omega_{\mathrm{J}}, \mathbb{C})$, which is a W*-algebra because Ω_{J} is hyperstonian, and note that $\mathrm{B}(C_{\mathrm{J}})$ is isomorphic to the Boolean algebra of clopen subsets of Ω_{J}. Let D_{J} be the W*-tensor product $A_{\mathrm{J}} \mathbin{\overline{\otimes}} C_{\mathrm{J}}$, which has Type J by the results referenced in Lemma 5-4.6. Since A_{J} is a factor, the centers of C_{J} and D_{J} are isomorphic [**46**, Proposition 2.6.7], and thus $\mathrm{B}(C_{\mathrm{J}}) \cong \mathrm{B}(D_{\mathrm{J}})$, *via* the map $e \mapsto 1 \otimes e$. Consequently, if L_{J} is the lattice of projections of D_{J}, the ultrafilter space of $\operatorname{Proj}\operatorname{Drng} L_{\mathrm{J}}$ is homeomorphic to Ω_{J}.

Since D_{J} is of Type J, it follows that $L_{\mathrm{J}}/{\sim}$ is of Type J, that is, $L_{\mathrm{J}}/{\sim} = (L_{\mathrm{J}}/{\sim})_{\mathrm{J}}$ in the notation of Definition 3-7.8. Set $q^{\mathrm{J}}_\alpha = p^{\mathrm{J}}_\alpha \otimes 1 \in D_{\mathrm{J}}$ for all ordinals $\alpha \leq \gamma$, and observe that the q^{J}_α are purely infinite projections with central cover 1, such that $q^{\mathrm{J}}_\alpha \lesssim q^{\mathrm{J}}_\beta$ for all ordinals $\alpha < \beta \leq \gamma$.

CLAIM. *For any ordinals $\alpha < \beta \leq \gamma$, we have $rq^{\mathrm{J}}_\beta \not\lesssim rq^{\mathrm{J}}_\alpha$ for all nonzero central projections r in D_{J}.*

PROOF OF CLAIM. For each point $x \in \Omega_{\mathrm{J}}$, let $\pi_x \colon D_{\mathrm{J}} \to A_{\mathrm{J}} \mathbin{\overline{\otimes}} \mathbb{C} \cong A_{\mathrm{J}}$ be the W*-algebra homomorphism obtained by tensoring the identity map on A_{J} with the evaluation map $f \mapsto f(x)$ from C_{J} to $\mathbb{C}$. Observe that $\pi_x(q^{\mathrm{J}}_\alpha) = p^{\mathrm{J}}_\alpha$ and $\pi_x(q^{\mathrm{J}}_\beta) = p^{\mathrm{J}}_\beta$. Moreover, $r = 1 \otimes e$ for some projection $e \in \mathbf{C}(\Omega_{\mathrm{J}}, \mathbb{C})$, and $\pi_x(r) = e(x) \in \{0, 1\}$. If $rq^{\mathrm{J}}_\beta \lesssim rq^{\mathrm{J}}_\alpha$, then $e(x)p^{\mathrm{J}}_\beta \lesssim e(x)p^{\mathrm{J}}_\alpha$ for all $x \in \Omega_{\mathrm{J}}$. Since $p^{\mathrm{J}}_\beta \not\lesssim p^{\mathrm{J}}_\alpha$, we must have $e(x) = 0$ for all $x \in \Omega_{\mathrm{J}}$, and thus $r = 0$. This contradiction establishes the claim. □ Claim.

We now apply Proposition 4-5.6, and conclude that there exist projections $r_{\mathrm{J}} \in L_{\mathrm{J}}$ for each J such that the dimension ranges of the intervals $[0, r_{\mathrm{J}}]$ have the following form:

$$\begin{aligned} [0, r_{\mathrm{I}}]/{\sim} &\cong \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \\ [0, r_{\mathrm{II}}]/{\sim} &\cong \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \\ [0, r_{\mathrm{III}}]/{\sim} &\cong \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma). \end{aligned}$$

Therefore the dimension range of the lattice of projections of the W*-algebra

$$A = r_{\mathrm{I}} D_{\mathrm{I}} r_{\mathrm{I}} \times r_{\mathrm{II}} D_{\mathrm{II}} r_{\mathrm{II}} \times r_{\mathrm{III}} D_{\mathrm{III}} r_{\mathrm{III}}$$

has the desired form. Note that each of the projections r_{J} is purely infinite, whence the projection $1 \in A$ is purely infinite, and consequently $[0, \Delta(A)] = V(A)$. Therefore, in view of Theorem 5-4.5, the present theorem is proved. □

COROLLARY 5-4.8. *Let S be a continuous dimension scale. Then S admits a lower embedding into the dimension range of the lattice of projections of some W*-algebra if and only if the ultrafilter space of* $\operatorname{Proj} S$ *is hyperstonian.*

PROOF. The sufficiency follows from Theorems 3-8.9 and 5-4.7, and the necessity from Theorem 5-4.5. □

In order to see that the projection lattices of AW*-algebras form a D-universal class of espaliers, we need an analogue of Theorem 5-4.7 in which Ω_{I}, Ω_{II}, Ω_{III} are arbitrary complete Boolean spaces and A is an AW*-algebra. However, there is no general theory of AW*-tensor products available to replace the W*-tensor products $A_{\mathrm{J}} \overline{\otimes} C_{\mathrm{J}}$ used in our proof. P. Ara has suggested that one might be able to use the *monotone complete tensor products* introduced by M. Hamana [**23, 24**] instead. (We thank him for making us aware of Hamana's work.) Rather than developing the necessary auxiliary results about monotone complete tensor products here, we complete the picture by taking a different route. Namely, we borrow the methods and results of G. Takeuti [**48**] and some of the subsequent results obtained in M. Ozawa [**45**]. These methods involve forcing, more specifically, the Scott-Solovay model $V^B$of B-valued set theory (also used in Section 5-1), for any complete Boolean algebra B.

We give a short summary of what we shall use from [**48, 45**]. If $\boldsymbol{A}$ is an AW*-algebra in V^B, the *bounded global section algebra* $\widetilde{\boldsymbol{A}}$ of $\boldsymbol{A}$ is the set of all $\boldsymbol{x} \in V^B$ such that $\|\boldsymbol{x} \in \boldsymbol{A}\| = 1$ and $\|\|\boldsymbol{x}\|_{\boldsymbol{A}} \leq \check{n} 1_{\boldsymbol{A}}\| = 1$ for some constant $n \in \mathbb{N}$, endowed with its canonical structure of AW*-algebra. For example, $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$ if and only if $\|\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}\| = 1$. The center of $\widetilde{\boldsymbol{A}}$ contains a copy of the bounded global section algebra $\widetilde{\mathbb{C}}$ of the complex numbers. Observe that $\widetilde{\mathbb{C}}$ is isomorphic to the algebra of continuous maps from the ultrafilter space of B to $\mathbb{C}$. In case $\boldsymbol{A}$ is an AW*-factor in V^B, the center of $\widetilde{\boldsymbol{A}}$ is exactly (the canonical copy of) $\widetilde{\mathbb{C}}$, see [**45**, Theorem 5]. In particular, for $u \in B$, the central idempotent of $\widetilde{\boldsymbol{A}}$ corresponding to u is the unique element $\boldsymbol{u} \in \widetilde{\boldsymbol{A}}$ such that $u = \|\boldsymbol{u} = 1\|$ while $\neg u = \|\boldsymbol{u} = 0\|$. By Lemma 5-4.4, the Boolean algebra of projections of $\widetilde{\boldsymbol{A}}$ is also isomorphic to B. Thus, letting L be the espalier of projections of $\widetilde{\boldsymbol{A}}$ and identifying the elements of B with the projections of $\operatorname{Drng} L$, we obtain that $\|\boldsymbol{a} \leq \boldsymbol{b}\|$ has the same meaning in the present paper and in [**48, 45**].

We apply this to the D-universality problem as follows.

THEOREM 5-4.9. *Let Ω_{I}, Ω_{II}, Ω_{III} be arbitrary complete Boolean spaces (possibly empty), and let γ be an arbitrary ordinal. Then there exists an AW*-algebra A such that*

$$V(A) \cong \mathbf{C}(\Omega_{\mathrm{I}}, \mathbb{Z}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{II}}, \mathbb{R}_\gamma) \times \mathbf{C}(\Omega_{\mathrm{III}}, \mathbf{2}_\gamma).$$

PROOF. By arguing as in the proof of Theorem 5-4.7, it suffices to prove that for every complete Boolean algebra B, every ordinal γ, and every $\mathrm{J} \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$, there exist an AW*-algebra A of type J with algebra of central idempotents isomorphic to B and a $(\gamma+1)$-sequence $(p_\alpha)_{\alpha \leq \gamma}$ of projections of central cover 1 such that $p_\alpha \lesssim p_\beta$ but $\|p_\beta \lesssim p_\alpha\| = 0$, for $\alpha < \beta \leq \gamma$.

By applying Lemma 5-4.6 within V^B, we obtain a factor $\boldsymbol{A}$ of type J in V^B and a B-valued name $\boldsymbol{p}$ such that the following statements hold in V^B (that is, they have Boolean value 1):

$$\boldsymbol{p} \text{ is a map from } \check{\gamma} \text{ to the purely infinite elements of } \boldsymbol{L},$$

$$\boldsymbol{p}(\boldsymbol{\alpha}) \lesssim \boldsymbol{p}(\boldsymbol{\beta}), \text{ for all } \boldsymbol{\alpha} < \boldsymbol{\beta} \leq \check{\gamma}, \tag{5-4.1}$$

$$\boldsymbol{p}(\boldsymbol{\beta}) \not\lesssim \boldsymbol{p}(\boldsymbol{\alpha}), \text{ for all } \boldsymbol{\alpha} < \boldsymbol{\beta} \leq \check{\gamma}, \tag{5-4.2}$$

where $\boldsymbol{L}$ denotes the espalier of projections of $\boldsymbol{A}$ within V^B.

Now let $A = \widetilde{\boldsymbol{A}}$ be the bounded global section algebra of $\boldsymbol{A}$. It follows from [**45**, Theorem 7] (see also [**48**, §2]) that A is Type J, furthermore, its algebra of central idempotents is isomorphic to B. For all $\alpha \leq \gamma$, let $\boldsymbol{p}_\alpha$ be the unique B-valued name such that $\|\boldsymbol{p}_\alpha = \boldsymbol{p}(\check{\alpha})\| = 1$. For $\alpha < \beta \leq \gamma$, it follows from (5-4.1), (5-4.2) that $\boldsymbol{p}_\alpha \lesssim \boldsymbol{p}_\beta$ and $\left\|\boldsymbol{p}_\beta \lesssim \boldsymbol{p}_\alpha\right\| = 0$. □

Therefore, we have obtained the following result, which, together with the other main results of the present section, completely elucidates the dimension theory of projections of W*- and AW*-algebras.

THEOREM 5-4.10. *The class of espaliers obtained from projection lattices of AW*-algebras is D-universal.*

5-5. Concluding remarks

The questions arising naturally from this work can be divided in two parts: namely, those where the theory reflects about itself, and those where it reflects about other topics.

In the first group, we shall mention the following. For given, "practical" examples, where we need to verify that a given structure is an espalier, the axiom (L7) is often a source of problems. Thus we may ask to what extent it is possible to remove Axiom (L7) from the definition of an espalier, thus defining "pre-espaliers" (see also Definition 5-1.3). But then, in order to extend a pre-espalier $(L, \leq, \perp, \sim)$ to an espalier, we need to define a new binary relation $\sim^*$ on L by letting $x \sim^* y$ hold, if there are decompositions $x = \oplus_{i\in I} x_i$ and $y = \oplus_{i\in I} y_i$ such that $x_i \sim y_i$, for all $i \in I$. However, proving the transitivity of the new relation $\sim^*$ leads to the verification of a common refinement property, see Lemma 5-1.4 for the Boolean case. This problem can be formulated as follows.

PROBLEM. Let $(L, \leq, \perp, \sim)$ be a structure satisfying all axioms from (L0) to (L8) with the possible exception of (L7), and let $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ be families of elements of L such that $\oplus_{i\in I} x_i = \oplus_{j\in J} y_j$. Are there families $(u_{i,j})_{(i,j)\in I\times J}$ and $(v_{i,j})_{(i,j)\in I\times J}$ of elements of L such that $x_i = \oplus_{j\in J} u_{i,j}$ (for all $i \in I$), $y_j = \oplus_{i\in I} v_{i,j}$ (for all $j \in J$), and $u_{i,j} \sim v_{i,j}$ (for all $(i,j) \in I \times J$)?

The second group of questions asks for constructing further classes of espaliers, within other areas of mathematics. Of course, isomorphism types of various structures are privileged, see, for example, B. Jónsson and A. Tarski's appendix in [**49**]. In another direction, one might ask about extensions of various results of cancellation or unique decomposition, known for finite structures (see [**37**, Chapter 5]) to infinite structures subjected to completeness conditions. This would in turn yield, for example, nontrivial cancellation results for further infinite structures, of which the main result of [**28**] about *σ-complete effect algebras* would be a prototype.

Expecting infinite generalizations of finite results *via* espaliers is reasonable as long as there are enough refinement theorems around, see, again, [**37**, Chapter 5]. Hence the Lovász cancellation theorems, see [**36**] or [**37**, Section 5.7], do not enter this category, as they are established by counting arguments, in contexts where refinement does not always hold. We do not know of any framework that could extend Lovász's results to infinite structures subjected to completeness conditions.

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Index