HOMOLOGICAL PROPERTIES OF QUANTIZED COORDINATE RINGS OF SEMISIMPLE GROUPS

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Abstract. We prove that the generic quantized coordinate ring $\mathcal{O}_q(G)$ is Auslander-regular, Cohen-Macaulay, and catenary for every connected semisimple Lie group G. This answers questions raised by Brown, Lenagan, and the first author. We also prove that under certain hypotheses concerning the existence of normal elements, a noetherian Hopf algebra is Auslander-Gorenstein and Cohen-Macaulay. This provides a new set of positive cases for a question of Brown and the first author.

0. Introduction

A guiding principle in the study of quantized coordinate rings has been that these algebras should enjoy noncommutative versions of the algebraic properties of their classical analogs. Moreover, based on the types of properties that have been established, one also conjectures that quantized coordinate rings should enjoy properties similar to the enveloping algebras of solvable Lie algebras. A property of the latter type is the *catenary* condition (namely, that all saturated chains of prime ideals between any two fixed primes should have the same length), which was established for a number of quantized coordinate rings by Lenagan and the first author [11]. However, among the quantized coordinate rings $\mathcal{O}_q(G)$ for semisimple Lie groups G, catenarity has remained an open question except for the case $G = SL_n$ [11, Theorem 4.5]. The first goal of the present paper is to establish catenarity for (the $\mathbb{C}$-form of) all the algebras $\mathcal{O}_q(G)$.

Following the principle indicated above, one expects the quantized coordinate rings of Lie groups to be homologically nice; in the noncommutative world, this means one looks for the *Auslander-regular* and *Cohen-Macaulay* conditions. In fact, Gabber's method for proving catenarity in enveloping algebras of solvable Lie algebras relies crucially on these properties, as does Lenagan and the first author's adaptation to quantum algebras. These conditions were verified for $\mathcal{O}_q(SL_n)$ by Levasseur and Stafford, but remained open for arbitrary $\mathcal{O}_q(G)$, although Brown and the first author were able to show that $\mathcal{O}_q(G)$ has finite global dimension [6, Proposition 2.7]. Our second goal here is to establish the Auslander-regular and Cohen-Macaulay conditions for general $\mathcal{O}_q(G)$.

Let G be a connected semisimple Lie group over $\mathbb{C}$. By $\mathcal{O}_q(G)$, we mean the subalgebra of the Hopf dual of $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G, generated by the coordinate functions of the type $\mathbf{1}$ highest weight modules whose weights

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lie in the lattice corresponding to a maximal torus of G. (See [7, Definition I.7.5], for instance.) We indicate that we consider the usual one-parameter version of this algebra by calling it the *standard* quantized coordinate ring of G. Since some of our work relies on properties established by Hodges-Levasseur-Toro [16] and Joseph [18], we must restrict attention to the *complex form* of $\mathcal{O}_q(G)$, that is, we construct $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$ over the base field $\mathbb{C}$. The assumption that q is *generic* means that q is a nonzero complex scalar which is not a root of unity; in [18], q is further restricted to be a scalar transcendental over the rationals.

We summarize our main results for $\mathcal{O}_q(G)$ as follows.

Theorem 0.1. *Let G be a connected semisimple Lie group over $\mathbb{C}$. Then the standard generic quantized coordinate ring $\mathcal{O}_q(G)$ (defined over $\mathbb{C}$) is catenary, and Tauvel's height formula holds. If q is transcendental over $\mathbb{Q}$, then $\mathcal{O}_q(G)$ is an Auslander-regular and Cohen-Macaulay domain with* $\operatorname{gldim} \mathcal{O}_q(G) = \operatorname{GKdim} \mathcal{O}_q(G) = \dim G$.

The third goal of the paper concerns the question

If H is a noetherian Hopf algebra, does H have finite injective dimension?

This was asked by Brown and the first author [6, 1.15] and Brown [5, 3.1]. Some partial results were given by Wu and the second author when H is PI [30, Theorem 0.1], and when H is graded with balanced dualizing complex [31, Theorem 1]. Here we give another partial answer, for which we need to state some hypotheses. Let A be a ring (for the first condition) or an algebra over a base field k (for the second and third). We say that A satisfies

- (H1) if A is noetherian and $\operatorname{Spec} A$ is *normally separated*, namely, for every two primes $\mathfrak{p} \subsetneq \mathfrak{q}$ of A, there is a normal regular element in the ideal $\mathfrak{q}/\mathfrak{p} \subset A/\mathfrak{p}$.
- (H2) if A has an exhaustive ascending $\mathbb{N}$-filtration such that the associated graded ring $\operatorname{gr} A$ is connected graded noetherian with *enough normal elements*, namely, every non-simple graded prime factor ring of $\operatorname{gr} A$ contains a homogeneous normal element of positive degree.
- (H3) if every maximal ideal of A is co-finite dimensional over k.

Our second result is the following.

Theorem 0.2. *Suppose A is a Hopf algebra satisfying* (H1,H2,H3). *Then A is Auslander-Gorenstein and Cohen-Macaulay and has a quasi-Frobenius classical quotient ring; further,* $\operatorname{Spec} A$ *is catenary and Tauvel's height formula holds.*

It is known that the quantized coordinate rings $\mathcal{O}_q(G)$ satisfy (H1) and (H2), but it is not known if they satisfy (H3). Therefore the proofs of Theorems 0.1 and 0.2 are similar in some steps and different in others.

Since our methods do not apply to quotient Hopf algebras of the $\mathcal{O}_q(G)$, we pose the following:

Question 0.3. Let G be a connected semisimple Lie group over $\mathbb{C}$, and I a Hopf ideal of the standard generic quantized coordinate ring $\mathcal{O}_q(G)$. Is the Hopf algebra $\mathcal{O}_q(G)/I$ Auslander-Gorenstein and Cohen-Macaulay?

Throughout, let k be a commutative base field. In treating general properties of algebras, the base field will be k; in particular, the unqualified term "algebra" will mean "k-algebra", and unmarked tensor products will be $\otimes_k$. In working with $\mathcal{O}_q(G)$, we will need to assume k is either $\mathbb{C}$ or a subfield of $\mathbb{C}$.

Let A be a ring or an algebra. Let A° denote the opposite ring of A, and A^{e} (when A is an algebra) the enveloping algebra $A \otimes A^\circ$. Since we usually work with left modules, the unqualified term "module" will mean "left module". Right A-modules will be identified, when convenient, with A°-modules, and (A, B)-bimodules with $(A \otimes B^\circ)$-modules. Let A-Mod and A°-Mod denote the categories of all left and right A-modules, respectively.

When working with an (A, B)-bimodule M, the key property is that M is finitely generated as both a left A-module and a right B-module (note that this is stronger than requiring M to be finitely generated as a bimodule). We can record this property by saying that M is *left-right finitely generated*. Since all our rings A and B will be noetherian, this is equivalent to saying that M is *left-right noetherian*, which we abbreviate to *noetherian*, following common usage. To summarize: a *noetherian bimodule* is a bimodule which is noetherian as both a left module and a right module.

1. Consequences of (H1)

Recall that a ring A satisfies (H1) if A is noetherian and $\operatorname{Spec} A$ is normally separated. It will be helpful to recall that these hypotheses imply that A satisfies the (left and right) strong second layer condition [14, Theorem 12.17]. The main result of this section is the following proposition. Here, modules are treated as complexes concentrated in degree 0, and we write $X \cong Y$ to indicate that complexes X and Y are *quasi-isomorphic*, that is, isomorphic in the derived category.

Proposition 1.1. *Suppose A and B are rings satisfying* (H1)*. Let X be a bounded complex of (A, B)-bimodules such that the bimodule $\mathrm{H}^i(X)$ is noetherian for all i.*

(a) *If $\operatorname{Tor}_i^A(A/\mathfrak{p}, X) = 0$ for all $i \neq 0$ and all maximal ideals $\mathfrak{p} \subset A$, then $X \cong \mathrm{H}^0(X)$ and $\mathrm{H}^0(X)$ is a projective A-module.*

(b) *If, in addition, $\operatorname{Tor}_0^A(A/\mathfrak{p}, X) \neq 0$ for all maximal ideals $\mathfrak{p} \subset A$, then $\mathrm{H}^0(X)$ is a progenerator in A-Mod.*

We need several lemmas to prove this proposition. Recall from [4, Definition 1] that a *bond* between two noetherian prime rings A and B is a nonzero noetherian (A, B)-bimodule which is torsionfree both as a left A-module and as a right B-module. The following lemma of the first author appears as [4, Theorem 10].

Lemma 1.2. *Let A and B be prime noetherian rings, and M a bond between them. If B is simple, then M is projective over A.*

Lemma 1.3. *Let A and B be noetherian simple rings, and M a nonzero noetherian (A, B)-bimodule. Then M is a progenerator for A-Mod and for B°-Mod.*

Proof. It follows from [14, Lemma 8.3] that M is torsionfree on each side. Thus, M is a bond, and so by Lemma 1.2, M is projective over A. Simplicity of A then implies that M is a generator over A. Since M is finitely generated over A, it is a progenerator. By symmetry, M is also a progenerator over B. □

Let A be any ring. Let $N(A)$ denote the set of all regular normal elements in A, and note that $N(A)$ is an Ore set. Then, let $S(A)$ denote the localization of A by inverting all elements in $N(A)$. If the set of all regular elements in A satisfies the left and right Ore conditions, then we use $Q(A)$ to denote the localization of A by inverting all regular elements. The ring $Q(A)$ is called the *classical quotient ring* of A. Let Kdim denote the Gabriel-Rentschler Krull dimension.

The following lemma is clear.

Lemma 1.4. *Suppose A is a prime ring satisfying* (H1)*. Then every nonzero ideal of A contains a regular normal element. Consequently, the localization $S(A)$ is simple.*

Lemma 1.5. *Suppose A and B are prime rings satisfying* (H1)*, and that M is a bond between them.*

(a) *$S(A) \otimes_A M \cong M \otimes_B S(B) \cong S(A) \otimes_A M \otimes_B S(B)$.*
(b) *A is simple if and only if B is simple.*
(c) *If $M \cong S(A) \otimes_A M$, then $A = S(A)$. In particular, A is simple.*

Proof. (a) Since M is a bond, left multiplication on M by any $d \in N(A)$ is injective. Hence, $S(A) \otimes_A M$ is a directed union of copies of M as a right B-module. This implies that we can make identifications

$$M \subset S(A) \otimes_A M \subset S(A) \otimes_A M \otimes_B S(B).$$

Let $e \in N(B)$, and let N be the bimodule M/Me. Since N is torsion as a right B-module, it follows from [14, Lemma 8.3] and Lemma 1.4 that there is an $f \in N(A)$ such that $fN = 0$. This implies that $S(A) \otimes_A N = 0$, and hence $S(A) \otimes_A M = S(A) \otimes_A Me$. So, the action of e on $S(A) \otimes_A M$ is invertible. This says that $S(A) \otimes_A M = S(A) \otimes_A M \otimes_B S(B)$. By symmetry, $M \otimes_B S(B) = S(A) \otimes_A M \otimes_B S(B)$.

(b) Since A and B satisfy the second layer condition, they have the same classical Krull dimension [14, Corollary 14.5]. Part (b) follows.

(c) Since $S(A)$ is simple and $M \cong S(A)\otimes_A M$ is a noetherian $(S(A), B)$-bimodule, there is a positive integer n such that $S(A)$ embeds in $M^{\oplus n}$ as a left $S(A)$-module [14, Lemma 8.1]. Consequently, $S(A)$ is a noetherian left A-module. This implies that, for each $e \in N(A)$, we have $Ae^{-i} = Ae^{-i+1}$ for some i. Thus $Ae^{-1} = A$, and hence e is invertible in A. Therefore $S(A) = A$. Simplicity follows from Lemma 1.4. □

Lemma 1.6. *Suppose A and B are rings satisfying* (H1)*, and A is prime. Let N be a nonzero noetherian (A, B)-bimodule.*

(a) *If $S(A) \otimes_A N$ is nonzero, then it is a progenerator for $S(A)$-*Mod*.*
(b) *Similarly, if A is simple, then N is a progenerator for A-*Mod*.*
(c) *If $N \cong S(A) \otimes_A N$, then $A = S(A)$ and A is simple.*

Proof. (a) First asssume that $S(A) \otimes_A N \neq 0$, and let T_1 denote the $N(A)$-torsion sub(bi)module of N. Since $S(A) \otimes_A N \cong S(A) \otimes_A (N/T_1)$, there is no loss of generality in assuming that N is $N(A)$-torsionfree. If T_2 is the torsion sub(bi)module of ${}_AN$, then $\mathrm{l.ann}_A(T_2)$ is nonzero by [14, Lemma 8.3]. Then $\mathrm{l.ann}_A(T_2)$ contains an element of $N(A)$ by Lemma 1.4, and since N is $N(A)$-torsionfree, we conclude that $T_2 = 0$. Thus, N is torsionfree as a left A-module.

Let $N_0 = 0 \subset N_1 \subset \cdots \subset N_m = N$ be a right affiliated series for N (cf. [14, Chapter 8]). The ideals $\mathfrak{q}_i := \mathrm{r.ann}_B(N_i/N_{i-1})$ are prime, and N_i/N_{i-1} is a torsionfree right $(B/\mathfrak{q}_i)$-module as well as a torsionfree left A-module [14, Propositions 8.7, 8.9]. It suffices to show that each $S(A) \otimes_A (N_i/N_{i-1})$ is a progenerator for $S(A)$-Mod. Hence, we need only consider the case when B is prime and N is a bond between A and B.

Now we are in the situation of Lemma 1.5, that is, we have $N' := S(A) \otimes_A N \cong S(A) \otimes_A N \otimes_B S(B)$. Then N' is a bond between $S(A)$ and $S(B)$. By Lemma 1.4, $S(A)$ and $S(B)$ are simple; thus, the conclusion of part (a) follows from Lemma 1.3.

(b) This is proved in the same manner as (a), with the help of Lemma 1.5(b).

(c) The isomorphism $N \cong S(A) \otimes_A N$ implies that N is $N(A)$-torsionfree, and then, as in the proof of (a), it follows that ${}_AN$ is torsionfree. Choose a right affiliated series for N, and let N' be its penultimate term. Then $\mathfrak{q} := \mathrm{r.ann}_B(N/N')$ is a prime ideal of B, and N/N' is a bond between A and $B/\mathfrak{q}$. Since N/N' is torsionfree as a left A-module, it is also $N(A)$-torsionfree, and so $S(A) \otimes_A (N/N') \neq 0$. On the other hand, since the natural map $N \to S(A) \otimes_A N$ is surjective, so is the natural map $N/N' \to S(A) \otimes_A (N/N')$. Consequently, $N/N' \cong S(A) \otimes_A (N/N')$, and therefore Lemma 1.5(c) implies that $A = S(A)$. $\square$

Lemma 1.7. *Let A, B, X satisfy the same hypotheses as* Proposition 1.1(a). *Then* $\mathrm{Tor}_i^A(N, X) = 0$ *for all noetherian A-bimodules N and all $i \neq 0$. As a consequence,* $X \cong \mathrm{H}^0(X)$.

Proof. Let N be a noetherian A-bimodule. By induction on the length of X, it is easy to see that each $\mathrm{Tor}_i^A(N, X)$ is a noetherian (A, B)-bimodule.

We will use induction on $\mathrm{rKdim}\, N = \mathrm{Kdim}\, N_A$. If $r\,\mathrm{Kdim}\, N < 0$, the bimodule $N = 0$, and the assertion is trivial. Now assume that $\mathrm{rKdim}\, N \geq 0$, and that $\mathrm{Tor}_i^A(N', X) = 0$ for all $i \neq 0$ and all noetherian A-bimodules N' with $\mathrm{rKdim}\, N' < \mathrm{rKdim}\, N$. We want to show that $\mathrm{Tor}_i^A(N, X) = 0$ for all $i \neq 0$.

By [14, Corollary 8.8], there exists a chain of sub-bimodules

$$N_0 = 0 \subset N_1 \subset \cdots \subset N_m = N$$

such that the annihilator ideals $\mathfrak{p}_j := \mathrm{l.ann}_A(N_j/N_{j-1})$ and $\mathfrak{q}_j := \mathrm{r.ann}_A(N_j/N_{j-1})$ are prime and N_j/N_{j-1} is a bond between $A/\mathfrak{p}_j$ and $A/\mathfrak{q}_j$. It suffices to show that $\mathrm{Tor}_i^A(N_j/N_{j-1}, X) = 0$ for all $i \neq 0$ and all j. Thus, there is no loss of generality in assuming that N is a bond between $A/\mathfrak{p}$ and $A/\mathfrak{q}$, for some prime ideals $\mathfrak{p}, \mathfrak{q} \subset A$. If either $\mathfrak{p}$ or $\mathfrak{q}$ is a maximal ideal, then both are maximal, by Lemma 1.5(b). Then, by Lemma 1.2, N is projective over $A/\mathfrak{q}$. Since $\mathrm{Tor}_i^A(A/\mathfrak{q}, X) = 0$ for all $i \neq 0$, it follows that $\mathrm{Tor}_i^A(N, X) = 0$ for all $i \neq 0$.

Now assume that $\mathfrak{p}$ and $\mathfrak{q}$ are both non-maximal, and let $A_1 := A/\mathfrak{p}$ and $A_2 := A/\mathfrak{q}$. For any $d \in N(A_1)$, one has $\mathrm{rKdim}\, N/dN < \mathrm{rKdim}\, N$. Applying $\mathrm{Tor}_i^A(-, X)$ to the short exact sequence

$$0 \to N \xrightarrow{l_d} N \to N/dN \to 0,$$

where l_d is the left multiplication map by d, we obtain a long exact sequence

$$\mathrm{Tor}_{i+1}^A(N/dN, X) \to \mathrm{Tor}_i^A(N, X) \xrightarrow{l_d} \mathrm{Tor}_i^A(N, X) \to \mathrm{Tor}_i^A(N/dN, X)$$

for each i. For $i > 0$ or $i < -1$, the two ends are zero by our induction hypothesis. Hence, $T := \mathrm{Tor}_i^A(N, X)$ is an $N(A_1)$-torsionfree A_1-module, and $T \cong S(A_1) \otimes_{A_1} T$. If $T \neq 0$, Lemma 1.6(c) implies that A_1 is simple, a contradiction. Therefore $T = 0$ as desired.

For $i = -1$, we have a long exact sequence

$$\mathrm{Tor}_0^A(N/dN, X) \to \mathrm{Tor}_{-1}^A(N, X) \xrightarrow{l_d} \mathrm{Tor}_{-1}^A(N, X) \to \mathrm{Tor}_{-1}^A(N/dN, X).$$

By the induction hypothesis, $\mathrm{Tor}_{-1}^A(N/dN, X) = 0$, and so the map l_d is surjective. Let $U := \mathrm{Tor}_{-1}^A(N, X)$, and let V be the $\mathcal{N}(A_1)$-torsion submodule of U. Since U is a bimodule, so is V. Now l_d acts on U/V bijectively. The argument for T shows that $U/V = 0$, and so U is $\mathcal{N}(A_1)$-torsion. Since U is a noetherian bimodule, there is a $d \in \mathcal{N}(A_1)$ such that $dU = 0$. But $U = dU$, so $U = 0$. Therefore $\mathrm{Tor}_i^A(N, X) = 0$ for all $i \neq 0$, as desired.

Finally, take $N = A$; then $\mathrm{H}^i(X) \cong \mathrm{Tor}_i^A(A, X) = 0$ for all $i \neq 0$. Therefore $X \cong \mathrm{H}^0(X)$. □

Now we are ready to prove Proposition 1.1.

Proof of Proposition 1.1. (a) By Lemma 1.7, X is quasi-isomorphic to the bimodule $\mathrm{H}^0(X)$. Since $\mathrm{Tor}_i^A(A/\mathfrak{p}, \mathrm{H}^0(X)) \cong \mathrm{Tor}_i^A(A/\mathfrak{p}, X) = 0$ for all $i \neq 0$ and all maximal ideals $\mathfrak{p} \subset A$, we may replace X by $\mathrm{H}^0(X)$. Thus, there is no loss of generality in assuming that X is a noetherian (A, B)-bimodule.

If ${}_AX$ is not projective, then it is not flat, and $\mathrm{Tor}_j^A(N, X) \neq 0$ for some $j > 0$ and some finitely generated right A-module N. By noetherian induction, we may assume that the ideal $\mathfrak{q} := \mathrm{r.ann}_A(N)$ is maximal for the existence of such an N, that is, $\mathrm{Tor}_i^A(M, X) = 0$ for any $i \neq 0$ and any finitely generated right A-module M whose annihilator properly contains $\mathfrak{q}$. By a second noetherian induction, we may assume that N is a minimal criminal, that is, $\mathrm{Tor}_i^A(N/L, X) = 0$ for any $i \neq 0$ and any nonzero submodule $L \subseteq N$. If there is a nonzero submodule $L \subseteq N$ with $\mathrm{r.ann}_A(L) \supsetneq \mathfrak{q}$, then $\mathrm{Tor}_j^A(L, X) = 0$ and $\mathrm{Tor}_j^A(N/L, X) = 0$, by our two induction hypotheses, which implies the contradiction $\mathrm{Tor}_j^A(N, X) = 0$. Thus, all nonzero submodules of N have annihilator $\mathfrak{q}$, that is, N is fully faithful as a right $A/\mathfrak{q}$-module. It follows that $\mathfrak{q}$ is a prime ideal of A [14, Proposition 3.12].

Set $A_2 := A/\mathfrak{q}$, and note that for any $c \in \mathcal{N}(A_2)$, the set $\mathrm{ann}_N(c)$ is a submodule of N, annililated by the nonzero ideal $cA_2 \subset A_2$. This forces $\mathrm{ann}_N(c) = 0$, since N is a fully faithful A_2-module. Thus, N is $\mathcal{N}(A_2)$-torsionfree. In particular, the natural map

$$N \longrightarrow N' := N \otimes_{A_2} S(A_2) = N \otimes_A S(A_2)$$

is injective. We identify N with its image in N'.

Each finitely generated submodule $M \subset N'/N$ is annihilated by some element of $\mathcal{N}(A_2)$, whence $\mathrm{r.ann}_A(M) \supsetneq \mathfrak{q}$. By our first noetherian induction, $\mathrm{Tor}_i^A(M, X) = 0$ for all $i \neq 0$, for any such M. Since $\mathrm{Tor}_i^A(-, X)$ commutes with direct limits, it follows that $\mathrm{Tor}_i^A(N'/N, X) = 0$ for all $i \neq 0$.

As a right A-module, $S(A_2)$ is the direct limit of the submodules $c^{-1}A_2$ for $c \in \mathcal{N}(A_2)$, and each $c^{-1}A_2 \cong A_2$. Since $\mathrm{Tor}_i^A(A_2, X) = 0$ for all $i \neq 0$, by Lemma 1.7, it follows that $\mathrm{Tor}_i^A(S(A_2), X) = 0$ for all $i \neq 0$. Now set $K_{-1} = X$, and choose short exact sequences of left A-modules

$$0 \to K_m \to F_m \to K_{m-1} \to 0$$

for all $m \geq 0$, where each F_m is free. Since

$$\mathrm{Tor}_1^A(S(A_2), K_{m-1}) \cong \mathrm{Tor}_{m+1}^A(S(A_2), X) = 0$$

for all $m \geq 0$, the induced sequences

$$0 \to S(A_2) \otimes_A K_m \to S(A_2) \otimes_A F_m \to S(A_2) \otimes_A K_{m-1} \to 0 \tag{E1.1}$$

are all exact. Further, $S(A_2) \otimes_A X \cong S(A_2) \otimes_{A/\mathfrak{q}} (X/\mathfrak{q}X)$, which is a projective left $S(A_2)$-module (possibly zero) by Lemma 1.6(b). Consequently, we see that the short exact sequences (E1.1) all split. These sequences remain split exact on tensoring with N, and hence we find that the induced sequences

$$0 \to N' \otimes_A K_m \to N' \otimes_A F_m \to N' \otimes_A K_{m-1} \to 0$$

are split exact. It follows that

$$\mathrm{Tor}^A_{m+1}(N', X) \cong \mathrm{Tor}^A_1(N', K_{m-1}) = 0$$

for all $m \geq 0$.

From the long exact sequence for Tor, we now have an exact sequence

$$0 = \mathrm{Tor}^A_{j+1}(N'/N, X) \to \mathrm{Tor}^A_j(N, X) \to \mathrm{Tor}^A_j(N', X) = 0.$$

However, this forces $\mathrm{Tor}^A_j(N, X) = 0$, contradicting our assumptions. Therefore X is indeed projective as a left A-module.

(b) As above, we may assume that X is a noetherian bimodule. Let I be the trace ideal of X in A, that is,

$$I = \sum_{f \in \mathrm{Hom}_A(X,A)} f(X).$$

For any maximal ideal $\mathfrak{p} \subset A$, we have $X/\mathfrak{p}X \neq 0$ by hypothesis. Since X is noetherian, $X/\mathfrak{p}X$ has a simple factor module, and so there exists a maximal left ideal $\mathfrak{m}$ of A, containing $\mathfrak{p}$, together with an epimorphism $f : X \to X/\mathfrak{p}X \to A/\mathfrak{m}$ of left A-modules. Since ${}_AX$ is projective, f lifts to an A-module homomorphism $g : X \to A$ such that $g(X) \nsubseteq \mathfrak{m}$. Consequently, $I \nsubseteq \mathfrak{p}$.

Since I is not contained in any maximal ideal of A, we conclude that $I = A$. Therefore X is a progenerator in A-Mod. □

We record the following analog of Proposition 1.1(a), in which the hypothesis $\mathrm{Tor}^A_i(A/\mathfrak{p}, X) = 0$ is only assumed for positive indices i. Since we will not use the result in this paper, we omit the proof, which is similar to that of Proposition 1.1(a). By definition, the *flat dimension* of a complex X is

$$\mathrm{flatdim}\, X := \sup\{i \mid \mathrm{Tor}^A_i(M, X) \neq 0 \text{ for all right } A\text{-modules } M\}.$$

Proposition 1.8. *Suppose A and B are rings satisfying* (H1). *Let X be a bounded complex of (A, B)-bimodules such that the bimodule $\mathrm{H}^i(X)$ is noetherian for all i. If $\mathrm{Tor}^A_i(A/\mathfrak{p}, X) = 0$ for all $i > 0$ and all maximal ideals $\mathfrak{p} \subset A$, then the flat dimension of X is at most* 0.

Finally, we note that $\mathcal{O}_q(G)$ satisfies (H1).

Theorem 1.9. *Let G be any connected semisimple Lie group over $\mathbb{C}$. Then the standard generic quantized coordinate ring $\mathcal{O}_q(G)$ is a noetherian domain of finite global dimension, and it satisfies* (H1).

Proof. See [6, Proposition 2.7] and [7, Theorems I.8.9, I.8.18, and II.9.19]. □

2. Consequences of (H2)

The main use of (H2) is to obtain the existence of an Auslander, Cohen-Macaulay, rigid dualizing complex. Since we will only use dualizing complexes as a tool in a few places, we intend to omit most of the definitions related to dualizing complexes and derived categories. For those readers who would like to see more details about this topic, we refer to the papers [27, 32, 33, 34]. However, we will review below the definitions of the Auslander and Cohen-Macaulay properties, since these two properties are the main point of Theorems 0.1 and 0.2.

Recall that a *connected graded* algebra is an $\mathbb{N}$-graded k-algebra $A = \bigoplus_{i\geq 0} A_i$ such that $A_0 = k$ and $\dim_k A_i < \infty$ for all i. A noetherian connected graded algebra A is said to *have enough normal elements* if every non-simple graded prime factor $A/\mathfrak{p}$ has a homogeneous normal element of positive degree. (Such a normal element is necessarily regular.) Recall that an algebra A satisfies (H2) if there is an ascending $\mathbb{N}$-filtration $F = \{F_iA\}_{i\geq 0}$ by k-subspaces such that

(i) $1 \in F_0A$,
(ii) $A = \bigcup_{i\geq 0} F_iA$,
(iii) $(F_iA)(F_jA) \subset F_{i+j}A$ for all i, j, and
(iv) the associated graded ring $\operatorname{gr} A := \bigoplus_{i=0}^{\infty} F_iA/F_{i-1}A$ is connected graded noetherian and has enough normal elements.

Condition (iv) implies that $\operatorname{gr} A$ is a finitely generated algebra with finite GK-dimension [33, Proposition 0.9], and consequently A satisfies those same properties.

In this section, we will recall several concepts which are essential to our work. First, let A be a noetherian ring. We say that A is *Gorenstein* if it has finite injective dimension on both sides. The standard Auslander-Gorenstein, Auslander-regular, and Cohen-Macaulay conditions are usually defined in terms of grades of modules, as follows. For any finitely generated left or right A-module, the *grade* or the *j-number* of M *with respect to* A is defined to be

$$j(M) := \inf\{n \mid \operatorname{Ext}^n_A(M, A) \neq 0\}.$$

The ring A is called *Auslander-Gorenstein* if it is Gorenstein and it satisfies the *Auslander condition*:

> For every finitely generated left (respectively, right) A-module M and every positive integer q, one has $j(N) \geq q$ for every finitely generated right (respectively, left) A-submodule $N \subseteq \operatorname{Ext}^q_A(M, A)$.

An *Auslander-regular* ring is a noetherian, Auslander-Gorenstein ring which has finite global dimension. The final condition requires A to be an algebra with finite GK-dimension. We say that A is *Cohen-Macaulay* (*with respect to* GKdim) if

$$j(M) + \operatorname{GKdim} M = \operatorname{GKdim} A$$

for every nonzero finitely generated left or right A-module M.

These conditions have analogs for rigid dualizing complexes. Thus, let R be a rigid dualizing complex over A, and let M be a finitely generated left or right A-module. The *grade* of M *with respect to* R is defined to be

$$j_R(M) := \inf\{n \mid \operatorname{Ext}^n_A(M, R) \neq 0\}.$$

We say that R is *Auslander* if

> For every finitely generated left (respectively, right) A-module M, every integer q, and every finitely generated right (respectively, left) A-submodule $N \subseteq \operatorname{Ext}_A^q(M,R)$, one has $j_R(N) \geq q$.

Now suppose that A is an algebra with finite GK-dimension. We say that R is *Cohen-Macaulay* (*with respect to* GKdim) if

$$j_R(M) + \operatorname{GKdim} M = 0$$

for every nonzero finitely generated left or right A-module M.

If A is Gorenstein, then any shift $A[n]$ is a dualizing complex over A. This leads to the following connection between the two versions of the above concepts, which is easily checked once one notes that $j_R(M) = j(M) - n$ for all finitely generated left or right A-modules M.

A noetherian algebra A with $\operatorname{GKdim} A = n < \infty$ *is Auslander-Gorenstein and Cohen-Macaulay if and only if*

(i) *A is Gorenstein, and*
(ii) *the dualizing complex $A[n]$ is Auslander and Cohen-Macaulay.*

Proposition 2.1. *Suppose A is an algebra satisfying* (H2)*.*

(a) *A has an Auslander, Cohen-Macaulay, rigid dualizing complex R. As a consequence, the Gelfand-Kirillov dimension of any finitely generated A-module is an integer.*
(b) *For any ideal $I \subset A$, the complex* $\operatorname{RHom}_A(A/I,R)$ *is an Auslander, Cohen-Macaulay, rigid dualizing complex over A/I.*

Proof. Part (a) is [33, Corollary 6.9(iii)], while part (b) follows from [33, Theorem 6.17, Proposition 3.9]. □

Given an algebra A with a dualizing complex R, define functors D and D° so that

$$D(M) = \operatorname{RHom}_A(M,R) \qquad \text{and} \qquad D^\circ(N) = \operatorname{RHom}_{A^\circ}(N,R)$$

for $M \in \mathsf{D}(A\text{-}\mathsf{Mod})$ and $N \in \mathsf{D}(A^\circ\text{-}\mathsf{Mod})$, where $\mathsf{D}(\mathsf{C})$ denotes the derived category of complexes over an abelian category C. These functors provide dualities between the subcategories $\mathsf{D}_f(A\text{-}\mathsf{Mod})$ and $\mathsf{D}_f(A^\circ\text{-}\mathsf{Mod})$ whose objects are the complexes with finitely generated cohomology modules [33, Proposition 1.3]. Let $(-)' = \operatorname{Hom}_k(-,k)$ denote the k-linear vector space dual.

We shall need the following relatively standard fact: *Suppose that A and B are noetherian rings, X a bounded complex of left* (*respectively, right*) *A-modules, and Y a bounded complex of* (A,B)*-bimodules* (*respectively,* (B,A)*-bimodules*)*. If the cohomology modules* $\mathrm{H}^i(X)$ *are finitely generated and the bimodules* $\mathrm{H}^i(Y)$ *are noetherian for all i, then the right* (*respectively, left*) *B-modules* $\operatorname{Ext}_A^i(X,Y)$ (*respectively,* $\operatorname{Ext}_{A^\circ}^i(X,Y)$) *are finitely generated.* By induction on the lengths of the complexes $\mathrm{H}^*(X)$ and $\mathrm{H}^*(Y)$, this can be reduced to the case where X and Y are (bi)modules. That case is well known.

Lemma 2.2. *Suppose A is an algebra satisfying* (H2)*, and let R be the dualizing complex given in* Proposition 2.1(a)*. Set* $D(-) := \operatorname{RHom}_A(-,R)$ *as above.*

(a) *$D(M) \cong M'$ for all finite dimensional A-modules M.*
(b) $\operatorname{Ext}_A^i(A/I,A)$ *is a noetherian* $(A/I,A)$*-bimodule for all ideals $I \subset A$ and all $i \geq 0$.*

Proof. (a) Let M be a finite dimensional A-module and $\mathfrak{p} := \operatorname{ann}_A(M)$. Then $A/\mathfrak{p}$ is finite dimensional over k. One has

$$D(M) \cong \operatorname{RHom}_A((A/\mathfrak{p}) \otimes^L_{A/\mathfrak{p}} M, R) \cong \operatorname{RHom}_{A/\mathfrak{p}}(M, D(A/\mathfrak{p})) =: (*)$$

[28, Theorem 10.8.7]. By Proposition 2.1(b) (or [33, Theorem 3.2] in this special case), $R_{A/\mathfrak{p}} := D(A/\mathfrak{p})$ is a rigid dualizing complex over $A/\mathfrak{p}$. Since $A/\mathfrak{p}$ is finite dimensional over k, we have $R_{A/\mathfrak{p}} \cong (A/\mathfrak{p})'$. Hence

$$(*) \cong \operatorname{RHom}_{A/\mathfrak{p}}(M, (A/\mathfrak{p})') \cong M'.$$

(b) Let I be an ideal of A and i a nonnegative integer. By the comments above, $\operatorname{Ext}^i_A(A/I, A)$ is finitely generated as a right A-module. By Proposition 2.1(b), $D(A/I)$ is a rigid dualizing complex over A/I. By duality,

$$\operatorname{Ext}^i_A(A/I, A) \cong \operatorname{Ext}^i_{A^\circ}(D(A), D(A/I)) \cong \operatorname{Ext}^i_{A^\circ}(R, D(A/I)).$$

The latter bimodule is finitely generated as a left (A/I)-module by the comments above. □

Part (a) of the lemma says that if M is an A-module finite dimensional over k, then

$$\operatorname{Ext}^i_A(M, R) = \begin{cases} M' & \text{if} \quad i = 0, \\ 0 & \text{if} \quad i \neq 0. \end{cases}$$

Let $\mathsf{D}^b_f(A\text{-}\mathsf{Mod})$ denote the derived category of bounded complexes Y of A-modules such that $\mathrm{H}^i(Y)$ is finitely generated over A for all i.

Lemma 2.3. *Suppose A is an algebra satisfying* (H2). *Let X and Y be objects in $\mathsf{D}^b_f(A\text{-}\mathsf{Mod})$ such that $\mathrm{H}^i(X)$ is finite dimensional over k for all i. Then $\operatorname{Ext}^i(X, Y)$ is finite dimensional over k for all i.*

Proof. By induction on the length of X, we may assume that X is a finite dimensional A-module M. By duality, $\operatorname{Ext}^i_A(M, Y) \cong \operatorname{Ext}^i_{A^\circ}(D(Y), D(M))$. By Lemma 2.2(a), $D(M) \cong M'$, which is finite dimensional; by the properties of the duality D, we have $D(Y) \in \mathsf{D}^b_f(A^\circ\text{-}\mathsf{Mod})$. So, it remains to show that $\operatorname{Ext}^i_{A^\circ}(Z, N)$ is finite dimensional when $Z \in \mathsf{D}^b_f(A^\circ\text{-}\mathsf{Mod})$ and N is a finite dimensional right A-module. But this follows by taking a bounded above projective resolution P of Z such that each term of P is finitely generated. □

A ring A, or its prime spectrum $\operatorname{Spec} A$, is called *catenary* if, for every pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$, all saturated chains of primes between $\mathfrak{p}$ and $\mathfrak{q}$ have same length. We say that *Tauvel's height formula* holds in an algebra A provided

$$\operatorname{height} \mathfrak{p} + \operatorname{GKdim} A/\mathfrak{p} = \operatorname{GKdim} A$$

for all prime ideals $\mathfrak{p} \subset A$. These properties are consequences of (H1) and (H2), as follows.

Theorem 2.4. [33, Theorems 0.1 and 2.23] *Suppose A is an algebra satisfying* (H1,H2).

(a) $\operatorname{Spec} A$ *is catenary.*
(b) *If, in addition, A is prime, then Tauvel's height formula holds.*

Even if A is not prime, but rather equidimensional in a suitable sense, Tauvel's height formula still holds.

Corollary 2.5. *Suppose A is an algebra satisfying* (H1,H2). *If* $\operatorname{GKdim} A/\mathfrak{p} = \operatorname{GKdim} A$ *for all minimal primes $\mathfrak{p}$, then Tauvel's height formula holds.*

Proof. Let $\mathfrak{q}$ be any prime, and let $\mathfrak{p}$ be a minimal prime such that $\mathfrak{p} \subset \mathfrak{q}$ and the length of a saturated chain between $\mathfrak{p}$ and $\mathfrak{q}$ is equal to the height of $\mathfrak{q}$. Since $A/\mathfrak{p}$ satisfies (H1) and (H2) also, Theorem 2.4(b) implies that

$$\operatorname{height} \mathfrak{q}/\mathfrak{p} + \operatorname{GKdim} A/\mathfrak{q} = \operatorname{GKdim} A/\mathfrak{p}.$$

By the choice of $\mathfrak{p}$, one has $\operatorname{height} \mathfrak{q}/\mathfrak{p} = \operatorname{height} \mathfrak{q}$. By hypothesis, $\operatorname{GKdim} A/\mathfrak{p} = \operatorname{GKdim} A$. Therefore,

$$\operatorname{height} \mathfrak{q} + \operatorname{GKdim} A/\mathfrak{q} = \operatorname{GKdim} A.$$

□

Next, we consider the quantized coordinate rings $\mathcal{O}_q(G)$.

Theorem 2.6. *If G is any connected semisimple Lie group over $\mathbb{C}$, then the standard generic quantized coordinate ring $\mathcal{O}_q(G)$ satisfies* (H2). *As a consequence,* $\operatorname{Spec} \mathcal{O}_q(G)$ *is catenary and Tauvel's height formula holds.*

Proof. By the proof of [7, Theorem I.8.18], $A := \mathcal{O}_q(G)$ has a set of generators satisfying the hypotheses of [7, Proposition I.8.17]. By the proof of [7, Proposition I.8.17], A has an $\mathbb{N}$-filtration such that $\operatorname{gr} A$ is a graded factor ring of a skew polynomial ring

$$\mathcal{O}_{\mathbf{q}}(\mathbb{C}^m) := \mathbb{C}\langle x_1, \dots, x_m \mid x_i x_j = q_{ij} x_j x_i \text{ for all } i, j\rangle,$$

for some matrix $\mathbf{q} = (q_{ij})$ of nonzero scalars, with $\deg x_i > 0$ for each i. It is clear that $\mathcal{O}_{\mathbf{q}}(\mathbb{C}^m)$ is connected graded noetherian, and that it has enough normal elements (namely, the images of $x_1, \dots, x_m$ in graded factor rings). Any graded factor ring of $\mathcal{O}_{\mathbf{q}}(\mathbb{C}^m)$ inherits these properties. Hence, A satisfies (H2).

The second assertion follows from Theorems 1.9 and 2.4. □

Here is one final consequence of (H2). Recall that an algebra A is called *universally noetherian* if $A \otimes B$ is noetherian for every noetherian algebra B [2, p.596].

Proposition 2.7. *If A is an algebra satisfying* (H2), *then it is universally noetherian.*

Proof. This follows from [2, Propositions 4.3, 4.9 and 4.10]. □

3. RELATIVELY PROJECTIVE MODULES

The hypothesis (H3) turns out to be very useful in dealing with modules satisfying the properties described in the following lemma. We leave the proof to the reader.

Lemma 3.1. *Let A be an algebra and M a finitely generated A-module. Then the following are equivalent:*

(a) *$M/\mathfrak{p}M$ is projective over $A/\mathfrak{p}$ for all co-finite dimensional ideals $\mathfrak{p} \subset A$.*
(b) $\operatorname{Hom}_A(M, -)$ *is an exact functor on finite dimensional A-modules.*
(c) $- \otimes_A M$ *is an exact functor on finite dimensional right A-modules.*

We shall say that a finitely generated module M over an algebra A is *relatively projective* (with respect to the category of finite dimensional A-modules) provided M satisfies the equivalent conditions of Lemma 3.1.

The hypothesis (H3), which says that every maximal ideal is co-finite dimensional, seems quite restrictive. Nonetheless, it holds in a number of interesting non-PI situations. Here are some examples.

Examples 3.2. **(a)** Let A be a finitely generated prime algebra. If A is not simple and $\operatorname{GKdim} A \leq 2$, then A satisfies (H3). To see this, we let $\mathfrak{p}$ be a maximal ideal of A. Since A is not simple, $\mathfrak{p} \neq 0$, and hence $\operatorname{GKdim} A/\mathfrak{p} \leq 1$. By [26], $A/\mathfrak{p}$ is PI. Now simple affine PI rings are finite dimensional, so $\mathfrak{p}$ is co-finite dimensional.

(b) If A is a finitely generated prime Hopf algebra with $\operatorname{GKdim} A \leq 2$, then A satisfies (H3). This follows from (a), since A cannot be simple unless the kernel of its counit is zero, in which case $\dim_k A = 1$. Of course, not all Hopf algebras satisfy (H3), as witnessed by many group algebras and enveloping algebras.

(c) If k is algebraically closed and $q \in k^\times$ is not a root of unity, then the quantized coordinate rings $\mathcal{O}_q(k^2)$ and $\mathcal{O}_q(SL_2(k))$ satisfy (H3) – just refer to the pictures of their prime spectra in [7, Diagrams II.1.2, II.1.3].

(d) Assume again that k is algebraically closed and $q \in k^\times$ is not a root of unity. We generalize the first example of (c) to the algebra

$$A = \mathcal{O}_q(k^n) := k\langle x_1, \dots, x_n \mid x_i x_j = q x_j x_i \text{ for all } i < j\rangle.$$

Given a maximal ideal $\mathfrak{p}$ in A, let $I = \langle x_i \mid x_i \in \mathfrak{p}\rangle$. It suffices to show that the maximal ideal $\mathfrak{p}/I$ in A/I is co-finite dimensional, and $A/I \cong \mathcal{O}_q(k^m)$ where $m = n - \operatorname{card}(I)$. Hence, there is no loss of generality in assuming that $x_i \notin \mathfrak{p}$ for all i.

Now the images of the x_i in $A/\mathfrak{p}$ are nonzero normal elements, hence invertible by simplicity. Thus, $A/\mathfrak{p} \cong B/\mathfrak{p}B$ where

$$B = \mathcal{O}_q((k^\times)^n) := A[x_1^{-1}, \dots, x_n^{-1}].$$

It is known that the maximal ideals of B are induced from the maximal ideals of its center [13, Corollary 1.5]. This center can be calculated using [13, Lemma 1.2] – if n is even, then $Z(B) = k$, while if n is odd, then $Z(B) = k[z^{\pm 1}]$ where $z := x_1 x_2^{-1} x_3 x_4^{-1} \cdots x_{n-1}^{-1} x_n$.

In case n is even, we obtain $\mathfrak{p}B = 0$ and $\mathfrak{p} = 0$, meaning that A itself is simple. This is only possible when $n = 0$, that is, $A = k$, and thus $\dim_k A/\mathfrak{p} = 1$ in this case.

In case n is odd, $\mathfrak{p}B = (z - \alpha)B$ for some $\alpha \in k^\times$, and thus the element

$$w := x_1 x_3 \cdots x_n - \alpha x_2 x_4 \cdots x_{n-1}$$

lies in $\mathfrak{p}B \cap A = \mathfrak{p}$. Since w generates $\mathfrak{p}B$, we have

$$\mathfrak{p} = \{a \in A \mid (x_1 x_2 \cdots x_n)^t a \in wA \text{ for some } t \in \mathbb{N}\},$$

from which it is easy to check that, in fact, $\mathfrak{p} = \langle w\rangle$. Now if $n \geq 3$, it would follow that $\mathfrak{p} \subseteq \langle x_1, x_2\rangle$, contradicting the maximality of $\mathfrak{p}$. Thus, $n = 1$, whence $\mathfrak{p} = \langle x_1 - \alpha\rangle \subset A = k[x_1]$, and again $\dim_k A/\mathfrak{p} = 1$.

Therefore all maximal ideals of A have codimension 1.

(e) In contrast to (d), multiparameter quantum affine spaces need not satisfy (H3). For instance, let $q \in k^\times$ be a non-root of unity, and let

$$A := k\langle x, y, z \mid xy = qyx,\ xz = q^{-1}zx,\ yz = qzy\rangle.$$

Then $xyz \in Z(A)$, and $A/\langle xyz - 1\rangle \cong \mathcal{O}_q((k^\times)^2)$, which is a simple algebra [22, Example 1.8.7(ii)]. Thus, $\langle xyz-1\rangle$ is a maximal ideal of A with infinite codimension. $\square$

Question 3.3. Let G be any connected semisimple Lie group over $\mathbb{C}$. Do all maximal ideals of the standard generic quantized coordinate ring $\mathcal{O}_q(G)$ have finite codimension?

The main result in this section is the following proposition.

Proposition 3.4. *Suppose that A is an algebra satisfying* (H1,H3)*, B a ring satisfying* (H1)*, and M a noetherian (A, B)-bimodule. If ${}_AM$ is relatively projective, then ${}_AM$ is projective. If, in addition, $M/\mathfrak{p}M \neq 0$* (*or, equivalently,* $\mathrm{Hom}_A(M, A/\mathfrak{p}) \neq 0$) *for all maximal ideals $\mathfrak{p}$ of A, then ${}_AM$ is a progenerator for A-*Mod.

We need some lemmas. Recall that a module M over an algebra A is said to be *locally finite dimensional* if every finitely generated submodule of M is finite dimensional.

Lemma 3.5. *Suppose A is an algebra satisfying* (H1)*. Then the injective hull of any finite dimensional left or right A-module is locally finite dimensional.*

Proof. Since the lemma is left-right symmetric, it suffices to prove it for right modules.

Let E be the injective hull of a finite dimensional right A-module M. Then $E = E_1 \oplus \cdots \oplus E_m$ for some uniform injective modules E_j, which are injective hulls of the finite dimensional modules $E_j \cap M$. Since it suffices to show that each E_j is locally finite dimensional, there is no loss of generality in assuming that E is uniform. Now M has an essential simple submodule M', and we may replace M by M', so we may assume that M is simple. As a consequence, $\mathfrak{p} := \mathrm{r.ann}_A(M)$ is a co-finite dimensional maximal ideal of A. Moreover, $\mathfrak{p}$ is the *assassinator* of E, that is, the unique associated prime.

Recall from Section 1 that (H1) implies the strong second layer condition. By [14, Corollary 12.8], each finitely generated submodule of E is annihilated by a product of primes from the right link closure of $\mathfrak{p}$. All such primes must have finite codimension, because finite codimension carries across links: if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals of A such that $\mathfrak{p}_1 \rightsquigarrow \mathfrak{p}_2$ via $(\mathfrak{p}_1 \cap \mathfrak{p}_2)/I$, then $A/\mathfrak{p}_1$ is finite dimensional if and only if $(\mathfrak{p}_1 \cap \mathfrak{p}_2)/I$ is finite dimensional, if and only if $A/\mathfrak{p}_2$ is finite dimensional. Thus, the annihilator of each finitely generated submodule of E has finite codimension. Therefore E is locally finite dimensional. $\square$

Lemma 3.6. *Suppose A is an algebra satisfying* (H1) *and M a finitely generated relatively projective A-module.*

(a) $\mathrm{Ext}^i_A(M, N) = 0$ *for all $i > 0$ and all finite dimensional A-modules N.*
(b) $\mathrm{Tor}^A_i(S, M) = 0$ *for all $i > 0$ and all finite dimensional right A-modules S.*

Proof. (a) Suppose
$$0 \to N \to I^0 \to I^1 \to I^2 \to \cdots$$
is a minimal injective resolution of N. The assertion follows if the complex
$$\mathrm{Hom}_A(M, I^0) \to \mathrm{Hom}_A(M, I^1) \to \mathrm{Hom}_A(M, I^2) \to \cdots$$

is exact. Since each I^i is locally finite dimensional by Lemma 3.5, it suffices to show that $\operatorname{Hom}_A(M,-)$ is exact on locally finite dimensional modules. Suppose

$$0 \to P \xrightarrow{f} Q \xrightarrow{g} R \to 0$$

is a short exact sequence of locally finite dimensional A-modules, and let $h : M \to R$ be a homomorphism. Then $R' = h(M)$ is a finitely generated submodule of R, and we can choose a finitely generated submodule $Q' \subseteq Q$ such that $g(Q') = R'$. Since Q' is finite dimensional, we obtain a short exact sequence

$$0 \to P' \xrightarrow{f'} Q' \xrightarrow{g'} R' \to 0$$

of finite dimensional A-modules, where $P' := f^{-1}(Q')$ and f', g' are the appropriate restrictions of f, g. When h is viewed as a homomorphism $M \to R'$, it lifts to a homomorphism $h' : M \to Q'$ because M is relatively projective. This provides the required lifting of h to $\operatorname{Hom}_A(M,Q)$, and proves that the sequence

$$0 \to \operatorname{Hom}_A(M,P) \xrightarrow{f^*} \operatorname{Hom}_A(M,Q) \xrightarrow{g^*} \operatorname{Hom}_A(M,R) \to 0$$

is exact, as desired.

(b) Let S be an arbitrary finite dimensional right A-module. Since k is injective as a module over itself, [10, Proposition VI.5.1] implies that

$$\operatorname{Hom}_k\big(\operatorname{Tor}_i^A(S,M),k\big) \cong \operatorname{Ext}_A^i\big(M,\operatorname{Hom}_k(S,k)\big)$$

for all i. By part (a), $\operatorname{Ext}_A^i\big(M,\operatorname{Hom}_k(S,k)\big) = 0$ for $i > 0$, and therefore we obtain $\operatorname{Tor}_i^A(S,M) = 0$ for $i > 0$. □

Now we are ready to prove Proposition 3.4.

Proof of Proposition 3.4. Let M be a noetherian (A,B)-bimodule which is relatively projective as an A-module. Since every maximal ideal of A is co-finite dimensional by (H3), Lemma 3.6(b) implies that $\operatorname{Tor}_i^A(A/\mathfrak{p},M) = 0$ for all $i \neq 0$ and all maximal ideals $\mathfrak{p}$ of A. By Proposition 1.1(a), ${}_AM$ is projective. If, in addition, $M/\mathfrak{p}M \neq 0$ for all maximal ideals $\mathfrak{p}$ of A, then ${}_AM$ is a progenerator by Proposition 1.1(b). □

4. Proof of Theorem 0.2

An algebra A is said to satisfy

(H4) if for all nonnegative integers n,
- (a) the functor $\operatorname{Ext}_A^n(-,A)$ is exact on finite dimensional A-modules;
- (b) $\operatorname{Ext}_A^n(S,A) \neq 0$ for some finite dimensional A-module S if and only if $\operatorname{Ext}_A^n(S,A) \neq 0$ for all finite dimensional A-modules S.

Note that (H4) is not left-right symmetric. Hence, when we come to combine (H4) with duality arguments, we shall need to apply right module versions of results from Sections 1–3.

We next record that Hopf algebras satisfy (H4), along with some other information. Note that if A is a Hopf algebra, the base field k can be identified with the trivial module $A/(\ker\epsilon)$, where $\epsilon : A \to k$ is the counit of A.

Lemma 4.1. *Let A be a Hopf algebra.*

(a) [6, Lemma 1.11] *For each nonnegative integer n, the following are equivalent:*
   (1) $\operatorname{Ext}_A^n(k,A) \neq 0$.

(2) $\operatorname{Ext}^n_A(S, A) \neq 0$ *for some finite dimensional A-module S.*
(3) $\operatorname{Ext}^n_A(S, A) \neq 0$ *for all nonzero finite dimensional A-modules S.*

(b) [29, Lemma 4.8] *The functors $\operatorname{Ext}^n_A(-, A)$, for $n \geq 0$, are exact on finite dimensional A-modules.*

(c) *As an algebra, A satisfies* (H4).

(d) *If M is a finite dimensional A-module, then*
$$\dim_k \operatorname{Ext}^n_A(M, A) = (\dim_k M)\big(\dim_k \operatorname{Ext}^n_A(k, A)\big)$$
for all $n \geq 0$.

Proof. Part (c) follows from (a),(b), and (d) follows from the proof of [29, Lemma 4.8]. □

Recall that an algebra A (respectively, a bimodule M) is called *residually finite dimensional* if the intersection of all co-finite dimensional ideals of A (respectively, sub-bimodules of M) is zero. Such algebras and bimodules appear in our context as follows.

Lemma 4.2. *If A is an algebra satisfying* (H1,H3)*, then all noetherian (A, A)-bimodules are residually finite dimensional. In particular, A is residually finite dimensional.*

Proof. Let M be a noetherian (A, A)-bimodule and $N \subseteq M$ a nonzero sub-bimodule. We show that M has a co-finite dimensional sub-bimodule not containing N.

Set $J := \operatorname{l.ann}_A(N)$. By [14, Lemma 8.1], A/J embeds in $N^{\oplus t}$ as left A-modules, for some $t \in \mathbb{N}$. Choose a maximal ideal $\mathfrak{m} \supseteq J$, and note that the quotient map $A/J \to A/\mathfrak{m}$ extends to a nonzero homomorphism $N^{\oplus t} \to E$, where E is the injective hull of the left A-module $A/\mathfrak{m}$. Hence, there exists a nonzero homomorphism $f : {}_AN \to E$, which extends to a homomorphism $g : {}_AM \to E$.

Now $A/\mathfrak{m}$ is finite dimensional by (H3), and so E is locally finite dimensional by Lemma 3.5. Hence, $g(M)$ is finite dimensional, and so the ideal $\mathfrak{q} := \operatorname{l.ann}_A g(M)$ has finite codimension in A. Since $g(\mathfrak{q}M) = 0$, we conclude that $\mathfrak{q}M$ is a co-finite dimensional sub-bimodule of M that does not contain N. □

Lemma 4.3. *Suppose A is a residually finite dimensional algebra, and M a finitely generated, relatively projective A-module. Assume that $M/\mathfrak{m}M \neq 0$, or equivalently,* $\operatorname{Hom}_A(M, A/\mathfrak{m}) \neq 0$*, for all co-finite dimensional maximal ideals $\mathfrak{m}$ of A. Then M is faithful over A. If, in addition, M is a noetherian (A, B)-bimodule for some noetherian ring B, then* $\operatorname{GKdim} M = \operatorname{GKdim} A$.

Proof. If, on the contrary, M is not faithful, then $\mathfrak{p} := \operatorname{l.ann}_A(M) \neq 0$. Since A is residually finite dimensional, it has a co-finite dimensional ideal $\mathfrak{q}$ such that $\mathfrak{p} \not\subseteq \mathfrak{q}$. Consider the short exact sequence
$$0 \to (\mathfrak{p} + \mathfrak{q})/\mathfrak{q} \to A/\mathfrak{q} \to A/(\mathfrak{p} + \mathfrak{q}) \to 0$$
and the associated sequence
$$0 \to \big((\mathfrak{p} + \mathfrak{q})/\mathfrak{q}\big) \otimes_A M \to (A/\mathfrak{q}) \otimes_A M \xrightarrow{f} \big(A/(\mathfrak{p} + \mathfrak{q})\big) \otimes_A M \to 0,$$
which is exact because M is relatively projective (Lemma 3.1). The map f is an isomorphism since $\mathfrak{p}M = 0$, and so $\big((\mathfrak{p}+\mathfrak{q})/\mathfrak{q}\big) \otimes_A M = 0$. Because $\mathfrak{p} \not\subseteq \mathfrak{q}$, there is an epimorphism $\big((\mathfrak{p}+\mathfrak{q})/\mathfrak{q}\big) \to S$ for some simple finite dimensional right A-module S,

whence $S \otimes_A M = 0$. But then $\mathfrak{m} = \mathrm{r.ann}_A(S)$ is a co-finite dimensional maximal ideal of A with $(A/\mathfrak{m}) \otimes_A M = 0$, which yields a contradiction to our hypotheses. Therefore M is faithful over A.

If M is a noetherian bimodule, then A can be embedded in $M^{\oplus t}$ for some t. Hence $\mathrm{GKdim}\, M = \mathrm{GKdim}\, A$. □

Lemma 4.4. *Let A be a noetherian algebra with $n := \mathrm{GKdim}\, A < \infty$, and assume that A has an Auslander, Cohen-Macaulay dualizing complex R.*

(a) $\mathrm{H}^i(R) = 0$ *for all $i > 0$ and all $i < -n$.*
(b) $\mathrm{H}^{-n}(R) \neq 0$, *and all nonzero left or right A-submodules of* $\mathrm{H}^{-n}(R)$ *have GK-dimension n.*
(c) $\mathrm{GKdim}\, \mathrm{H}^{-i}(R) \leq i$ *(on each side) for $0 \leq i \leq n$.*

Proof. Note that the Cohen-Macaulay hypothesis implies that $n = -j_R(A)$, and so n is an integer. Since A is a projective module, $\mathrm{RHom}_A(A, R) \cong \mathrm{Hom}_A(A, R) \cong R$. Hence,

$$\mathrm{Ext}^q_A(A, R) \cong \mathrm{H}^q(R)$$

for all $q \in \mathbb{Z}$. Since $j_R(A) = -n$, it follows that $\mathrm{H}^{-n}(R) \neq 0$ and $\mathrm{H}^i(R) = 0$ for all $i < -n$.

If $\mathrm{H}^i(R) \neq 0$ for some i, then because $\mathrm{H}^i(R)$ is a noetherian bimodule, the Auslander hypothesis implies that $j_R\big(\mathrm{H}^i(R)\big) \geq i$. Combined with the Cohen-Macaulay condition, this yields

$$\mathrm{GKdim}\, \mathrm{H}^i(R) = -j_R\big(\mathrm{H}^i(R)\big) \leq -i,$$

and similarly on the right. Part (c) follows, as well as the inequality $0 \leq -i$. Hence, $\mathrm{H}^i(R) = 0$ for all $i > 0$, and part (a) is proved.

By [33, Theorem 2.10], the canonical dimension Cdim_R is an exact dimension function. Since

$$\mathrm{Cdim}_R(M) = -j_R(M) = \mathrm{GKdim}\, M$$

for all finitely generated A-modules M, it follows that GKdim is an exact dimension function. Thus, [33, Theorem 2.14(1)] implies that $\mathrm{H}^{-n}(R)_A$ is GKdim-pure of GK-dimension n. By symmetry, ${}_A\,\mathrm{H}^{-n}(R)$ is GKdim-pure, completing the proof of (b). □

Recall from Morita theory that if A is a ring, Ω a progenerator in A°-Mod, and $B := \mathrm{End}_{A^\circ}(\Omega)$, then Ω is an invertible (B, A)-bimodule (with inverse $\mathrm{Hom}_A(\Omega, A)$).

Proposition 4.5. *Suppose A is an algebra satisfying* (H1,H2,H3,H4)*, with $n := \mathrm{GKdim}\, A < \infty$.*

(a) *A has a rigid dualizing complex R, and* $\mathrm{H}^i(R) = 0$ *for all $i \neq -n$.*
(b) $\mathrm{H}^{-n}(R)$ *is an invertible (A, A)-bimodule, and* $R \cong \mathrm{H}^{-n}(R)[n]$.
(c) $\mathrm{Ext}^i_A(S, A) = 0$ *for all $i \neq n$ and all finite dimensional A-modules S.*
(d) *If A is a Hopf algebra of finite global dimension, then* $\mathrm{gldim}\, A = n$.

Proof. (a) By Proposition 2.1(a), A has an Auslander, Cohen-Macaulay, rigid dualizing complex R. By [27, Proposition 8.2], rigid dualizing complexes over A are unique up to isomorphism.

Set $n_0 = \max\{i \in \mathbb{Z} \mid \mathrm{H}^i(R) \neq 0\}$ and $\Omega = \mathrm{H}^{n_0}(R)$. By Lemma 4.4, $-n \leq n_0 \leq 0$. There is a truncated complex $\widetilde{R}$, quasi-isomorphic to R, such that $\widetilde{R}^i = 0$ for all $i > n_0$. For any right A-module M, we may take an injective resolution I of

M and compute $\operatorname{Ext}^i_{A^\circ}(R,M) \cong \operatorname{Ext}_{A^\circ}(\widetilde{R},M) \cong \mathrm{H}^i \operatorname{Hom}_{A^\circ}(\widetilde{R},I)$ for all i. Since I vanishes in negative degree, the complex $\operatorname{Hom}_{A^\circ}(\widetilde{R},I)$ vanishes in degrees less than $-n_0$, and hence $\operatorname{Ext}^i_{A^\circ}(R,M) = 0$ for all $i < -n_0$. In degrees $-n_0$ and $-n_0+1$, the complex $\operatorname{Hom}_{A^\circ}(\widetilde{R},I)$ has the form

$$\operatorname{Hom}_{A^\circ}(\widetilde{R}^{n_0}, I^0) \to \operatorname{Hom}_{A^\circ}(\widetilde{R}^{n_0}, I^1) \oplus \operatorname{Hom}_{A^\circ}(\widetilde{R}^{n_0-1}, I^0),$$

from which we see that $\operatorname{Ext}^{-n_0}_{A^\circ}(R,M) \cong \operatorname{Hom}_{A^\circ}(\Omega, M)$.

We now restrict attention to finite dimensional right A-modules, denoted by S. By duality and Lemma 2.2(a),

$$\operatorname{Ext}^i_{A^\circ}(R,S) \cong \operatorname{Ext}^i_A(S', D(R)) \cong \operatorname{Ext}^i_A(S', A) \tag{E4.1}$$

for all i, where $S' = \operatorname{Hom}_k(S,k)$. Since these isomorphisms are natural, and since $\operatorname{Ext}^{-n_0}_A(-,A)$ is exact on finite dimensional A-modules by (H4), we see that $\operatorname{Hom}_{A^\circ}(\Omega,-) \cong \operatorname{Ext}^{-n_0}_{A^\circ}(R,-)$ is exact on finite dimensional right A-modules. Hence, Ω is relatively projective as a right A-module, and thus projective, by Proposition 3.4.

Because of Lemma 4.2, Ω is residually finite dimensional, and so $\operatorname{Hom}_{A^\circ}(\Omega, S_0)$ is nonzero for some finite dimensional right A-module S_0. Since

$$\operatorname{Hom}_{A^\circ}(\Omega,S) \cong \operatorname{Ext}^{-n_0}_{A^\circ}(R,S) \cong \operatorname{Ext}^{-n_0}_A(S',A)$$

for all finite dimensional right A-modules S, it follows from (H4) that $\operatorname{Hom}_{A^\circ}(\Omega,S)$ is nonzero for all finite dimensional right A-modules S. Proposition 3.4 and Lemmas 4.2, 4.3 now imply that Ω_A is a progenerator and $\operatorname{GKdim}\Omega_A = n$. By Lemma 4.4(c), $n_0 = -n$, and thus $\mathrm{H}^i(R) = 0$ for all $i > -n$. Combined with Lemma 4.4(a), this establishes part (a).

(b) Since R has nonzero cohomology only in degree $-n$, it is quasi-isomorphic to $\mathrm{H}^{-n}(R)[n] = \Omega[n]$.

We have already proved that Ω is a progenerator for A°-Mod. By definition of a dualizing complex, the canonical algebra homomorphism

$$A \to \operatorname{RHom}_{A^\circ}(R,R) \cong \operatorname{RHom}_{A^\circ}(\Omega[n]_A, \Omega[n]_A) \cong \operatorname{End}_{A^\circ}(\Omega_A)$$

is an isomorphism. Therefore, it follows from Morita theory that Ω is invertible as an (A,A)-bimodule.

(c) If M is a finitely generated right A-module and I an injective resolution of M, then

$$\operatorname{RHom}_{A^\circ}(R,M) \cong \operatorname{RHom}_{A^\circ}(\Omega[n],M) \cong \operatorname{Hom}_{A^\circ}(\Omega[n],I).$$

The latter complex vanishes in degrees less than n, and is exact in degrees greater than n, because Ω_A is projective. Hence, $\operatorname{Ext}^i_{A^\circ}(R,M) = 0$ for all $i \neq n$. It now follows from (E4.1) that $\operatorname{Ext}^i_A(T,A) = 0$ for all $i \neq n$ and all finite dimensional A-modules T.

(d) If A is a Hopf algebra with finite global dimension d, then the trivial A-module k has projective dimension d [6, Corollary 1.4]. Since A is noetherian, $\operatorname{Ext}^d_A(k,A) \neq 0$, and therefore $d = n$ by part (c). □

Lemma 4.6. *Let A be an algebra satisfying* (H1,H2)*, and R a rigid dualizing complex over A. If $R \cong \Omega[n]$ where $n = \operatorname{GKdim} A$ and Ω is an invertible (A,A)-bimodule, then A is Auslander-Gorenstein and Cohen-Macaulay, and has a quasi-Frobenius classical quotient ring. Moreover,* $\operatorname{Spec} A$ *is catenary, and Tauvel's height formula holds.*

Proof. By Proposition 2.1(a), A has an Auslander, Cohen-Macaulay, rigid dualizing complex, and this complex must be quasi-isomorphic to R by uniqueness of rigid dualizing complexes [27, Proposition 8.2]. Since $R \cong \Omega \otimes_A A[n]$, the ring A is Auslander-Gorenstein and Cohen-Macaulay by [34, Proposition 4.3]. Catenarity of $\operatorname{Spec} A$ follows from Theorem 2.4. By [1, Theorem 6.1(2)(3)], A has a quasi-Frobenius classical quotient ring, and $\operatorname{GKdim} A/\mathfrak{p} = \operatorname{GKdim} A$ for all minimal primes $\mathfrak{p}$ of A. Thus, by Corollary 2.5, Tauvel's height formula holds. □

Theorem 4.7. *Let A be an algebra satisfying* (H1,H2,H3,H4)*.*

(a) *A is Auslander-Gorenstein and Cohen-Macaulay, and has a quasi-Frobenius classical quotient ring.*
(b) $\operatorname{Spec} A$ *is catenary, and Tauvel's height formula holds.*
(c) *There exists an invertible (A, A)-bimodule Ω such that $\Omega[n]$ is a rigid dualizing complex over A, where $n = \operatorname{GKdim} A$.*
(d) *If A is regular, then* $\operatorname{gldim} A = \operatorname{GKdim} A$*.*

Proof. Recall from the discussion at the beginning of Section 2 that (H2) implies $\operatorname{GKdim} A < \infty$. Parts (a), (b), (c) follow from Proposition 4.5 and Lemma 4.6.

(d) The Auslander-regular and Cohen-Macaulay conditions imply that $\operatorname{gldim} A \leq \operatorname{GKdim} A$. There exists a nonzero finite dimensional A-module S because of (H3), and if $d = \operatorname{projdim}_A S$, then $\operatorname{Ext}^d_A(S, A) \neq 0$. Proposition 4.5(c) implies that $d = \operatorname{GKdim} A$, and thus $\operatorname{GKdim} A \leq \operatorname{gldim} A$. □

Corollary 4.8. *If A is an affine PI Hopf algebra satisfying* (H2)*, then statements* (a)–(d) *of* Theorem 4.7 *hold.*

Proof. The affine PI hypothesis implies that A satisfies (H1,H3), and (H4) follows from Lemma 4.1(c). □

The corollary recovers [29, Theorem 0.3].

Theorem 0.2 is an immediate consequence of Theorem 4.7, as follows.

Proof of Theorem 0.2. Let A be a Hopf algebra satisfying (H1,H2,H3). By Lemma 4.1(c), A satisfies (H4). Thus, the desired conclusions follow from Theorem 4.7. □

5. Proof of Theorem 0.1

We say an algebra A of finite GK-dimension satisfies

(H5) provided that, for every maximal ideal $\mathfrak{p}$ of A and every nonnegative integer i, one has
$$\operatorname{Ext}^i_A(A/\mathfrak{p}, A) \neq 0$$
if and only if $i = \operatorname{GKdim} A - \operatorname{GKdim} A/\mathfrak{p}$.

Note that by our conventions, $\operatorname{Ext}^i_A(A/\mathfrak{p}, A)$ refers to an Ext-group of the *left* A-modules $A/\mathfrak{p}$ and A.

Condition (H5) is necessary for our main result, as the following lemma shows.

Lemma 5.1. *If A is an Auslander-Gorenstein, Cohen-Macaulay algebra with finite* GK*-dimension, then A satisfies* (H5)*.*

Proof. Let $n := \operatorname{GKdim} A$, let $\mathfrak{p}$ be a maximal ideal of A, and set $d := \operatorname{GKdim} A/\mathfrak{p}$. The Cohen-Macaulay condition implies that $j(A/\mathfrak{p}) = n - d$, and thus we have $\operatorname{Ext}^{n-d}_A(A/\mathfrak{p}, A) \neq 0$ and $\operatorname{Ext}^i_A(A/\mathfrak{p}, A) = 0$ for all $i < n - d$.

Suppose that, for some i, the $(A/\mathfrak{p}, A)$-bimodule $N := \mathrm{Ext}^i_A(A/\mathfrak{p}, A) \neq 0$. Since N is finitely generated as a right A-module, $j(N_A) \geq i$ by the Auslander condition. Combined with the Cohen-Macaulay condition, we thus find that $\mathrm{GKdim}(N_A) \leq n - i$. By [22, Proposition 8.3.14(ii)], $\mathrm{GKdim}({}_{A/\mathfrak{p}}N) \leq \mathrm{GKdim}(N_A)$ (this result does not require N to be finitely generated on the left). Since $A/\mathfrak{p}$ is a simple ring, N must be faithful as a left $(A/\mathfrak{p})$-module. The finite generation of N on the right then implies that $A/\mathfrak{p}$ embeds in $N^{\oplus t}$, as left $(A/\mathfrak{p})$-modules, for some $t \in \mathbb{N}$. Consequently, $\mathrm{GKdim}({}_{A/\mathfrak{p}}N) = d$, whence $d \leq n - i$. Thus $i \leq n - d$, and therefore $i = n - d$ by the previous paragraph. □

Theorem 5.2. *Let A be an algebra satisfying* (H1,H2,H5).

(a) *There exists an invertible (A, A)-bimodule Ω such that $\Omega[n]$ is a rigid dualizing complex over A, where $n = \mathrm{GKdim}\, A$.*
(b) *A is Auslander-Gorenstein and Cohen-Macaulay, and has a quasi-Frobenius classical quotient ring.*
(c) $\mathrm{Spec}\, A$ *is catenary, and Tauvel's height formula holds.*

Proof. (a) By Proposition 2.1(a), A has an Auslander, Cohen-Macaulay, rigid dualizing complex R.

Fix a maximal ideal $\mathfrak{p}$ of A. Let $m := \mathrm{GKdim}\, A/\mathfrak{p}$ and $n := \mathrm{GKdim}\, A$, and let $M_\mathfrak{p}$ denote the complex over $A/\mathfrak{p}$ given by $\mathrm{RHom}_A(A/\mathfrak{p}, R)$. By Proposition 2.1(b), $M_\mathfrak{p}$ is an Auslander, Cohen-Macaulay, rigid dualizing complex over $A/\mathfrak{p}$. Since $A/\mathfrak{p}$ is simple, it follows from [34, Theorem 0.2] that $M_\mathfrak{p} = \Omega_\mathfrak{p}[m]$ where $\Omega_\mathfrak{p}$ is an invertible $(A/\mathfrak{p}, A/\mathfrak{p})$-bimodule. By duality,

$$\mathrm{Ext}^i_{A^\circ}(R[-n], \Omega_\mathfrak{p}) = \mathrm{Ext}^{i+n-m}_{A^\circ}(R, M_\mathfrak{p}) \cong \mathrm{Ext}^{i+n-m}_A(A/\mathfrak{p}, A),$$

which is nonzero if and only if $i = 0$ (by hypothesis (H5)). By Hom-$\otimes$ adjunction in $\mathsf{D}^b(A^\circ\text{-}\mathsf{Mod})$, we have

$$\mathrm{Ext}^i_{A^\circ}(R[-n], \Omega_\mathfrak{p}) \cong \mathrm{Hom}_{(A/\mathfrak{p})^\circ}(\mathrm{Tor}^A_i(R[-n], A/\mathfrak{p}), \Omega_\mathfrak{p}).$$

Since $\mathrm{Tor}^A_i(R[-n], A/\mathfrak{p})$ is projective over $A/\mathfrak{p}$ (Lemma 1.6(b)), it follows that $\mathrm{Tor}^A_i(R[-n], A/\mathfrak{p}) \neq 0$ if and only if $i = 0$.

By Proposition 1.1, $R[-n]$ is isomorphic to a bimodule, say Ω, and Ω is a progenerator for A°-Mod. As in the proof of Proposition 4.5(b), the natural map $A \to \mathrm{End}_{A^\circ}(\Omega)$ is an isomorphism, and Morita theory implies that Ω is invertible.

(b)(c) These properties follow from part (a) and Lemma 4.6. □

For future use, we record the following consequence of Lemma 5.1 and Theorem 5.2.

Corollary 5.3. *If A is an Auslander-Gorenstein, Cohen-Macaulay algebra satisfying* (H1,H2)*, then there exists an invertible (A, A)-bimodule Ω such that $\Omega[n]$ is a rigid dualizing complex over A, where $n = \mathrm{GKdim}\, A$.*

In the rest of this section, we verify the hypothesis (H5) for standard generic quantized coordinate rings $\mathcal{O}_q(G)$. Since this relies on results from [18], we will need to assume that q is transcendental over $\mathbb{Q}$.

Lemma 5.4. *Suppose $A_1, \ldots, A_n$ are connected graded noetherian algebras having enough normal elements.*

(a) *The algebra $A := A_1 \otimes \cdots \otimes A_n$ is connected graded noetherian and has enough normal elements.*

(b) *If σ is a graded automorphism of A, then $A[t;\sigma]$ is connected graded noetherian with enough normal elements, where t is assigned degree* 1*.*

(c) *Let $C := A[t_1;\sigma_1]\cdots[t_m;\sigma_m]$ be an iterated skew polynomial ring, where the σ_i are graded automorphisms and the t_i have degree* 1*. Then C is connected graded, universally noetherian, and has enough normal elements.*

(d) *If the A_i are Gorenstein* (*respectively, regular*)*, then the algebra C above is Auslander-Gorenstein* (*respectively, Auslander-regular*) *and Cohen-Macaulay.*

Proof. (a) It follows from [2, Propositions 4.3, 4.9 and 4.10] that every connected graded noetherian algebra with enough normal elements is universally noetherian. As a consequence, A is universally noetherian by induction on n. It is clear that A is connected graded (with respect to the standard tensor product grading). To show that A has enough normal elements, we can proceed by induction on n, so it suffices to consider the case $n = 2$. Let P be a non-maximal graded prime ideal of $A_1 \otimes A_2$. Since $A_1 \otimes 1$ commutes with $1 \otimes A_2$, the preimage Q_i of P in A_i is prime. Since we can pass to the algebra $A/(Q_1 \otimes A_2 + A_1 \otimes Q_2)$, there is no loss of generality in assuming that A_1 and A_2 are graded prime, that $P \cap (A_1 \otimes 1) = 0$, and that $P \cap (1 \otimes A_2) = 0$. Since A/P is not simple, A is not finite dimensional, so A_1 and A_2 cannot both be finite dimensional. Say A_1 is infinite dimensional, so that it is not simple. Then there is a homogeneous normal element $x_1 \in A_1$ with positive degree, whence the image of $x_1 \otimes 1$ in A/P is a homogeneous normal element, with positive degree because it is nonzero. The proof is symmetric in case A_2 is infinite dimensional. This shows that $A_1 \otimes A_2$ has enough normal elements.

(b) It is clear that $A[t;\sigma]$ is connected graded noetherian. Let P be a non-maximal graded prime ideal of $A[t;\sigma]$. If $t \notin P$, then the image of t in $A[t;\sigma]/P$ is a homogeneous normal element of degree 1. Otherwise, $A[t;\sigma]/P \cong A/P'$ for some non-maximal graded prime ideal P' of A. The assertion follows.

(c) Using (a), (b), and induction, we see that C is connected graded noetherian with enough normal elements. The universally noetherian property follows from the comment at the beginning of the proof of (a).

(d) Recall that the global dimension of a connected graded noetherian algebra equals the projective dimension of its trivial module (e.g., [23, Chapter 1, Corollary 8.7]). For A_1 and A_2, this means

$$\begin{aligned}\operatorname{gldim} A_1 \otimes A_2 &= \operatorname{projdim}_{A_1 \otimes A_2} k \\ &\le \operatorname{projdim}_{A_1} k + \operatorname{projdim}_{A_2} k = \operatorname{gldim} A_1 + \operatorname{gldim} A_2.\end{aligned}$$

Hence, if A_1 and A_2 are regular, then so is $A_1 \otimes A_2$.

Next, we assume that A_1 and A_2 are Gorenstein of injective dimensions d_1 and d_2 respectively. Since A_1 and A_2 have enough normal elements, they are Artin-Schelter Gorenstein [36, Theorems 0.2 and 0.3] in the following sense:

$$\operatorname{Ext}^i_{A_j}(k, A_j) = \begin{cases} k & \text{if } i = d_j \\ 0 & \text{if } i \neq d_j \end{cases}$$

for $j = 1, 2$. It follows from the Künneth formula that

$$\begin{aligned}\mathrm{Ext}^n_{A_1\otimes A_2}(k, A_1 \otimes A_2) &= \bigoplus_{i=0}^{n} \mathrm{Ext}^i_{A_1}(k, A_1) \otimes \mathrm{Ext}^{n-i}_{A_2}(k, A_2)\\ &= \begin{cases} k & \text{if } n = d_1 + d_2 \\ 0 & \text{if } n \neq d_1 + d_2 \end{cases}.\end{aligned}$$

This means that $\mathrm{k.injdim}\, A_1 \otimes A_2 = d_1 + d_2$, where k. injdim is the k-injective dimension defined in [17]. By [3, Corollary 8.12], $A_1 \otimes A_2$ satisfies the condition χ, and its cohomological dimension $\mathrm{cd}(A_1 \otimes A_2)$ is bounded by its Krull dimension. Since $A_1 \otimes A_2$ has enough normal elements, its Krull dimension is finite [33, Proposition 0.9], and thus $\mathrm{cd}(A_1 \otimes A_2) < \infty$. Hence, we can apply [17, Theorem 4.5] to obtain that $\mathrm{injdim}\, A_1 \otimes A_2 = \mathrm{k.injdim}\, A_1 \otimes A_2$. Therefore $A_1 \otimes A_2$ is Gorenstein.

It follows by induction that if the A_i are Gorenstein (respectively, regular), then so is A. Either property then passes to C (e.g., use [20, Theorem 3.6(1)] and [22, Theorem 5.3(iii)] inductively). The assertion now follows from (c) and [36, Theorem 0.2]. □

Lemma 5.5. *Let A be a noetherian connected graded Gorenstein (respectively, regular) algebra with enough normal elements. Let $C := A[t_1; \sigma_1] \cdots [t_m; \sigma_m]$ be an iterated skew polynomial ring, where the σ_i are graded automorphisms and the t_i have degree 1. Let $D := C[t_1^{-1}, \ldots, t_m^{-1}] = A[t_1^{\pm 1}; \sigma_1] \cdots [t_m^{\pm 1}; \sigma_m]$. Suppose that t_j is an eigenvector for σ_i, for all $i > j$. Then:*

(a) *D is Auslander-Gorenstein (respectively, Auslander-regular) and Cohen-Macaulay.*
(b) *D has a filtration such that* $\mathrm{gr}\, D$ *is a noetherian, connected graded algebra with enough normal elements.*

Proof. Assume that A is regular. The Gorenstein case is completely parallel, except that one has to check the Gorenstein case of [21, Lemma], to feed into [7, Lemmas I.15.4, II.9.11].

(a) By Lemma 5.4, C is connected graded noetherian, Auslander-regular, and Cohen-Macaulay. Observe that each t_i is regular and normal in C; moreover, $t_i C_j = C_j t_i$ for all i, j, where the C_j are the homogeneous components of C. Hence, the element $t := t_1 t_2 \cdots t_m$ is regular normal, and $tC_j = C_j t$ for all j. By [7, Lemmas II.9.11(b)], $C[t^{-1}] = D$ is Auslander-regular and Cohen-Macaulay.

(b) Let t be as above; then there exists a graded automorphism σ of C such that $tc = \sigma(c)t$ for all $c \in C$. Set $E := C[T; \sigma^{-1}]$, which is a connected graded noetherian algebra with enough normal elements, by Lemma 5.4, where $\deg(T) = 1$. Give E the filtration induced from its grading. Since $E/\langle tT - 1\rangle \cong D$, the required properties pass from E to D. □

Theorem 5.6. *Let G be any connected semisimple Lie group over $\mathbb{C}$, and $A := \mathcal{O}_q(G)$ the standard quantized coordinate ring. Assume that q is transcendental over $\mathbb{Q}$. There exists a left and right Ore set $\mathcal{C} \subset A$ such that*

(a) *$(A/I)\mathcal{C}^{-1} \neq 0$ for all proper ideals I of A.*
(b) *The algebra $B := A\mathcal{C}^{-1}$ is Auslander-regular and Cohen-Macaulay.*
(c) *B satisfies* (H2).
(d) $\mathrm{GKdim}\, B = \dim G$.

Proof. We first work over the field $F := \mathbb{Q}(q) \subset \mathbb{C}$ and then extend scalars. Let $\mathfrak{g}$ be the Lie algebra of G, and write $\check{U}_F$ for the simply connected quantized enveloping algebra of $\mathfrak{g}$ over F. Following the notation of [7, §I.6.3], we write E_i, F_i, $K_\lambda^{\pm 1}$ for the standard generators of $\check{U}_F$, where $i = 1, \dots, n = \operatorname{rank}(\mathfrak{g})$ and λ runs through the weight lattice P. Write U_F^+ and U_F^- for the respective subalgebras of $\check{U}_F$ generated by the E_i and the F_i. Since the quantum Serre relations for the E_i and the F_i are homogeneous, U_F^+ and U_F^- are connected graded F-algebras, where the E_i and F_i are homogeneous elements of degree 1.

Write A_F for the version of $\mathcal{O}_q(G)$ defined over F, and let $\mathcal{C}$ denote the multiplicative subset of A_F generated by the coordinate functions $c^\mu_{\mu,\mu}$ (in the notation of [18, §9.1.1]), for $\mu \in P^+$. By [18, Lemma 9.1.10], $\mathcal{C}$ is an Ore set in A_F. Moreover,

$$A_F\mathcal{C}^{-1} \cong (U_F^+ \otimes_F U_F^-)[K_\lambda^{-1} \otimes K_\lambda \mid \lambda \in P] \subset \check{U}_F \otimes_F \check{U}_F$$

[18, Lemma 9.2.13, Proposition 9.2.14], and $\mathcal{C}$ is disjoint from all prime ideals of A_F [18, Corollary 9.3.9]. It follows that $\mathcal{C}$ is disjoint from all proper ideals I of A_F, so that $(A_F/I)\mathcal{C}^{-1} \neq 0$.

We now identify A with $A_F \otimes_F \mathbb{C}$ and A_F with the F-subalgebra $A_F \otimes 1$. Then $\mathcal{C}$ becomes an Ore set in A, and since every proper ideal of A contracts to a proper ideal of A_F, we see that (a) holds. Extending scalars, the isomorphism above yields

$$B = A\mathcal{C}^{-1} \cong B' := (U^+ \otimes_{\mathbb{C}} U^-)[K_\lambda^{-1} \otimes K_\lambda \mid \lambda \in P] \subset \check{U} \otimes_{\mathbb{C}} \check{U},$$

where $U^\pm := U_F^\pm \otimes_F \mathbb{C}$ and $\check{U} := \check{U}_F \otimes_F \mathbb{C}$.

As with $U_F^\pm$, the $\mathbb{C}$-algebras $U^\pm$ are connected graded and noetherian. The basic relations for $\check{U}$ show that the inner automorphism induced by any K_λ on $\check{U}$ restricts to graded automorphisms of U^+ and U^-. Since P is a free abelian group of rank n, we see that B' is an iterated skew-Laurent extension of the form

$$B' = (U^+ \otimes_{\mathbb{C}} U^-)[t_1^{\pm 1}; \sigma_1] \cdots [t_n^{\pm 1}; \sigma_n], \tag{E5.1}$$

for some graded automorphisms σ_i, where $\deg(t_j) = 1$ for all j and $\sigma_i(t_j) = t_j$ for $i > j$. Therefore, properties (b) and (c) will follow from Lemmas 5.4 and 5.5 once we know that U^+ and U^- are regular and have enough normal elements. Since $U^+ \cong U^-$, we need only deal with U^+.

Ringel has shown that U_F^+ is isomorphic to the twisted generic Hall algebra $\mathcal{H}$ [24] (see also [15, Theorem 3]), and that $\mathcal{H}$ is an iterated skew polynomial ring of the form

$$\mathcal{H} = F[X_1][X_2; \tau_2, \delta_2] \cdots [X_m; \tau_m, \delta_m]$$

where the following hold [25, Theorems 2,3]:

(1) The number m of iterations equals the number of positive roots for the Lie algebra $\mathfrak{g}$ of G.
(2) Each τ_i and each δ_i is F-linear.
(3) X_j is an eigenvector of τ_i for all $i > j$.
(4) Each $\delta_i\tau_i = q^{r_i}\tau_i\delta_i$ for some positive integer r_i.

Consequently,

$$U^+ \cong U_F^+ \otimes_F \mathbb{C} \cong \mathbb{C}[X_1][X_2; \widehat{\tau}_2, \widehat{\delta}_2] \cdots [X_m; \widehat{\tau}_m, \widehat{\delta}_m]$$

with corresponding properties. Hence, U^+ is regular [22, Theorem 7.5.3], and all its prime ideals are completely prime [12, Theorem 2.3].

The existence of a large supply of normal elements was established by Caldero [9] for the $\mathbb{C}(\widehat{q})$-form of U^+, where $\widehat{q}$ is an indeterminate over $\mathbb{C}$. Rather than reworking all his proofs for U^+, we can obtain what we need from his results with some base change arguments.

First, extend $\{q\}$ to a transcendence base X for $\mathbb{C}$ over $\mathbb{Q}$, and let K denote the algebraic closure of $\mathbb{Q}(X \setminus \{q\})$ within $\mathbb{C}$. Then q is transcendental over K, and $K \cong \mathbb{C}$ (as abstract fields) because X is infinite. Hence, there is an isomorphism $K(q) \to \mathbb{C}(\widehat{q})$ sending $q \mapsto \widehat{q}$. Consequently, the algebra $U^+_{K(q)} := U^+_F \otimes_F K(q)$ is isomorphic, as a ring, to the $\mathbb{C}(\widehat{q})$-form of U^+ (that is, the $\mathbb{C}(\widehat{q})$-algebra given by generators $E_1, \dots, E_n$ satisfying the quantum Serre relations). Caldero's result, [9, Corollaire 3.2], thus shows that every ideal of $U^+_{K(q)}$ has a normalizing sequence of generators.

Let $\mathfrak{p}$ be a non-maximal graded prime ideal of U^+. Since $\mathfrak{p}$ is completely prime, the contraction $\mathfrak{p}_1 := \mathfrak{p} \cap U^+_{K(q)}$ is a graded prime ideal of $U^+_{K(q)}$. Let $\mathfrak{m}_1$ denote the (unique) graded maximal ideal of $U^+_{K(q)}$, and observe that $\mathfrak{m}_1 U^+$ is the graded maximal ideal of U^+. Consequently, $\mathfrak{m}_1 \not\subseteq \mathfrak{p}$, and so $\mathfrak{p}_1$ is properly contained in $\mathfrak{m}_1$. Since $\mathfrak{m}_1$ has a normalizing sequence of generators, there exists a nonzero element $c \in \mathfrak{m}_1/\mathfrak{p}_1$ which is normal in $U^+_{K(q)}/\mathfrak{p}_1$. A leading term argument, using the fact that $U^+_{K(q)}/\mathfrak{p}_1$ is a domain, shows that the highest degree term of c is normal, so there is no loss of generality in assuming that c is homogeneous. Since $c \in \mathfrak{m}_1/\mathfrak{p}_1$, its degree is positive. Thus, finally, the image of c in $U^+/\mathfrak{p}$ is a homogeneous normal element of positive degree.

Therefore U^+ has enough normal elements, as required. Statements (b) and (c) are now proved.

Since $U^- \cong U^+$, the algebra $U^+ \otimes_{\mathbb{C}} U^-$ can be written as an iterated skew polynomial extension of U^+, and so

$$U^+ \otimes_{\mathbb{C}} U^- \cong \mathbb{C}[X_1][X_2; \widehat{\tau}_2, \widehat{\delta}_2] \cdots [X_m; \widehat{\tau}_m, \widehat{\delta}_m][Y_1][Y_2; \widetilde{\tau}_2, \widetilde{\delta}_2] \cdots [Y_m; \widetilde{\tau}_m, \widetilde{\delta}_m],$$

where

(5) Each $\widehat{\tau}_i$, $\widetilde{\tau}_i$, $\widehat{\delta}_i$, and $\widetilde{\delta}_i$ is $\mathbb{C}$-linear.
(6) X_j is an eigenvector of $\widehat{\tau}_i$ for all $i > j$.
(7) Y_j is an eigenvector of $\widetilde{\tau}_i$ for all $i > j$, and X_j is fixed by $\widetilde{\tau}_i$ for all i, j.

In view of (5)–(7), it follows from [7, Lemma II.9.7] that

$$\operatorname{GKdim}(U^+ \otimes_{\mathbb{C}} U^-) = 2m.$$

Now return to equation (E5.1), and consider the intermediate iterations

$$B'_j := (U^+ \otimes_{\mathbb{C}} U^-)[t_1^{\pm 1}; \sigma_1] \cdots [t_j^{\pm 1}; \sigma_j].$$

Recall that $U^+ \otimes_{\mathbb{C}} U^-$ is generated by the elements $E_i \otimes 1$ and $1 \otimes F_i$, each of which is an eigenvector for the automorphisms σ_j, and that t_l is fixed by σ_j when $l < j$. Hence, B'_{j-1} is generated by a finite set of σ_j-eigenvectors, from which we see that σ_j is *locally algebraic* in the sense of [19], meaning that for each $b \in B'_{j-1}$, the forward orbit $\{\sigma_j^t(b) \mid t \geq 0\}$ spans a finite dimensional subspace of B'_{j-1}. Consequently, it follows from [19, Proposition 1] that $\operatorname{GKdim} B'_j = 1 + \operatorname{GKdim} B_{j-1}$. Hence,

$$\operatorname{GKdim} B = \operatorname{GKdim} B' = 2m + n = \dim \mathfrak{g} = \dim G$$

(recall (1) above), and part (d) is proved. □

Lemma 5.7. *Let A, $\mathcal{C}$, and B be as in* Theorem 5.6.

(a) *Any noetherian (A, A)-bimodule M is $\mathcal{C}$-torsionfree as a left and a right A-module. Hence, if $M \neq 0$, then $M\mathcal{C}^{-1} \neq 0$ and $\mathcal{C}^{-1}M \neq 0$.*
(b) $\operatorname{GKdim}(A/\mathfrak{p})\mathcal{C}^{-1} = \operatorname{GKdim} A/\mathfrak{p}$ *for all prime ideals* $\mathfrak{p}$ *of* A.
(c) $\operatorname{gldim} A = \operatorname{GKdim} A = \operatorname{gldim} B = \operatorname{GKdim} B < \infty$.
(d) *Let $n = \operatorname{gldim} A$. If S is any nonzero finite dimensional left or right A-module, then $\operatorname{Ext}^n_A(S, A) \neq 0$, and $\operatorname{Ext}^i_A(S, A) = 0$ for all $i \neq n$.*

Proof. (a) Let N be the left $\mathcal{C}$-torsion submodule of M. Then N is a sub-bimodule, and so $N = x_1A + \cdots + x_tA$ for some elements $x_1, \dots, x_t$. There exists $c \in \mathcal{C}$ such that $cx_i = 0$ for all i, and thus $c \in \operatorname{l.ann}_A(N)$. Since $\mathcal{C}$ is disjoint from all proper ideals of A (Theorem 5.6(a)), it follows that $\operatorname{l.ann}_A(N) = A$ and $N = 0$. Thus M is $\mathcal{C}$-torsionfree as a left A-module. By symmetry, it is also $\mathcal{C}$-torsionfree as a right A-module.

(b) If $\mathfrak{p}$ is a prime ideal of A, then by part (a), all elements of $\mathcal{C}$ are regular modulo $\mathfrak{p}$. (This also follows directly from the fact that $\mathcal{C} \cap \mathfrak{p} = \varnothing$, by [14, Lemma 10.19].) Hence, $\mathfrak{p}B$ is a prime ideal of B which contracts to $\mathfrak{p}$ [14, Theorem 10.20]. Consequently, $A/\mathfrak{p}$ embeds in $B/\mathfrak{p}B \cong (A/\mathfrak{p})\mathcal{C}^{-1}$, and these prime rings have isomorphic Goldie quotient rings, call them $Q_1 = \operatorname{Fract} A/\mathfrak{p}$ and $Q_2 = \operatorname{Fract}(A/\mathfrak{p})\mathcal{C}^{-1}$. By Theorems 2.6 and 5.6, A and B satisfy (H2), hence so do $A/\mathfrak{p}$ and $(A/\mathfrak{p})\mathcal{C}^{-1}$. It now follows from [35, Theorem 6.9(a)] that $\operatorname{Htr} Q_1 = \operatorname{GKdim} A/\mathfrak{p}$ and $\operatorname{Htr} Q_2 = \operatorname{GKdim}(A/\mathfrak{p})\mathcal{C}^{-1}$. Since $Q_1 \cong Q_2$, part (b) is proved.

(c) We have $\operatorname{GKdim} A = \operatorname{GKdim} B$ by part (b), $\operatorname{gldim} B \leq \operatorname{gldim} A$ because B is a localization of A, and $\operatorname{gldim} B \leq \operatorname{GKdim} B$ because B is Auslander-regular and Cohen-Macaulay.

Set $n := \operatorname{GKdim} B$, and let k denote the trivial module for A. We have $k\mathcal{C}^{-1} \neq 0$ by part (a), from which we see that $k\mathcal{C}^{-1} = k$, that is, the A-module structure on k extends to a B-module structure. Since B is Cohen-Macaulay, $j({}_Bk) = n$, and thus $\operatorname{Ext}^n_B(k, B) \neq 0$. Hence, $n \leq \operatorname{projdim}_B k \leq \operatorname{gldim} B$, and we have

$$\operatorname{gldim} B = \operatorname{projdim}_B k = n.$$

As noted in the proof of [6, Proposition 2.7], it follows from the flatness of B over A that $\operatorname{projdim}_A k \leq n$, and thus $\operatorname{gldim} A \leq n$ by [6, Corollary 1.4(c)]. Therefore

$$\operatorname{gldim} A = \operatorname{GKdim} A = \operatorname{projdim}_A k = n.$$

(d) By symmetry, it is enough to prove this for left modules. In view of [6, Lemma 1.11] (i.e., Lemma 4.1(a)), it suffices to consider the case when S is the trivial module k.

We proved above that $\operatorname{projdim}_A k = n$, whence $\operatorname{Ext}^n_A(k, A) \neq 0$, and also that $j({}_Bk) = n$, whence $\operatorname{Ext}^i_B(k, B) = 0$ for all $i \neq n$. It now follows using [8, Proposition 1.6] that

$$\operatorname{Ext}^i_A(k, A)\mathcal{C}^{-1} \cong \operatorname{Ext}^i_{A\mathcal{C}^{-1}}(k\mathcal{C}^{-1}, A\mathcal{C}^{-1}) = 0$$

for all $i \neq n$. However, $\operatorname{Ext}^i_A(k, A)$ is finite dimensional by Lemma 2.3, and so is noetherian as an (A, A)-bimodule. Thus, it follows from part (a) that $\operatorname{Ext}^i_A(k, A) = 0$ for all $i \neq n$. □

Lemma 5.8. *Let G be any connected semisimple Lie group over $\mathbb{C}$, and $A := \mathcal{O}_q(G)$ the standard quantized coordinate ring. If q is transcendental over $\mathbb{Q}$, then A satisfies* (H5)*, and* $\operatorname{GKdim} A = \dim G$.

Proof. Let $\mathcal{C}$ and B be as in Theorem 5.6, and set

$$n = \operatorname{gldim} A = \operatorname{GKdim} A = \operatorname{gldim} B = \operatorname{GKdim} B = \dim G$$

(recall Lemma 5.7(c) and Theorem 5.6(d)). Fix a maximal ideal $\mathfrak{p} \subset A$, and set $d := \operatorname{GKdim} A/\mathfrak{p}$ and $\mathfrak{q} := \mathfrak{p}B$. Then $\mathfrak{q}$ is a maximal ideal of B by localization theory [14, Theorem 10.20], and $\operatorname{GKdim} B/\mathfrak{q} = d$ by Lemma 5.7(b). Since B is Auslander-regular and Cohen-Macaulay, Lemma 5.1 implies that for nonnegative integers i, we have $\operatorname{Ext}^i_B(B/\mathfrak{q}, B) \neq 0$ if and only if $i = n - d$.

According to Lemma 2.2(b), each $\operatorname{Ext}^i_A(A/\mathfrak{p}, A)$ is a noetherian (A, A)-bimodule. Moreover, $\operatorname{Ext}^i_A(A/\mathfrak{p}, A)\mathcal{C}^{-1} \cong \operatorname{Ext}^i_B(B/\mathfrak{q}, B)$ by [8, Proposition 1.6]. Taking account of Lemma 5.7(a), we conclude that $\operatorname{Ext}^i_A(A/\mathfrak{p}, A) \neq 0$ if and only if $i = n - d$, as desired. □

Now we are ready to prove Theorem 0.1.

Proof of Theorem 0.1. By Theorems 1.9, 2.6 and Lemmas 5.7, 5.8, $\mathcal{O}_q(G)$ satisfies the hypotheses (H1,H2,H5), and $\operatorname{gldim} \mathcal{O}_q(G) = \operatorname{GKdim} \mathcal{O}_q(G) = \dim G$. Theorem 0.1 thus follows from Theorems 2.6 and 5.2. □

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