

COMMUTATION RELATIONS FOR ARBITRARY QUANTUM MINORS

K. R. GOODEARL

ABSTRACT. Complete sets of commutation relations for arbitrary pairs of quantum minors are computed, with explicit coefficients in closed form.

INTRODUCTION

The title of this paper begins with what may seem a misnomer – the term *commutation relation*, in current usage, does not refer to a commutativity condition, $xy = yx$, but has evolved to encompass various “skew commutativity” conditions that have proved to be useful replacements for commutativity. Older types of commutation relations include conditions of the form $xy - yx = z$, used in defining Weyl algebras and enveloping algebras. In quantized versions of classical algebras, relations such as $xy = qyx$ (known as *q-commutation*) appear, along with mixtures of both types. Thus, it has become common to refer to any equation of the form $xy = \lambda yx + z$, where λ is a nonzero scalar, as a *commutation relation for x and y* . One important use of such a relation, especially in enveloping algebras, is that if the algebra supports a filtration such that $\deg(z) < \deg(x) + \deg(y)$, then the images of x and y in the associated graded algebra, call them \tilde{x} and \tilde{y} , commute up to a scalar: $\tilde{x}\tilde{y} = \lambda\tilde{y}\tilde{x}$. Similarly, the cosets of x and y modulo the ideal generated by z commute up to λ . Such coset relations are key ingredients in the work of Soibelman [28], Hodges-Levasseur [9, 10], Joseph [12], and others on quantized coordinate rings.

In many quantized algebras, the available commutation relations are homogeneous and quadratic, of the form $xy = \lambda yx + \sum_i \mu_i x_i y_i$ (where λ and the μ_i are nonzero scalars). Relations of this type are particularly important in establishing a (noncommutative) standard basis of monomials in generators that include the elements x, y, x_i, y_i . Namely, if the generators are ordered in such a way that each $x_i \leq y_i$ but $x > y$, then the given relation allows one to rewrite monomials involving xy as linear combinations of monomials closer to standard form. For example, noncommutative standard bases have been constructed by Lakshmibai and Reshetikhin [18, 19] for quantized coordinate rings of flag varieties and Schubert schemes, by the author and Lenagan [5] for quantum matrix algebras, and by Lenagan and Rigal [23] for quantum Grassmannians and quantum determinantal rings.

2000 *Mathematics Subject Classification.* 16W35, 20G42.

This research was partially supported by a grant from the National Science Foundation.

In order to work effectively with quantized coordinate rings of matrices, Grassmannians, special or general linear groups, and related algebras, one needs explicit commutation relations for quantum minors and related elements. Such relations have often been derived for special cases as needed, either by induction on the size of the minors, using quantum Laplace relations, as in Parshall-Wang [25] and Taft-Towber [29], or by applying the quasitriangular structure of $U_q(\mathfrak{sl}_n(k))$ (that is, its universal R-matrix) to coordinate functions in $\mathcal{O}_q(SL_n(k))$, as in the work of Lakshmibai-Reshetikhin [18, 19], Soibelman [28], and Hodges-Levasseur [9, 10]. Along the former line, the most complete results to date were obtained by Fioresi [3, 4], who developed an algorithm which yields a commutation relation for any pair of quantum minors. This algorithm is an iterative procedure, in which certain products of quantum minors may appear multiple times; explicit coefficients are produced, but are not expressed as closed formulas. Via the quasitriangular approach, general commutation relations for pairs of coordinate functions in quantized coordinate rings $\mathcal{O}_q(G)$, where G is a semisimple Lie group, have been derived in special cases (e.g., see [18, 19, 28, 9, 10]), not all with explicit coefficients. (Quantum minors in $\mathcal{O}_q(SL_n(k))$ are special coordinate functions.) Perhaps the largest group of explicit commutation relations obtained in this way appeared in Hodges-Levasseur-Toro [11] (cf. also [2]). However, to make these fully explicit, canonical elements for the Rosso-Tanisaki Killing form on $U_q(\mathfrak{sl}_n(k))$ had to be computed.

Here we introduce a new method – new only in the sense that it has apparently not been used for this purpose before – with which we derive complete commutation relations for arbitrary pairs of quantum minors, with explicit coefficients in closed form. Our method is dual to the quasitriangular approach, as it relies on the coquasitriangular (or braided) bialgebra structure on the quantized coordinate ring of $n \times n$ matrices. Representation-theoretically, the two approaches are based on equivalent information, in that a quasitriangular (respectively, coquasitriangular) structure on a bialgebra encodes braiding isomorphisms $V \otimes W \xrightarrow{\cong} W \otimes V$ for finite dimensional modules (respectively, comodules) V and W . To record such isomorphisms, one typically requires formulas for matrix entries. However, in the case of a coquasitriangular bialgebra A , the above isomorphism information is stored more compactly, in a bilinear form \mathbf{r} on A – the braiding isomorphism for left A -comodules V and W is then given by the formula

$$v \otimes w \mapsto \sum_{(v),(w)} \mathbf{r}(v_0, w_0) w_1 \otimes v_1,$$

where we have used the Sweedler notation $v \mapsto \sum_{(v)} v_0 \otimes v_1$ for the comodule structure map $V \rightarrow A \otimes V$, and similarly for W . The resulting commutation relations are equations with values of \mathbf{r} as coefficients, namely

$$(0.1) \quad \sum_{(a),(b)} \mathbf{r}(a_1, b_1) a_2 b_2 = \sum_{(a),(b)} \mathbf{r}(a_2, b_2) b_1 a_1$$

for $a, b \in A$, where now the Sweedler notation is used for the comultiplication map $A \rightarrow A \otimes A$.

When A is the bialgebra $\mathcal{O}_q(M_n(k))$ and $a = [I|J]$ and $b = [M|N]$ are quantum minors (see below for notation), equation (0.1) becomes

$$(0.2) \quad \sum_{\substack{|S|=|I| \\ |T|=|M|}} \mathbf{r}([I|S], [M|T])[S|J][T|N] = \sum_{\substack{|S|=|J| \\ |T|=|N|}} \mathbf{r}([S|J], [T|N])[M|T][I|S].$$

Observe that $[I|J][M|N]$ occurs on the left hand side of (0.2) when $S = I$ and $T = M$, while $[M|N][I|J]$ occurs on the right when $S = J$ and $T = N$. As we shall see, the coefficients for these terms, namely $\mathbf{r}([I|I], [M|M])$ and $\mathbf{r}([J|J], [N|N])$, are nonzero (in fact, they are powers of q). Thus, to obtain explicit commutation relations for $[I|J]$ and $[M|N]$, we only need to compute the values $\mathbf{r}([I|S], [M|T])$ and $\mathbf{r}([S|J], [T|N])$. This is precisely what we do in the paper – see especially Theorems 4.6 and 5.2. Additional relations follow from these by various symmetries, or by investing quantum Laplace relations. (Quantum Plücker relations in quantum Grassmannians can also be used for this purpose.) See Theorems 5.7, 6.3 and Corollaries 5.3, 5.8, 6.4.

Our notation and conventions are collected in Section 1. In particular, the relations we use for $\mathcal{O}_q(M_n(k))$ are displayed in (1.6), so that the reader may compare with other papers in which q is replaced by q^{-1} or q^2 . Our computations of the values of the form \mathbf{r} on pairs of quantum minors occupy Sections 2 and 4; the intermediate Section 3 provides a first set of commutation relations to illustrate our methods. The general commutation relations are derived in Sections 5 and 6, and we conclude by using these relations, in Section 7, to evaluate the standard Poisson bracket on pairs of classical minors.

1. NOTATION AND CONVENTIONS

Fix a positive integer n , a base field k , and a nonzero scalar $q \in k^\times$. We work within the standard single-parameter quantized coordinate ring of $n \times n$ matrices over k , which we denote $\mathcal{O}_q(M_n(k))$, as defined in §1.2 below. We use the abbreviation

$$(1.1) \quad \widehat{q} = q - q^{-1},$$

since this scalar appears in numerous formulas.

1.1. R-matrix. The standard R-matrix of type A_{n-1} can be presented in the form

$$(1.2) \quad R = q \sum_{i=1}^n e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1 \\ i \neq j}}^n e_{ii} \otimes e_{jj} + \widehat{q} \sum_{\substack{i,j=1 \\ i > j}}^n e_{ij} \otimes e_{ji}$$

[26, Equation (1.5), p. 200]. We view R as a linear automorphism of $k^n \otimes k^n$, which acts on the standard basis vectors $x_i \otimes x_j$ according to the following formula, using

the conventions of [14]:

$$(1.3) \quad R(x_l \otimes x_m) = \sum_{i,j=1}^n R_{lm}^{ij} x_i \otimes x_j.$$

The entries of the $n^2 \times n^2$ matrix R_{lm}^{ij} are as follows (cf. [14, Equation (9.13), p. 309]):

$$(1.4) \quad \begin{array}{llll} R_{ii}^{ii} = q & (\text{all } i) & R_{ij}^{ij} = 1 & (i \neq j) \\ R_{ji}^{ij} = \widehat{q} & (i > j) & R_{lm}^{ij} = 0 & (\text{otherwise}). \end{array}$$

1.2. Generators, relations, and grading. The algebra $A = \mathcal{O}_q(M_n(k))$ is obtained from (1.4) by the Faddeev-Reshetikhin-Takhtadzhyan construction, namely as the k -algebra $A(R)$ presented by generators X_{ij} (for $i, j = 1, \dots, n$) and relations

$$(1.5) \quad \sum_{s,t=1}^n R_{st}^{ij} X_{sl} X_{tm} = \sum_{s,t=1}^n X_{jt} X_{is} R_{lm}^{st}$$

for all $i, j, l, m = 1, \dots, n$. (See [26, Definition 1, p. 197] and [14, §9.1.1]. We have written X_{ij} for the generators labelled t_{ij} in [26] and u_j^i in [14].) As is well known, the relations (1.5) are equivalent to

$$(1.6) \quad \begin{array}{ll} X_{ij} X_{lj} = q X_{lj} X_{ij} & (i < l) \\ X_{ij} X_{im} = q X_{im} X_{ij} & (j < m) \\ X_{ij} X_{lm} = X_{lm} X_{ij} & (i < l, j > m) \\ X_{ij} X_{lm} - X_{lm} X_{ij} = \widehat{q} X_{im} X_{lj} & (i < l, j < m) \end{array}$$

(cf. [14, Equations (9.17), p. 310]). Some authors define quantum matrices using relations as in (1.6) but with q replaced by q^{-1} ; thus, the algebras they define match what we would label $\mathcal{O}_{q^{-1}}(M_n(k))$. See [21, p. 3317] or [25, Equations (3.5a), p. 37], for example. In comparing our work with those papers, we must be careful to interchange q and q^{-1} . However, \widehat{q} is defined to be $q^{-1} - q$ in [25, p. 38], and so we do not change \widehat{q} when carrying over results from that paper.

Because of the homogeneity of the relations (1.6), A carries a natural $(\mathbb{Z}^n \times \mathbb{Z}^n)$ -grading, such that each X_{ij} is homogeneous of degree (ϵ_i, ϵ_j) , where $\epsilon_1, \dots, \epsilon_n$ are the standard basis elements for \mathbb{Z}^n .

1.3. Coquasitriangular structure. We follow [8, Section 1] in defining a *coquasitriangular bialgebra* (also called a *bialgebra with braiding structure* [21, Theorem 2.7] or a *cobraided bialgebra* [13, Definition VIII.5.1]) to be a bialgebra B equipped

with a convolution-invertible bilinear form $\mathbf{r} : B \otimes B \rightarrow k$ such that

$$(1.7)(i) \quad \sum_{(a),(b)} \mathbf{r}(a_1, b_1) a_2 b_2 = \sum_{(a),(b)} \mathbf{r}(a_2, b_2) b_1 a_1$$

$$(1.7)(ii) \quad \mathbf{r}(ab, c) = \sum_{(c)} \mathbf{r}(a, c_1) \mathbf{r}(b, c_2)$$

$$(1.7)(iii) \quad \mathbf{r}(a, bc) = \sum_{(a)} \mathbf{r}(a_1, c) \mathbf{r}(a_2, b)$$

$$(1.7)(iv) \quad \mathbf{r}(a, 1) = \mathbf{r}(1, a) = \varepsilon(a)$$

for all $a, b, c \in B$, where we have written $\mathbf{r}(x \otimes y)$ as $\mathbf{r}(x, y)$ for convenience, and have used the Sweedler notation for comultiplication in the form $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$. Condition (1.7)(iv) is redundant by [14, Proposition 10.2(ii), p. 333]. Thus, the above definition agrees with [13, Definition VIII.5.1], [14, Definition 10.1, pp. 331-2], and [20, Definition 7.3.1], but not with the conditions in [21, Theorem 2.7]. However, the latter conditions match those of (1.7)(i)–(iv) if one uses the form $\langle - | - \rangle$ given by $\langle a | b \rangle = \mathbf{r}(b, a)$.

By [14, Theorem 10.7, p. 337], whenever R is an invertible R-matrix satisfying the original form of the quantum Yang-Baxter equation, the FRT-algebra $A(R)$ is coquasitriangular with respect to the form \mathbf{r} determined by

$$(1.8) \quad \mathbf{r}(X_{ij}, X_{lm}) = R_{jm}^{il}$$

for all i, j, l, m . (By the *original QYBE*, we mean the equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ [26, Equation (0.7), p. 195], as opposed to the form exhibiting the braid relation, namely $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$.) Note that, in view of (1.8), if we put $a = X_{il}$ and $b = X_{jm}$ into (1.7)(i), we recover the relations (1.5).

It is well known that the R-matrix given in (1.2) satisfies the original QYBE (e.g., [14, §8.1.2, pp. 246-7 and Equation (8.60), p. 270]). Consequently:

1.4. Theorem. *The algebra $A = \mathcal{O}_q(M_n(k))$ is a coquasitriangular bialgebra with respect to the bilinear form $\mathbf{r} : A \otimes A \rightarrow k$ determined by the following conditions:*

$$(1.9) \quad \begin{array}{lll} \mathbf{r}(X_{ii}, X_{ii}) = q & (\text{all } i) & \mathbf{r}(X_{ii}, X_{jj}) = 1 \quad (i \neq j) \\ \mathbf{r}(X_{ij}, X_{ji}) = \widehat{q} & (i > j) & \mathbf{r}(X_{ij}, X_{lm}) = 0 \quad (\text{otherwise}). \quad \square \end{array}$$

1.5. Quantum minors. We write $[I|J]$ for the quantum minor in A with row index set I and column index set J ; this minor is just the quantum determinant in the subalgebra $k\langle X_{ij} \mid i \in I, j \in J \rangle$, which is naturally isomorphic to $\mathcal{O}_q(M_{|I|}(k))$. Specifically, if we write the elements of I and J in ascending order, say

$$I = \{i_1 < \cdots < i_t\} \quad J = \{j_1 < \cdots < j_t\},$$

then

$$(1.10) \quad \begin{aligned} [I|J] &= \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} X_{i_{\sigma(1)}, j_1} X_{i_{\sigma(2)}, j_2} \cdots X_{i_{\sigma(t)}, j_t} \\ &= \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} X_{i_1, j_{\sigma(1)}} X_{i_2, j_{\sigma(2)}} \cdots X_{i_t, j_{\sigma(t)}}, \end{aligned}$$

where $\ell(\sigma)$ denotes the length of the permutation $\sigma \in S_t$ as a product of simple transpositions $(l, l+1)$ (cf. [14, equations (9.18) and (9.20), pp. 311-312], [25, p. 43]). Note that $[I|J]$ is homogeneous of degree

$$(\epsilon_{i_1} + \cdots + \epsilon_{i_t}, \epsilon_{j_1} + \cdots + \epsilon_{j_t})$$

with respect to the grading of §1.2.

Comultiplication of quantum minors is given by the rule

$$(1.11) \quad \Delta([I|J]) = \sum_{\substack{K \subseteq \{1, \dots, n\} \\ |K|=|I|}} [I|K][K|J]$$

(e.g., [14, Proposition 9.7(ii), p. 312]).

1.6. Transpose and anti-transpose. As observed in [25, Proposition 3.7.1(1)], there is a k -algebra automorphism τ on A such that $\tau(X_{ij}) = X_{ji}$ for all i, j . We refer to τ as the *transpose automorphism*. There is also a k -algebra anti-automorphism τ_2 on A sending $X_{ij} \mapsto X_{n+1-i, n+1-j}$ for all i, j [25, Proposition 3.7.1(2)]. This proposition also shows that τ is a coalgebra anti-automorphism while τ_2 is a coalgebra automorphism, that is,

$$\Delta \circ \tau = \phi \circ (\tau \otimes \tau) \circ \Delta \quad \Delta \circ \tau_2 = (\tau_2 \otimes \tau_2) \circ \Delta,$$

where ϕ is the *flip* automorphism on $A \otimes A$, sending $a \otimes b \mapsto b \otimes a$ for all $a, b \in A$. Hence,

$$\Delta \tau(a) = \sum_{(a)} \tau(a_2) \otimes \tau(a_1) \quad \Delta \tau_2(a) = \sum_{(a)} \tau_2(a_1) \otimes \tau_2(a_2)$$

for $a \in A$. Consequently, when writing out $\Delta \tau(a)$ and $\Delta \tau_2(a)$ in Sweedler notation we may take

$$(1.12) \quad \begin{aligned} \tau(a)_1 &= \tau(a_2) & \tau(a)_2 &= \tau(a_1) \\ \tau_2(a)_1 &= \tau_2(a_1) & \tau_2(a)_2 &= \tau_2(a_2). \end{aligned}$$

We recall from [25, Lemma 4.3.1] that

$$(1.13) \quad \tau([I|J]) = [J|I] \quad \tau_2([I|J]) = [\omega_0 I | \omega_0 J]$$

for all quantum minors $[I|J]$ in A , where ω_0 is the longest element of S_n , that is, the permutation $i \mapsto n+1-i$.

As discussed in [25, Remark 3.7.2], there is an isomorphism (of bialgebras) $\mathcal{O}_q(M_n(k)) \rightarrow \mathcal{O}_{q^{-1}}(M_n(k))$ that sends $X_{ij} \mapsto X'_{n+1-i, n+1-j}$ for all i, j , where the $X'_{\bullet, \bullet}$ are the standard generators for $\mathcal{O}_{q^{-1}}(M_n(k))$. Let us call this isomorphism

β , and let us use the notation $[I|J]'$ for quantum minors in $\mathcal{O}_{q^{-1}}(M_n(k))$. It was shown in [7, proof of Corollary 5.9] that

$$(1.14) \quad \beta([I|J]) = [\omega_0 I | \omega_0 J]'$$

for all quantum minors $[I|J]$ in A .

1.7. Lemma. *The form \mathbf{r} satisfies $\mathbf{r}(a, b) = \mathbf{r}(\tau(b), \tau(a)) = \mathbf{r}(\tau_2(b), \tau_2(a))$ for all $a, b \in A$. In particular,*

$$(1.15) \quad \mathbf{r}([I|J], [M|N]) = \mathbf{r}([N|M], [J|I]) = \mathbf{r}([\omega_0 M | \omega_0 N], [\omega_0 I | \omega_0 J])$$

for all quantum minors $[I|J]$ and $[M|N]$ in A .

Proof. Set $\mathbf{r}'(a, b) = \mathbf{r}(\tau(b), \tau(a))$ and $\mathbf{r}''(a, b) = \mathbf{r}(\tau_2(b), \tau_2(a))$ for all $a, b \in A$, and note from (1.9) that $\mathbf{r}'(X_{ij}, X_{lm}) = \mathbf{r}''(X_{ij}, X_{lm}) = \mathbf{r}(X_{ij}, X_{lm})$ for all i, j, l, m . To prove that \mathbf{r}' and \mathbf{r}'' coincide with \mathbf{r} , it suffices to show that these forms agree on all monomials in the X_{ij} . This will be clear by induction on the lengths of the monomials once we show that \mathbf{r}' and \mathbf{r}'' satisfy (1.7)(ii) and (1.7)(iii). These identities are routine with the aid of (1.12); we give one sample:

$$\begin{aligned} \mathbf{r}'(ab, c) &= \mathbf{r}(\tau(c), \tau(a)\tau(b)) = \sum_{(\tau(c))} \mathbf{r}(\tau(c)_1, \tau(b)) \mathbf{r}(\tau(c)_2, \tau(a)) \\ &= \sum_{(c)} \mathbf{r}(\tau(c_2), \tau(b)) \mathbf{r}(\tau(c_1), \tau(a)) = \sum_{(c)} \mathbf{r}'(b, c_2) \mathbf{r}'(a, c_1) \\ &= \sum_{(c)} \mathbf{r}'(a, c_1) \mathbf{r}'(b, c_2) \end{aligned}$$

for all $a, b, c \in A$. □

1.8. Definition of quantities $\ell(S; T)$. Many formulas concerning quantum minors require powers of q or $-q$ whose exponents are quantities which might be called the number of inversions between two sets. We follow [24] in defining

$$(1.16) \quad \ell(S; T) = |\{(s, t) \in S \times T \mid s > t\}|$$

for any subsets $S, T \subseteq \{1, \dots, n\}$.

1.9. Quantum Laplace relations. We shall need the following q -Laplace relations from [24, Proposition 1.1], for index sets $I, J \subseteq \{1, \dots, n\}$ of the same cardinality. If I_1, I_2 are nonempty subsets of I with $|I_1| + |I_2| = |I|$, then

$$(1.17) \quad \sum_{\substack{J=J_1 \sqcup J_2 \\ |J_i|=|I_i|}} (-q)^{\ell(J_1; J_2)} [I_1|J_1][I_2|J_2] = \begin{cases} (-q)^{\ell(I_1; I_2)} [I|J] & (I_1 \cap I_2 = \emptyset) \\ 0 & (I_1 \cap I_2 \neq \emptyset), \end{cases}$$

while if J_1, J_2 are nonempty subsets of J with $|J_1| + |J_2| = |J|$, then

$$(1.18) \quad \sum_{\substack{I=I_1 \sqcup I_2 \\ |I_i|=|J_i|}} (-q)^{\ell(I_1; I_2)} [I_1|J_1][I_2|J_2] = \begin{cases} (-q)^{\ell(J_1; J_2)} [I|J] & (J_1 \cap J_2 = \emptyset) \\ 0 & (J_1 \cap J_2 \neq \emptyset). \end{cases}$$

Observe that (1.17) holds trivially in case I_1 or I_2 is empty, and that (1.18) holds trivially in case J_1 or J_2 is empty

Reduction formulas for values of the form \mathbf{r} can be obtained by combining (1.17) and (1.18) with (1.7)(ii)(iii). For example, if $J = J_1 \sqcup J_2$, then (1.18) together with (1.7)(ii) yields

$$(1.19) \quad (-q)^{\ell(J_1; J_2)} \mathbf{r}([I|J], [M|N]) = \sum_{I=I_1 \sqcup I_2} \sum_L (-q)^{\ell(I_1; I_2)} \mathbf{r}([I_1|J_1], [M|L]) \mathbf{r}([I_2|J_2], [L|N])$$

for all $[M|N]$.

1.10. Some further notation. To simplify notation for operations on index sets, we often omit braces from singletons – in particular, we write

$$(1.20) \quad I \setminus i = I \setminus \{i\} \quad I \sqcup l = I \sqcup \{l\} \quad I \setminus i \sqcup l = (I \setminus \{i\}) \sqcup \{l\}$$

for $i \in I$ and $l \notin I$. The Kronecker delta symbol will be applied to index sets as well as to individual indices – thus, $\delta(I, J) = 1$ when $I = J$ while $\delta(I, J) = 0$ when $I \neq J$. In the case of an index versus an index set, the Kronecker symbol will be used to indicate membership, that is, $\delta(i, I) = 1$ means $i \in I$ while $\delta(i, I) = 0$ means $i \notin I$.

Finally, we shall need the following partial order on index sets of the same cardinality. If I and J are t -element subsets of $\{1, \dots, n\}$, write their elements in ascending order, say

$$I = \{i_1 < i_2 < \dots < i_t\} \quad J = \{j_1 < j_2 < \dots < j_t\},$$

and then define

$$(1.21) \quad I \leq J \quad \iff \quad i_l \leq j_l \text{ for } l = 1, \dots, t.$$

2. INITIAL COMPUTATIONS

Throughout this section, let i and j denote indices in $\{1, \dots, n\}$, and let I, J, M, N denote index sets contained in $\{1, \dots, n\}$, with $|I| = |J|$ and $|M| = |N|$.

2.1. Lemma. $\mathbf{r}(X_{ii}, [I|J]) = \mathbf{r}([I|J], X_{ii}) = q^{\delta(i, I)} \delta(I, J)$.

Proof. Write $I = \{i_1 < \dots < i_t\}$ and $J = \{j_1 < \dots < j_t\}$, and note using (1.10) and (1.7)(ii) that

$$(2.1) \quad \mathbf{r}([I|J], X_{ii}) = \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} \sum_{l_1, \dots, l_{t-1}}^n \mathbf{r}(X_{i_1 j_{\sigma(1)}}, X_{i l_1}) \mathbf{r}(X_{i_2 j_{\sigma(2)}}, X_{l_1 l_2}) \cdots \mathbf{r}(X_{i_t j_{\sigma(t)}}, X_{l_{t-1} i}).$$

In view of (1.9), a nonzero term can occur in the second summation of (2.1) only when $i \leq l_1 \leq l_2 \leq \dots \leq l_{t-1} \leq i$, that is, when $l_1 = \dots = l_{t-1} = i$. Hence, (2.1) reduces to

$$(2.2) \quad \mathbf{r}([I|J], X_{ii}) = \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} \mathbf{r}(X_{i_1 j_{\sigma(1)}}, X_{ii}) \mathbf{r}(X_{i_2 j_{\sigma(2)}}, X_{ii}) \cdots \mathbf{r}(X_{i_t j_{\sigma(t)}}, X_{ii}).$$

In (2.2), a nonzero term can occur in the sum only when $i_s = j_{\sigma(s)}$ for $s = 1, \dots, t$. Since the i_s and j_s are arranged in ascending order, this situation only happens when $I = J$ and $\sigma = \text{id}$. Thus, $\mathbf{r}([I|J], X_{ii}) = 0$ when $I \neq J$, and

$$\mathbf{r}([I|I], X_{ii}) = \mathbf{r}(X_{i_1 i_1}, X_{ii}) \mathbf{r}(X_{i_2 i_2}, X_{ii}) \cdots \mathbf{r}(X_{i_t i_t}, X_{ii}) = q^{\delta(i, I)}.$$

The formula for $\mathbf{r}(X_{ii}, [I|J])$ follows via Lemma 1.7. \square

2.2. Lemma. $\mathbf{r}(X_{ij}, -) \equiv 0$ when $i < j$, and $\mathbf{r}(-, X_{ij}) \equiv 0$ when $i > j$.

Proof. Consider any monomial $a = X_{i(1),j(1)} X_{i(2),j(2)} \cdots X_{i(t),j(t)} \in A$. Then by (1.7)(ii),

$$\mathbf{r}(a, X_{ij}) = \sum_{l_1, \dots, l_{t-1}}^n \mathbf{r}(X_{i(1),j(1)}, X_{il_1}) \mathbf{r}(X_{i(2),j(2)}, X_{l_1 l_2}) \cdots \mathbf{r}(X_{i(t),j(t)}, X_{l_{t-1} j}).$$

If some term $\mathbf{r}(X_{i(1),j(1)}, X_{il_1}) \mathbf{r}(X_{i(2),j(2)}, X_{l_1 l_2}) \cdots \mathbf{r}(X_{i(t),j(t)}, X_{l_{t-1} j})$ does not vanish, we must have $i \leq l_1 \leq \cdots \leq l_{t-1} \leq j$. This shows that $\mathbf{r}(-, X_{ij})$ can fail to vanish only when $i \leq j$. The first statement of the lemma follows via Lemma 1.7. \square

2.3. Corollary. $\mathbf{r}([I|J], -) \equiv 0$ when $I \not\geq J$, and $\mathbf{r}(-, [I|J]) \equiv 0$ when $I \not\leq J$.

Proof. Write $I = \{i_1 < \cdots < i_t\}$ and $J = \{j_1 < \cdots < j_t\}$, and suppose that $\mathbf{r}([I|J], c) \neq 0$ for some $c \in A$. Then by (1.10) and (1.7)(ii),

$$\sum_{(c)} \mathbf{r}(X_{i_1 j_{\sigma(1)}}, c_1) \mathbf{r}(X_{i_2 j_{\sigma(2)}}, c_2) \cdots \mathbf{r}(X_{i_t j_{\sigma(t)}}, c_t) \neq 0$$

for some $\sigma \in S_t$. Lemma 2.2 then implies that $i_s \geq j_{\sigma(s)}$ for $s = 1, \dots, t$.

First, $i_1 \geq j_{\sigma(1)} \geq j_1$. Now let $1 < s \leq t$. If $\sigma(s) \geq s$, then $i_s \geq j_{\sigma(s)} \geq j_s$. If $\sigma(s) < s$, then $\sigma(u) \geq s$ for some $u < s$, whence $i_s > i_u \geq j_{\sigma(u)} \geq j_s$. Thus, $i_s \geq j_s$ for all s , and therefore $I \geq J$. Similarly, if $\mathbf{r}(-, [I|J])$ does not vanish, then $I \leq J$. \square

2.4. Proposition. *If $i < j$, then*

$$(2.3)(i) \quad \mathbf{r}([I|J], X_{ij}) = \widehat{q}(-q)^{|[1,i] \cap J| - |[1,j] \cap I|} \delta(i, J) \delta(j, I) \delta(I \setminus j, J \setminus i)$$

$$(2.3)(ii) \quad = \widehat{q}(-q)^{-|(i,j) \cap I \cap J|} \delta(i, J) \delta(j, I) \delta(I \setminus j, J \setminus i).$$

Proof. Note first that (2.3)(ii) follows from (2.3)(i). For if the right hand side of (2.3)(i) is nonzero, then $I = (I \cap J) \sqcup j$ and $J = (I \cap J) \sqcup i$, whence $[1, i] \cap J = [1, i] \cap I \cap J = [1, i] \cap I \cap J$ and $[1, j] \cap I = [1, j] \cap I \cap J$.

We induct on $|I|$, the case $|I| = 1$ being clear from (1.9). Now assume that $|I| > 1$, and suppose that $\mathbf{r}([I|J], X_{ij}) \neq 0$.

Choose $s \in I$, and write $I = I_1 \sqcup I_2$ with $I_1 = \{s\}$ and $I_2 = I \setminus \{s\}$. The q -Laplace relation (1.17) yields

$$(2.4) \quad (-q)^{|[1,s] \cap I|} [I|J] = \sum_{t \in J} (-q)^{|[1,t] \cap J|} X_{st} [I \setminus s | J \setminus t].$$

For each $t \in J$, we have

$$(2.5) \quad \mathbf{r}(X_{st}[I \setminus s | J \setminus t], X_{ij}) = \sum_{l=1}^n \mathbf{r}(X_{st}, X_{il}) \mathbf{r}([I \setminus s | J \setminus t], X_{lj}).$$

Since $\mathbf{r}([I | J], X_{ij}) \neq 0$, we must have $\mathbf{r}(X_{st}, X_{il}) \mathbf{r}([I \setminus s | J \setminus t], X_{lj}) \neq 0$ for some $l \in \{1, \dots, n\}$ and $t \in J$.

Suppose that $i \notin J$. Then $t \neq i$, and so because $\mathbf{r}(X_{st}, X_{il}) \neq 0$, we must have $t = s$ and $l = i$. Then $\mathbf{r}([I \setminus s | J \setminus s], X_{ij}) \neq 0$, which contradicts the induction hypothesis because $i \notin J \setminus s$. Therefore $i \in J$.

Next, suppose that $j \notin I \setminus s$. If $l < j$, we would have $\mathbf{r}([I \setminus s | J \setminus t], X_{lj}) = 0$ by the induction hypothesis. Since $\mathbf{r}(-, X_{lj})$ would vanish if $l > j$, we must have $l = j$. Now $\mathbf{r}(X_{st}, X_{ij}) \neq 0$, and so $s = j$ and $t = i$. Thus, either $j \in I \setminus s$ or $j = s$, so in any case we conclude that $j \in I$.

We may now assume that $s = j$. Since $j \notin I \setminus j$, we have $\mathbf{r}([I \setminus j | J \setminus t], X_{ij}) = 0$ for all $t \in J$ by the induction hypothesis. On the other hand, $\mathbf{r}(X_{jt}, X_{il}) = 0$ for $l \neq i, j$, and $\mathbf{r}(X_{jt}, X_{ij}) = 0$ for $t \neq i$. Hence, the right hand side of (2.5) vanishes when $t \neq i$, and it equals $\widehat{q} \mathbf{r}([I \setminus j | J \setminus i], X_{jj})$ when $t = i$. Combining (2.4) and (2.5) thus yields

$$(2.6) \quad (-q)^{|[1, j] \cap I|} \mathbf{r}([I | J], X_{ij}) = (-q)^{|[1, i] \cap J|} \widehat{q} \mathbf{r}([I \setminus j | J \setminus i], X_{jj}).$$

Since the left hand side of (2.6) is nonzero by assumption, Lemma 2.1 implies that $I \setminus j = J \setminus i$ and $\mathbf{r}([I \setminus j | J \setminus i], X_{jj}) = 1$. The formula (2.3)(i) follows, and the induction step is established. \square

2.5. Corollary. *If $i > j$, then*

$$(2.7)(i) \quad \mathbf{r}(X_{ij}, [I | J]) = \widehat{q} (-q)^{|[1, j] \cap I| - |[1, i] \cap J|} \delta(i, J) \delta(j, I) \delta(I \setminus j, J \setminus i)$$

$$(2.7)(ii) \quad = \widehat{q} (-q)^{-|(j, i] \cap I \cap J|} \delta(i, J) \delta(j, I) \delta(I \setminus j, J \setminus i).$$

Proof. Apply Lemma 1.7 to Proposition 2.4. \square

2.6. Proposition. $\mathbf{r}([I | I], [M | N]) = \mathbf{r}([M | N], [I | I]) = q^{|I \cap M|} \delta(M, N)$.

Proof. This is parallel to the proof of Lemma 2.1. Write $M = \{m_1 < \dots < m_t\}$ and $N = \{n_1 < \dots < n_t\}$, and note that

$$(2.8) \quad \mathbf{r}([M | N], [I | I]) = \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} \mathbf{r}(X_{m_1 n_{\sigma(1)}} X_{m_2 n_{\sigma(2)}} \cdots X_{m_t n_{\sigma(t)}}, [I | I]),$$

while for each $\sigma \in S_t$ we have

$$(2.9) \quad \mathbf{r}(X_{m_1 n_{\sigma(1)}} X_{m_2 n_{\sigma(2)}} \cdots X_{m_t n_{\sigma(t)}}, [I | I]) = \sum_{L_1, \dots, L_{t-1}} \mathbf{r}(X_{m_1 n_{\sigma(1)}}, [I | L_1]) \mathbf{r}(X_{m_2 n_{\sigma(2)}}, [L_1 | L_2]) \cdots \mathbf{r}(X_{m_t n_{\sigma(t)}}, [L_{t-1} | I]).$$

Consider the right hand side of (2.9). By Corollary 2.3, a nonzero term can occur in that sum only when $I \leq L_1 \leq \dots \leq L_{t-1} \leq I$, and so only when all the $L_s = I$.

Thus,

$$(2.10) \quad \mathbf{r}([M|N], [I|I]) = \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} \mathbf{r}(X_{m_1 n_{\sigma(1)}}, [I|I]) \mathbf{r}(X_{m_2 n_{\sigma(2)}}, [I|I]) \cdots \mathbf{r}(X_{m_t n_{\sigma(t)}}, [I|I]).$$

In view of Lemma 2.2 and Corollary 2.5, $\mathbf{r}(X_{ij}, [I|I]) = 0$ for all $i \neq j$. Hence, a nonzero term can occur in the right hand side of (2.10) only when $m_s = n_{\sigma(s)}$ for all s , that is, only when $M = N$ and $\sigma = \text{id}$. Therefore $\mathbf{r}([M|N], [I|I]) = 0$ when $M \neq N$, while

$$\mathbf{r}([M|M], [I|I]) = \mathbf{r}(X_{m_1 m_1}, [I|I]) \mathbf{r}(X_{m_2 m_2}, [I|I]) \cdots \mathbf{r}(X_{m_t m_t}, [I|I]) = q^{|I \cap M|},$$

in view of Lemma 2.1. The formula for $\mathbf{r}([I|I], [M|N])$ follows via Lemma 1.7. \square

3. INITIAL COMMUTATION RELATIONS

We now use the computations of $\mathbf{r}(-, -)$ obtained so far to derive some commutation relations, both to illustrate the method and to doublecheck the results against known relations in the literature. As in the previous section, let i and j denote indices in $\{1, \dots, n\}$, and let I, J, M, N denote index sets contained in $\{1, \dots, n\}$, with $|I| = |J|$ and $|M| = |N|$.

3.1. Direct application of (1.7)(i). If we set $a = X_{ij}$ and $b = [I|J]$ in (1.7)(i), we obtain

$$(3.1) \quad \sum_{l, L} \mathbf{r}(X_{il}, [I|L]) X_{lj} [L|J] = \sum_{l, L} \mathbf{r}(X_{lj}, [L|J]) [I|L] X_{il}.$$

We claim that (3.1) reduces to

$$(3.2) \quad q^{\delta(i, I)} X_{ij} [I|J] + (1 - \delta(i, I)) \widehat{q} \sum_{\substack{l \in I \\ l < i}} (-q)^{-|(l, i) \cap I|} X_{lj} [I \setminus l \sqcup i | J] = q^{\delta(j, J)} [I|J] X_{ij} + (1 - \delta(j, J)) \widehat{q} \sum_{\substack{l \in J \\ l > j}} (-q)^{-|(j, l) \cap J|} [I|J \setminus l \sqcup j] X_{il}.$$

According to Lemma 2.2 and Corollary 2.3, $\mathbf{r}(X_{il}, [I|L]) = 0$ unless $i \geq l$ and $I \leq L$. By Lemma 2.1, $\mathbf{r}(X_{ii}, [I|L]) = 0$ unless $L = I$, and $\mathbf{r}(X_{ii}, [I|I]) = q^{\delta(i, I)}$. When $i > l$, Corollary 2.5 shows that $\mathbf{r}(X_{il}, [I|L])$ is nonzero only when $i \in L$, $l \in I$, and $I \setminus l = L \setminus i$. In such cases, $i \notin I$ and $L = I \setminus l \sqcup i$, and the exponent of $-q$ that appears in (2.7)(ii) is $-|(l, i) \cap I \cap L| = -|(l, i) \cap I|$. Thus, the left hand sides of (3.1) and (3.2) agree.

Similarly, $\mathbf{r}(X_{lj}, [L|J]) = 0$ unless $l \geq j$ and $L \leq J$, while $\mathbf{r}(X_{jj}, [L|J]) = 0$ unless $L = J$, and $\mathbf{r}(X_{jj}, [J|J]) = q^{\delta(j, J)}$. When $l > j$, Corollary 2.5 shows that $\mathbf{r}(X_{lj}, [L|J])$ is nonzero only when $l \in J$, $j \in L \setminus J$, and $L = J \setminus l \sqcup j$. In such cases, the exponent of $-q$ that appears in (2.7)(ii) is $-|(j, l) \cap L \cap J| = -|(j, l) \cap J|$. Therefore, the right hand sides of (3.1) and (3.2) agree. This establishes (3.2).

3.2. Application of the transpose automorphism. There are several ways to obtain a second commutation relation of a similar kind to (3.2). First, we could set $a = [I|J]$ and $b = X_{ij}$ in (1.7)(i) and proceed as above. Alternatively, we could apply the automorphism τ , the anti-automorphism τ_2 , or the isomorphism β of §1.6 to (3.2) itself. As we shall see in §3.4 below, the first three ways are equivalent, up to some relabelling. The use of β is discussed in §3.5.

Among the first three alternatives above, the most convenient choice is to apply the transpose automorphism τ to (3.2). If we do this, and then relabel the terms by interchanging $i \leftrightarrow j$ and $I \leftrightarrow J$, we obtain

$$(3.3) \quad \begin{aligned} q^{\delta(j,J)} X_{ij}[I|J] + (1 - \delta(j,J)) \widehat{q} \sum_{\substack{l \in J \\ l < j}} (-q)^{-|(l,j) \cap J|} X_{il}[I|J \setminus l \sqcup j] = \\ q^{\delta(i,I)} [I|J] X_{ij} + (1 - \delta(i,I)) \widehat{q} \sum_{\substack{l \in I \\ l > i}} (-q)^{-|(i,l) \cap I|} [I \setminus l \sqcup i | J] X_{lj}. \end{aligned}$$

3.3. Some known cases. We now compare some cases of (3.2) and (3.3) with the literature.

When $i \in I$ and $j \in J$, (3.2) and (3.3) both yield $qX_{ij}[I|J] = q[I|J]X_{ij}$, the well known fact that X_{ij} and $[I|J]$ commute in that case. (This is just the centrality of the quantum determinant in the subalgebra $k\langle X_{st} \mid s \in I, t \in J \rangle$.) If $i \in I$ and $j \notin J$, then (3.2) yields

$$(3.4) \quad qX_{ij}[I|J] = [I|J]X_{ij} + \widehat{q} \sum_{\substack{l \in J \\ l > j}} (-q)^{-|(j,l) \cap J|} [I|J \setminus l \sqcup j] X_{il}.$$

Multiply (3.4) by q^{-1} , and note that $q^{-1}(-q)^{-|(j,l) \cap J|} = -(-q)^{-|[j,l] \cap J|}$. With this modification, (3.4) recovers [5, Lemma A.1(b)] (which is the second equation of [25, Lemma 4.5.1(2)], rewritten in present notation). Similarly, consider the case that $i \notin I$ and $j \in J$. Then (3.3) yields

$$(3.5) \quad qX_{ij}[I|J] = [I|J]X_{ij} + \widehat{q} \sum_{\substack{l \in I \\ l > i}} (-q)^{-|(i,l) \cap I|} [I \setminus l \sqcup i | J] X_{lj}.$$

We again multiply by q^{-1} , and note that $q^{-1}(-q)^{-|(i,l) \cap I|} = -(-q)^{-|[i,l] \cap I|}$. Thus, (3.5) recovers [6, Lemma A.2(c), Equation (A.3)] (which is the second equation of [25, Lemma 4.5.1(4)] in present notation).

Finally, let us consider the case when $i \notin I$ and $j \notin J$. We may assume that $I \sqcup i = J \sqcup j = \{1, \dots, n\}$. If we write $\widehat{s} = \{1, \dots, n\} \setminus \{s\}$ for $s = 1, \dots, n$, then (3.2) yields

$$(3.6) \quad X_{ij}[\widehat{i} | \widehat{j}] + \widehat{q} \sum_{\substack{l \in I \\ l < i}} (-q)^{l+1-i} X_{lj}[\widehat{l} | \widehat{j}] = [\widehat{i} | \widehat{j}] X_{ij} + \widehat{q} \sum_{\substack{l \in J \\ l > j}} (-q)^{j+1-l} [\widehat{i} | \widehat{l}] X_{il}.$$

Multiplying (3.6) by q^{-1} and then interchanging $q \leftrightarrow q^{-1}$ recovers the fourth equation of [25, Lemma 5.1.2].

3.4. Remark. As mentioned above, (3.3) could also have been obtained by setting $a = [I|J]$ and $b = X_{ij}$ in (1.7)(i) and proceeding as with (3.2). In fact, interchanging any choice of a and b in (1.7)(i) has the same effect as applying τ , as follows.

First, apply τ to (1.7)(i), and use (1.12) for both a and b . This yields

$$(3.7) \quad \sum_{(a),(b)} \mathbf{r}(a_1, b_1) \tau(a)_1 \tau(b)_1 = \sum_{(a),(b)} \mathbf{r}(a_2, b_2) \tau(b)_2 \tau(a)_2.$$

Invoking Lemma 1.7, and setting $a' = \tau(a)$ and $b' = \tau(b)$, (3.7) becomes

$$(3.8) \quad \sum_{(a'),(b')} \mathbf{r}(b'_2, a'_2) a'_1 b'_1 = \sum_{(a'),(b')} \mathbf{r}(b'_1, a'_1) b'_2 a'_2.$$

Equation (3.8) is nothing but (1.7)(i) with a and b replaced by b' and a' , respectively.

Similarly, applying the anti-automorphism τ_2 to (1.7)(i) and relabelling again recovers (1.7)(i) with a and b interchanged.

3.5. Two further commutation relations. Each case of commutation relations for X_{ij} and $[I|J]$ derived in [25] has four subcases – two pairs in which one equation of each pair is obtained from the other by inserting a q -Laplace relation. Two commutation relations from each group of four correspond to our equations (3.2) and (3.3). It is more efficient to derive the remaining two by applying the isomorphism β of §1.6, as follows. For that purpose, set $A' = \mathcal{O}_{q^{-1}}(M_n(k))$, and recall the notation X'_{ij} and $[I|J]'$ for generators and quantum minors in A' .

First, consider the relation (3.2) in A' , but replace i, j, I, J by $\tilde{i}, \tilde{j}, \tilde{I}, \tilde{J}$, respectively. The result is

$$(3.9) \quad \begin{aligned} & q^{-\delta(\tilde{i}, \tilde{I})} X'_{\tilde{i}\tilde{j}} [\tilde{I}|\tilde{J}]' + (1 - \delta(\tilde{i}, \tilde{I})) (-\hat{q}) \sum_{\substack{\tilde{l} \in \tilde{I} \\ \tilde{l} < \tilde{i}}} (-q)^{|\tilde{l}, \tilde{i}) \cap \tilde{I}|} X'_{\tilde{i}\tilde{j}} [\tilde{I} \setminus \tilde{l} | \tilde{i} | \tilde{J}]' = \\ & q^{-\delta(\tilde{j}, \tilde{J})} [\tilde{I}|\tilde{J}]' X'_{\tilde{i}\tilde{j}} + (1 - \delta(\tilde{j}, \tilde{J})) (-\hat{q}) \sum_{\substack{\tilde{l} \in \tilde{J} \\ \tilde{l} > \tilde{j}}} (-q)^{|\tilde{j}, \tilde{l}) \cap \tilde{J}|} [\tilde{I}|\tilde{J} \setminus \tilde{l} | \tilde{j}]' X'_{\tilde{i}\tilde{j}}. \end{aligned}$$

Now set

$$\begin{aligned} \tilde{i} &= \omega_0(i) & \tilde{j} &= \omega_0(j) & \tilde{l} &= \omega_0(l) \\ \tilde{I} &= \omega_0(I) & \tilde{J} &= \omega_0(J) \end{aligned}$$

and apply β^{-1} to (3.9). This yields

$$(3.10) \quad \begin{aligned} & q^{-\delta(i, I)} X_{ij} [I|J] + (\delta(i, I) - 1) \hat{q} \sum_{\substack{l \in I \\ l > i}} (-q)^{|(i, l) \cap I|} X_{lj} [I \setminus l | i | J] = \\ & q^{-\delta(j, J)} [I|J] X_{ij} + (\delta(j, J) - 1) \hat{q} \sum_{\substack{l \in J \\ l < j}} (-q)^{|(l, j) \cap J|} [I|J \setminus l | j] X_{il}. \end{aligned}$$

Similarly, the relation (3.3) in A' can be written

$$(3.11) \quad \begin{aligned} q^{-\delta(\tilde{j}, \tilde{J})} X'_{\tilde{i}\tilde{j}} [\tilde{I} | \tilde{J}]' + (1 - \delta(\tilde{j}, \tilde{J})) (-\hat{q}) \sum_{\substack{\tilde{l} \in \tilde{J} \\ \tilde{l} < \tilde{j}}} (-q)^{|\tilde{l}, \tilde{j}) \cap \tilde{J}|} X'_{\tilde{i}\tilde{l}} [\tilde{I} | \tilde{J} \setminus \tilde{l} \sqcup \tilde{j}]' = \\ q^{-\delta(\tilde{i}, \tilde{I})} [\tilde{I} | \tilde{J}]' X'_{\tilde{i}\tilde{j}} + (1 - \delta(\tilde{i}, \tilde{I})) (-\hat{q}) \sum_{\substack{\tilde{l} \in \tilde{I} \\ \tilde{l} > \tilde{i}}} (-q)^{|\tilde{i}, \tilde{l}) \cap \tilde{I}|} [\tilde{I} \setminus \tilde{l} \sqcup \tilde{i} | \tilde{J}]' X'_{\tilde{l}\tilde{j}}. \end{aligned}$$

Applying β^{-1} to (3.11) as above, we conclude that

$$(3.12) \quad \begin{aligned} q^{-\delta(j, J)} X_{ij} [I | J] + (\delta(j, J) - 1) \hat{q} \sum_{\substack{l \in J \\ l > j}} (-q)^{|(j, l) \cap J|} X_{il} [I | J \setminus l \sqcup j] = \\ q^{-\delta(i, I)} [I | J] X_{ij} + (\delta(i, I) - 1) \hat{q} \sum_{\substack{l \in I \\ l < i}} (-q)^{|(l, i) \cap I|} [I \setminus l \sqcup i | J] X_{lj}. \end{aligned}$$

3.6. Quasicommutation. Elements $a, b \in A$ are said to *quasicommute* or *q-commute* provided they commute up to a power of q , that is, $ab = q^m ba$ for some integer m . The relations (1.6) say that two of the standard generators for A which have the same row (or column) indices must quasicommute, and it is natural to expect other instances of this in A . From the results above, we can recover the quasicommutation relations for quantum minors given by Krob and Leclerc [16]. These apply to certain quantum minors whose row (or column) index sets are disjoint. Cases allowing non-disjoint index sets were obtained by Leclerc and Zelevinsky by investing quantum Plücker relations [22, Lemmas 2.1–2.3]. Building on the results of [22], Scott determined exactly which pairs of quantum minors quasicommute, and calculated the corresponding relations [27, Theorems 1,2]. We recover some other cases of his results in Corollary 5.5 below.

First, consider X_{ij} and $[M | N]$, with $i \in M$. If $j < \min(N)$, then either (3.3) or (3.10) implies that $X_{ij}[M | N] = q[M | N]X_{ij}$, while if $j > \max(N)$, then by either (3.2) or (3.12), $X_{ij}[M | N] = q^{-1}[M | N]X_{ij}$. Of course, if $j \in N$, then $X_{ij}[M | N] = [M | N]X_{ij}$.

Now suppose that $I \subseteq M$ and that J and N are *separated* in the following sense: there is a partition $J = J' \sqcup J''$ such that

$$\max(J') < \min(N) \leq \max(N) < \min(J'').$$

Each of the generators $X_{i_{\sigma(t)}, j_t}$ occurring in (1.10) quasicommutates with $[M | N]$ as in the previous paragraph, whence

$$X_{i_{\sigma(1)}, j_1} X_{i_{\sigma(2)}, j_2} \cdots X_{i_{\sigma(t)}, j_t} [M | N] = q^{|J'| - |J''|} [M | N] X_{i_{\sigma(1)}, j_1} X_{i_{\sigma(2)}, j_2} \cdots X_{i_{\sigma(t)}, j_t}$$

for all $\sigma \in S_t$. Consequently,

$$(3.13) \quad [I | J][M | N] = q^{|J'| - |J''|} [M | N][I | J]$$

under the present hypotheses. This recovers [16, Lemma 3.7] (after interchanging q and q^{-1}). In fact, (3.13) holds when $I \subseteq M$ and J and N are *weakly separated*

in the sense of [22], meaning that there is a partition $J \setminus N = J' \sqcup J''$ such that $\max(J') < \min(N \setminus J) \leq \max(N \setminus J) < \min(J'')$ [22, Lemma 2.1].

Applying τ to (3.13) and relabelling, we find that

$$(3.14) \quad [I|J][M|N] = q^{|I'| - |I''|} [M|N][I|J]$$

when $J \subseteq N$ and $I = I' \sqcup I''$ with $\max(I') < \min(M) \leq \max(M) < \min(I'')$.

4. COMPUTATION OF $\mathbf{r}([I|J], [M|N])$

Throughout this section, let I, J, M, N denote index sets contained in the interval $\{1, \dots, n\}$, with $|I| = |J|$ and $|M| = |N|$. Our goal is to develop a formula for $\mathbf{r}([I|J], [M|N])$.

4.1. Lemma. *If $\mathbf{r}([I|J], [M|N]) \neq 0$, then $I \cap M = J \cap N$ and $I \cup M = J \cup N$.*

Proof. We induct on $|I|$, starting with the case $[I|J] = X_{ij}$. If $i = j$, Lemma 2.1 implies that $M = N$, and the conclusion is clear. If $i \neq j$, then $i > j$ by Lemma 2.2, whence Corollary 2.5 implies that $i \in N, j \in M$, and $M \setminus j = N \setminus i$. Consequently, $I \cap M = J \cap N = \emptyset$ and $I \cup M = J \cup N$.

Now suppose that $|I| \geq 2$. If $I = J$, then Proposition 2.6 implies that $M = N$, and we are done. Hence, we may assume that $I \neq J$. Since $|I| = |J|$, there must exist an element $j \in J \setminus I$. Set $J = J_1 \sqcup J_2$ with $J_1 = \{j\}$ and $J_2 = J \setminus j$, and write (1.19) in the form

$$(4.1) \quad \pm q^{\bullet} \mathbf{r}([I|J], [M|N]) = \sum_{i \in I} \sum_L \pm q^{\bullet} \mathbf{r}(X_{ij}, [M|L]) \mathbf{r}([I \setminus i | J \setminus j], [L|N]).$$

Since $\mathbf{r}([I|J], [M|N]) \neq 0$, (4.1) implies that

$$(4.2) \quad \mathbf{r}(X_{ij}, [M|L]) \mathbf{r}([I \setminus i | J \setminus j], [L|N]) \neq 0$$

for some $i \in I$ and some L .

Note that $i \neq j$, because $j \notin I$. Equation (4.2) and Lemma 2.2 now show that $i > j$, and then Corollary 2.5 implies that $i \in L, j \in M$, and $L \setminus i = M \setminus j$. Consequently, $i \notin M$ and $j \notin L$, while $L = (L \cap M) \sqcup i$ and $M = (L \cap M) \sqcup j$. Since the second factor of (4.2) is nonzero, our induction implies that $(I \setminus i) \cap L = (J \setminus j) \cap N$ and $(I \setminus i) \cup L = (J \setminus j) \cup N$. Now

$$I \cup (L \cap M) = (I \setminus i) \cup i \cup (L \cap M) = (I \setminus i) \cup L = (J \setminus j) \cup N,$$

and so $I \cup M = I \cup (L \cap M) \cup j = J \cup N$. Since $j \notin I \cup L$, we see from the equation $(I \setminus i) \cup L = (J \setminus j) \cup N$ that $j \notin N$. Consequently,

$$I \cap M = I \cap (M \setminus j) = I \cap (L \setminus i) = (I \setminus i) \cap L = (J \setminus j) \cap N = J \cap N.$$

This establishes the induction step. \square

4.2. Lemma. *Assume that $I \cap M = J \cap N$ and $I \cup M = J \cup N$.*

(a) $I \setminus J = N \setminus M$ and $J \setminus I = M \setminus N$.

(b) $\mathbf{r}([I|J], [M|N]) = q^{|I \cap M|} (-q)^{\ell(I; J \cap N) - \ell(J; I \cap M)} \mathbf{r}([I \setminus M | J \setminus N], [M|N])$.

Proof. (a) This follows easily from the hypotheses.

(b) Write $J = J_1 \sqcup J_2$ with $J_1 = J \setminus N$ and $J_2 = J \cap N = I \cap M$, and recall equation (1.19). We focus first on the term on the right hand side of (1.19) with $I_2 = J_2$ and $L = N$, in which case $I_1 = I \setminus M$. For this term, we have

$$(4.3) \quad \begin{aligned} (-q)^{\ell(I_1; I_2)} \mathbf{r}([I_1 | J_1], [M | L]) \mathbf{r}([I_2 | J_2], [L | N]) = \\ (-q)^{\ell(I \setminus M; J \cap N)} q^{|I \cap M|} \mathbf{r}([I \setminus M | J \setminus N], [M | N]), \end{aligned}$$

in view of Proposition 2.6. We claim that all other terms on the right hand side of (1.19) vanish.

Suppose that $\mathbf{r}([I_1 | J_1], [M | L]) \mathbf{r}([I_2 | J_2], [L | N]) \neq 0$ for some I_1, I_2, L . Lemma 4.1 implies that $I_2 \cap L = J_2 \cap N = J_2$, and then because $|I_2| = |J_2|$, we must have $I_2 = J_2$. Consequently, Proposition 2.6 implies that $L = N$, verifying the claim. Equations (1.19) and (4.3) thus yield

$$(4.4) \quad \begin{aligned} (-q)^{\ell(J \setminus N; I \cap M)} \mathbf{r}([I | J], [M | N]) = \\ (-q)^{\ell(I \setminus M; J \cap N)} q^{|I \cap M|} \mathbf{r}([I \setminus M | J \setminus N], [M | N]). \end{aligned}$$

Finally, we have

$$\begin{aligned} \ell(I; J \cap N) &= \ell(I \setminus M; J \cap N) + \ell(I \cap M; J \cap N) \\ \ell(J; I \cap M) &= \ell(J \setminus N; I \cap M) + \ell(J \cap N; I \cap M), \end{aligned}$$

and since $I \cap M = J \cap N$, we obtain

$$(4.5) \quad \ell(I \setminus M; J \cap N) - \ell(J \setminus N; I \cap M) = \ell(I; J \cap N) - \ell(J; I \cap M).$$

Part (b) follows from (4.4) and (4.5). \square

4.3. Lemma. *Assume that $I \cap M = J \cap N = \emptyset$ and $I \cup M = J \cup N$. Then*

$$\mathbf{r}([I | J], [M | N]) = (-q)^{\ell(I \cup N; I \setminus J) - \ell(J \cup M; J \setminus I)} \mathbf{r}([I \setminus J | J \setminus I], [M \setminus N | N \setminus M]).$$

Proof. Write $J = J_1 \sqcup J_2$ with $J_1 = I \cap J$ and $J_2 = J \setminus I$, and recall (1.19). Consider the term with $I_1 = J_1$ and $L = M$, in which case $I_2 = I \setminus J$. Since $I_1 \cap M = \emptyset$, Proposition 2.6 implies that $\mathbf{r}([I_1 | J_1], [M | L]) = 1$. Thus, for this term of (1.19), we have

$$(4.6) \quad \begin{aligned} (-q)^{\ell(I_1; I_2)} \mathbf{r}([I_1 | J_1], [M | L]) \mathbf{r}([I_2 | J_2], [L | N]) = \\ (-q)^{\ell(I \cap J; I \setminus J)} \mathbf{r}([I \setminus J | J \setminus I], [M | N]). \end{aligned}$$

We next claim that all other terms on the right hand side of (1.19) vanish. Hence, suppose that $\mathbf{r}([I_1 | J_1], [M | L]) \mathbf{r}([I_2 | J_2], [L | N]) \neq 0$ for some I_1, I_2, L . Lemma 4.1 implies that $I_2 \cap L = J_2 \cap N = \emptyset$ and $I_2 \cup L = J_2 \cup N = (J \setminus I) \cup N$, from which it follows that $I_2 = N \setminus L$. Now $I_2 \cap J \subseteq N \cap J = \emptyset$, and so $I_2 \subseteq I \setminus J$. Since also

$$|I_2| = |J_2| = |J \setminus I| = |I \setminus J|,$$

we must have $I_2 = I \setminus J$. Consequently, $I_1 = J_1$, and then Proposition 2.6 implies that $L = M$. This verifies the claim. As a result, (1.19) and (4.6) combine to yield

$$(4.7) \quad \mathbf{r}([I | J], [M | N]) = (-q)^{\ell(I \cap J; I \setminus J) - \ell(I \cap J; J \setminus I)} \mathbf{r}([I \setminus J | J \setminus I], [M | N]).$$

Note that $(I \setminus J) \cap M = (J \setminus I) \cap N = \emptyset$ and $(I \setminus J) \cup M = M \cup N = (J \setminus I) \cup N$. Hence, (4.7) also holds with I, J, M, N replaced by $N, M, J \setminus I, I \setminus J$, respectively. That is,

$$(4.8) \quad \mathbf{r}([N|M], [J \setminus I | I \setminus J]) = (-q)^{\ell(N \cap M; N \setminus M) - \ell(N \cap M; M \setminus N)} \mathbf{r}([N \setminus M | M \setminus N], [J \setminus I | I \setminus J]).$$

In view of Lemma 1.7, (4.8) can be rewritten as

$$(4.9) \quad \mathbf{r}([I \setminus J | J \setminus I], [M|N]) = (-q)^{\ell(N \cap M; N \setminus M) - \ell(N \cap M; M \setminus N)} \mathbf{r}([I \setminus J | J \setminus I], [M \setminus N | N \setminus M]).$$

Combining (4.7) and (4.9), we obtain

$$(4.10) \quad \mathbf{r}([I|J], [M|N]) = (-q)^\lambda \mathbf{r}([I \setminus J | J \setminus I], [M \setminus N | N \setminus M]),$$

where (recalling Lemma 4.2(a))

$$(4.11) \quad \begin{aligned} \lambda &= \ell(I \cap J; I \setminus J) - \ell(I \cap J; J \setminus I) + \ell(N \cap M; N \setminus M) - \ell(N \cap M; M \setminus N) \\ &= \ell((I \cap J) \sqcup (M \cap N); I \setminus J) - \ell((I \cap J) \sqcup (M \cap N); J \setminus I). \end{aligned}$$

Next, observe that

$$I \cup N = (I \setminus J) \sqcup (I \cap J) \sqcup (M \cap N) \quad J \cup M = (J \setminus I) \sqcup (I \cap J) \sqcup (M \cap N).$$

Because $|I \setminus J| = |J \setminus I|$, we have $\ell(I \setminus J; I \setminus J) = \ell(J \setminus I; J \setminus I)$, and therefore

$$(4.12) \quad \lambda = \ell(I \cup N; I \setminus J) - \ell(J \cup M; J \setminus I).$$

Equations (4.10) and (4.12) establish the lemma. \square

In view of Lemmas 4.1–4.3, it only remains to calculate $\mathbf{r}([I|J], [M|N])$ in case

$$(I \cup N) \cap (J \cup M) = \emptyset \quad I \cup M = J \cup N,$$

whence $I = N$ and $J = M$. Further, because of Corollary 2.3, we may assume that $I > J$. In these cases, certain sums of powers of $-q$ appear in $\mathbf{r}([I|J], [M|N])$, and we introduce the following notation to deal with them.

4.4. Definition of $\xi_q(I; J)$. Recall that for $d \in \mathbb{N}$, the $(-q)$ -integer $[d]_{-q}$ is given by

$$\begin{aligned} [d]_{-q} &= \frac{(-q)^d - (-q)^{-d}}{(-q) - (-q)^{-1}} = (-q)^{d-1} + (-q)^{d-3} + \cdots + (-q)^{-(d-1)} \\ &= (-q)^{1-d} (1 + q^2 + q^4 + \cdots + q^{2d-2}). \end{aligned}$$

Hence, $1 + q^2 + q^4 + \cdots + q^{2d-2} = (-q)^{d-1} [d]_{-q}$.

We next define a scalar $\xi_q(I; J)$, for index sets $I \geq J$, as follows. First set $m = |I|$ and write $I = \{r_1 < \cdots < r_m\}$. Then set $d_l = |[1, r_l] \cap J| - l + 1$ for $l = 1, \dots, m$, noting that $d_l \geq 1$ because $J \leq I$. Finally, define

$$\xi_q(I; J) = [d_1]_{-q} [d_2]_{-q} \cdots [d_m]_{-q},$$

with the convention that $\xi_q(\emptyset; \emptyset) = 1$. When $I \cap J = \emptyset$, as in the next lemma, each $d_l = \ell(r_l; J) - l + 1$. Note that $[d]_{-q^{-1}} = [d]_{-q}$ for all $d \in \mathbb{N}$, whence $\xi_{q^{-1}}(I; J) = \xi_q(I; J)$.

4.5. Lemma. *If $I > J$ and $I \cap J = \emptyset$, then*

$$(4.13) \quad \mathbf{r}([I|J], [J|I]) = \widehat{q}^{|I|} (-q)^{\ell(J;I) - \ell(I;I)} \xi_q(I; J).$$

Proof. Set $m = |I| = |J|$, write $I = \{r_1 < \dots < r_m\}$, and set $d_l = \ell(r_l; J) - l + 1$ for $l = 1, \dots, m$ as in §4.4.

We proceed by induction on m . If $m = 1$, then $J = \{j\}$ for some $j < r_1$, whence $\ell(J; I) = \ell(I; I) = 0$. Moreover, $d_1 = 1$ and so $\xi_q(I; J) = 1$. By (1.9), $\mathbf{r}([I|J], [J|I]) = \mathbf{r}(X_{r_1 j}, X_{j r_1}) = \widehat{q}$, which verifies (4.13) in this case.

Now suppose that $m > 1$. Write $I = I_1 \sqcup I_2$ with $I_1 = \{r_1\}$ and $I_2 = \{r_2, \dots, r_m\}$. Since $\ell(I_1; I_2) = 0$, equation (1.17) implies that

$$[I|J] = \sum_{j \in J} (-q)^{\ell(j; J \setminus j)} X_{r_1 j} [I_2 | J \setminus j].$$

Applying (1.7)(ii), we obtain

$$(4.14) \quad \mathbf{r}([I|J], [J|I]) = \sum_{j \in J} \sum_L (-q)^{|[1, j] \cap J|} \mathbf{r}(X_{r_1 j}, [J|L]) \mathbf{r}([I_2 | J \setminus j], [L|I]).$$

According to Lemma 2.2 and Corollary 2.5, a nonzero term can occur on the right hand side of (4.14) only if $r_1 > j$ and $r_1 \in L$, as well as $J \setminus j = L \setminus r_1$, in which case

$$\mathbf{r}(X_{r_1 j}, [J|L]) = \widehat{q} (-q)^{|[1, j] \cap J| - |[1, r_1] \cap L|}.$$

Now $|[1, r_1] \cap L| = |[1, r_1] \cap (L \setminus r_1)| = |[1, r_1] \cap (J \setminus j)| = d_1 - 1$, and so

$$(4.15) \quad \mathbf{r}(X_{r_1 j}, [J|L]) = \widehat{q} (-q)^{1 + |[1, j] \cap J| - d_1}.$$

Next, note that $L = J \setminus j \sqcup r_1$, whence $L \cap I = \{r_1\}$. Consequently, $I_2 \cap L = (J \setminus j) \cap I = \emptyset$ and $I_2 \cup L = I \cup L = (J \setminus j) \cup I$. Lemma 4.3 now implies that

$$(4.16) \quad \mathbf{r}([I_2 | J \setminus j], [L|I]) = (-q)^\lambda \mathbf{r}([I_2 | J \setminus j], [J \setminus j | I_2])$$

where

$$(4.17) \quad \begin{aligned} \lambda &= \ell(I; I_2) - \ell(L; J \setminus j) \\ &= \ell(I_2; I_2) - \ell(J \setminus j; J \setminus j) + \ell(r_1; I_2) - \ell(r_1; J \setminus j) = -d_1 + 1. \end{aligned}$$

Combining equations (4.14)–(4.17), we obtain

$$(4.18) \quad \mathbf{r}([I|J], [J|I]) = \widehat{q} \sum_{\substack{j \in J \\ j < r_1}} (-q)^{2 + 2|[1, j] \cap J| - 2d_1} \mathbf{r}([I_2 | J \setminus j], [J \setminus j | I_2]).$$

It remains to compute $\mathbf{r}([I_2 | J \setminus j], [J \setminus j | I_2])$ for $j \in J$ with $j < r_1$. Observe that $I_2 > J \setminus j$ for any such j , so that our induction hypothesis will apply. Now

$$\begin{aligned} \ell(J \setminus j; I_2) &= \ell(J; I_2) = \ell(J; I) - \ell(J; r_1) = \ell(J; I) - m + d_1 \\ \ell(I_2; I_2) &= \ell(I; I_2) = \ell(I; I) - m + 1, \end{aligned}$$

whence $\ell(J \setminus j; I_2) - \ell(I_2; I_2) = \ell(J; I) - \ell(I; I) + d_1 - 1$. For $l = 1, \dots, m - 1$, observe that

$$\ell(r_{l+1}; J \setminus j) - l + 1 = \ell(r_{l+1}; J) - l = d_{l+1},$$

and consequently $\xi_q(I_2; J \setminus j) = [d_2]_{-q} [d_3]_{-q} \cdots [d_m]_{-q}$. Thus, our induction hypothesis implies that

$$(4.19) \quad \mathbf{r}([I_2 | J \setminus j], [J \setminus j | I_2]) = \widehat{q}^{m-1} (-q)^{\ell(J; I) - \ell(I; I) + d_1 - 1} [d_2]_{-q} [d_3]_{-q} \cdots [d_m]_{-q}.$$

Inserting (4.19) in (4.18), we obtain

$$(4.20) \quad \mathbf{r}([I | J], [J | I]) = \widehat{q}^m (-q)^{\ell(J; I) - \ell(I; I) + 1 - d_1} [d_2]_{-q} [d_3]_{-q} \cdots [d_m]_{-q} \sum_{\substack{j \in J \\ j < r_1}} q^{2|[1, j] \cap J|}.$$

The summation appearing in (4.20) is just $\sum_{t=1}^{d_1} q^{2(t-1)} = (-q)^{d_1-1} [d_1]_{-q}$, whence

$$(4.21) \quad [d_2]_{-q} [d_3]_{-q} \cdots [d_m]_{-q} \sum_{\substack{j \in J \\ j < r_1}} q^{2|[1, j] \cap J|} = (-q)^{d_1-1} \xi_q(I; J).$$

Equations (4.20) and (4.21) establish (4.13), completing the induction step. \square

4.6. Theorem. *Let $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$.*

(a) *If $\mathbf{r}([I | J], [M | N]) \neq 0$, then*

$$(4.22) \quad I \geq J; \quad I \cap M = J \cap N; \quad I \cup M = J \cup N.$$

(b) *If conditions (4.22) hold, then*

$$(4.23) \quad \mathbf{r}([I | J], [M | N]) = q^{|I \cap M|} \widehat{q}^{|I \setminus J|} (-q)^\lambda \xi_q(I \setminus J; J \setminus I), \quad \text{where} \\ \lambda = \ell((J \setminus N) \cup (M \setminus I); I \setminus J) - \ell((J \setminus N) \cup (M \setminus I); J \setminus I).$$

Proof. (a) Corollary 2.3 and Lemma 4.1.

(b) Recall from Lemma 4.2 that $I \setminus J = N \setminus M$ and $J \setminus I = M \setminus N$. If $I = J$, then we must have $M = N$. In this case, $\mathbf{r}([I | J], [M | N]) = q^{|I \cap M|}$ by Proposition 2.6, and we are done. Now assume that $I \neq J$, and note that $I \setminus J > J \setminus I$. We shall need the observations that

$$\begin{aligned} (I \setminus M) \cup N &= I \cup N & (J \setminus N) \cup M &= J \cup M \\ (I \setminus M) \setminus (J \setminus N) &= I \setminus J & (J \setminus N) \setminus (I \setminus M) &= J \setminus I. \end{aligned}$$

Applying, successively, Lemmas 4.2, 4.3, and 4.5, we obtain

$$(4.24) \quad \mathbf{r}([I | J], [M | N]) = q^{|I \cap M|} \widehat{q}^{|I \setminus J|} (-q)^\lambda \xi_q(I \setminus J; J \setminus I),$$

where

$$\begin{aligned} \lambda &= \ell(I; J \cap N) - \ell(J; I \cap M) + \ell(I \cup N; I \setminus J) \\ &\quad - \ell(J \cup M; J \setminus I) + \ell(J \setminus I; I \setminus J) - \ell(I \setminus J; I \setminus J). \end{aligned}$$

Observe that $(I \cup N) \sqcup (J \setminus I) = J \cup N = I \cup M = (J \cup M) \sqcup (I \setminus J)$, whence

$$(4.25) \quad \ell(I \cup N; I \setminus J) - \ell(J \cup M; J \setminus I) + \ell(J \setminus I; I \setminus J) - \ell(I \setminus J; I \setminus J) = \\ \ell(J \cup M; I \setminus J) - \ell(J \cup M; J \setminus I).$$

Next, observe that $I \setminus N = J \setminus M$ and $N \setminus I = M \setminus J$. Moreover,

$$\begin{aligned} I \cup M &= I \cup M \cup N = I \sqcup (N \setminus I) \sqcup (M \setminus N) \\ J \cup N &= J \cup M \cup N = J \sqcup (M \setminus J) \sqcup (N \setminus M), \end{aligned}$$

and consequently

$$\begin{aligned}\ell(I; J \cap N) + \ell(N \setminus I; J \cap N) + \ell(M \setminus N; J \cap N) &= \ell(I \cup M; J \cap N) \\ \ell(J; I \cap M) + \ell(M \setminus J; I \cap M) + \ell(N \setminus M; I \cap M) &= \ell(J \cup N; I \cap M).\end{aligned}$$

It follows that

$$\begin{aligned}(4.26) \quad \ell(I; J \cap N) - \ell(J; I \cap M) &= \ell(N \setminus M; I \cap M) - \ell(M \setminus N; J \cap N) \\ &= |N \setminus M| \cdot |I \cap M| - \ell(I \cap M; N \setminus M) \\ &\quad - |M \setminus N| \cdot |J \cap N| + \ell(J \cap N; M \setminus N) \\ &= \ell(I \cap M; J \setminus I) - \ell(I \cap M; I \setminus J).\end{aligned}$$

Finally, since

$$(J \cup M) \setminus (I \cap M) = (J \setminus (J \cap N)) \cup (M \setminus (I \cap M)) = (J \setminus N) \cup (M \setminus I),$$

we conclude from (4.25) and (4.26) that

$$(4.27) \quad \lambda = \ell((J \setminus N) \cup (M \setminus I); I \setminus J) - \ell((J \setminus N) \cup (M \setminus I); J \setminus I).$$

In view of (4.24) and (4.27), the theorem is proved. \square

4.7. Example. Let $[I|J] = [45678|12345]$ and $[M|N] = [123459|456789]$, where we have omitted commas between elements of the index sets. It is clear that $I \geq J$; moreover, $I \cap M = \{4, 5\} = J \cap N$ and $I \cup M = \{1, \dots, 9\} = J \cup N$. Hence, conditions (4.22) hold. Now $I \setminus J = \{6, 7, 8\}$ and $J \setminus I = \{1, 2, 3\}$, while $(J \setminus N) \cup (M \setminus I) = \{1, 2, 3, 9\}$, whence

$$\ell((J \setminus N) \cup (M \setminus I); I \setminus J) - \ell((J \setminus N) \cup (M \setminus I); J \setminus I) = 3 - 6 = -3.$$

Since all the elements of $I \setminus J$ are greater than all the elements of $J \setminus I$, we have

$$\xi_q(I \setminus J; J \setminus I) = [3]_{-q}[2]_{-q}[1]_{-q} = (q^2 + 1 + q^{-2})(-q - q^{-1}).$$

Thus, we conclude from (4.23) that

$$\mathbf{r}([I|J], [M|N]) = q^2 \widehat{q}^3 (-q)^{-3} (q^2 + 1 + q^{-2})(-q - q^{-1}).$$

5. GENERAL COMMUTATION RELATIONS

Now that we have formulas for the value of the braiding form \mathbf{r} on pairs of quantum minors, commutation relations follow readily from property (1.7)(i). The following notation for certain index sets and exponents will be helpful in displaying the results. Recall the quantities $\ell(-; -)$ and $\xi_q(-; -)$ from §1.8 and §4.4.

5.1. Definitions of index sets $\{<X||Y\}$ and $\{>X||Y\}$ and numerical quantities $\mathcal{L}(S, X, Y)$ and $\mathcal{L}^\natural(T, X, Y)$. For any subsets X and Y of $\{1, \dots, n\}$, define

$$(5.1) \quad \begin{aligned}\{<X||Y\} &= \{S \subseteq X \cup Y \mid X \cap Y \subseteq S; |S| = |X|; S < X\} \\ \{>X||Y\} &= \{T \subseteq X \cup Y \mid X \cap Y \subseteq T; |T| = |X|; T > X\}.\end{aligned}$$

In Section 6, we shall need index sets $\{\leq X||Y\}$ and $\{\geq X||Y\}$, defined in the same manner. For any set $S \subseteq X \cup Y$ such that $X \cap Y \subseteq S$, set

$$(5.2) \quad S^\natural = S_{X,Y}^\natural = (X \cap Y) \sqcup ((X \cup Y) \setminus S).$$

Note that if $S \in \{<X||Y\}$ or $S \in \{>X||Y\}$, then $|S^\natural| = |Y|$. Finally, for $S \in \{<X||Y\}$ and $T \in \{>X||Y\}$, define

$$(5.3) \quad \begin{aligned} \mathcal{L}(S, X, Y) &= \ell((S \setminus S^\natural) \cup (Y \setminus X); X \setminus S) - \ell((S \setminus S^\natural) \cup (Y \setminus X); S \setminus X) \\ \mathcal{L}^\natural(T, X, Y) &= \ell((T^\natural \setminus T) \cup (X \setminus Y); T \setminus X) - \ell((T^\natural \setminus T) \cup (X \setminus Y); X \setminus T). \end{aligned}$$

For example, suppose that $X = \{2, 3, 4, 6\}$ and $Y = \{1, 3, 5\}$. Then $\{<X||Y\}$ consists of those 4-element subsets S of $\{1, \dots, 6\}$ such that $3 \in S$ and $S < X$. There are six such sets:

$$\{1, 2, 3, 4\}, \quad \{1, 2, 3, 5\}, \quad \{1, 2, 3, 6\}, \quad \{1, 3, 4, 5\}, \quad \{1, 3, 4, 6\}, \quad \{2, 3, 4, 5\}.$$

Similarly, $\{>X||Y\}$ consists of those 4-element subsets T of $\{1, \dots, 6\}$ such that $3 \in T$ and $T > X$. There are two: $\{3, 4, 5, 6\}$ and $\{2, 3, 5, 6\}$. Finally, consider the set $S = \{1, 2, 3, 4\} \in \{<X||Y\}$. Then $S^\natural = \{3, 5, 6\}$, and so

$$\mathcal{L}(S, X, Y) = \ell(\{1, 2, 4, 5\}; \{6\}) - \ell(\{1, 2, 4, 5\}; \{1\}) = 0 - 3.$$

5.2. Theorem. *Let $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$. Then*

$$(5.4) \quad \begin{aligned} q^{|I \cap M|} [I|J][M|N] + q^{|I \cap M|} \sum_{S \in \{<I||M\}} \lambda_S [S|J][S^\natural|N] = \\ q^{|J \cap N|} [M|N][I|J] + q^{|J \cap N|} \sum_{T \in \{>J||N\}} \mu_T [M|T^\natural][I|T], \end{aligned}$$

where

$$(5.5) \quad \begin{aligned} \lambda_S &= \widehat{q}^{|I \setminus S|} (-q)^{\mathcal{L}(S, I, M)} \xi_q(I \setminus S; S \setminus I) \\ \mu_T &= \widehat{q}^{|T \setminus J|} (-q)^{\mathcal{L}^\natural(T, J, N)} \xi_q(T \setminus J; J \setminus T) \end{aligned}$$

for $S \in \{<I||M\}$ and $T \in \{>J||N\}$.

Proof. Taking $a = [I|J]$ and $b = [M|N]$ in (1.7)(i), we obtain

$$(5.6) \quad \sum_{\substack{|S|=|I| \\ |S'|=|M|}} \mathbf{r}([I|S], [M|S']) [S|J][S'|N] = \sum_{\substack{|T|=|J| \\ |T'|=|N|}} \mathbf{r}([T|J], [T'|N]) [M|T'] [I|T].$$

In view of Corollary 2.3 and Lemma 4.1, the left hand summation in (5.6) can be restricted to index sets S and S' such that

$$(5.7) \quad \begin{aligned} |S| &= |I| & I &\geq S \\ I \cap M &= S \cap S' & I \cup M &= S \cup S'. \end{aligned}$$

Proposition 2.6 shows that the coefficient of the term with $S = I$ and $S' = M$ is $q^{|I \cap M|}$, and that the terms with $S = I$ and $S' \neq M$ vanish.

The index sets S and S' such that $S \neq I$ and (5.7) hold are precisely those for which $S \in \{<I||M\}$ and $S' = S^\natural$. For these index sets, Theorem 4.6 shows that

$$\mathbf{r}([I|S], [M|S']) = q^{|I \cap M|} \lambda_S.$$

Thus, the left hand side of (5.6) reduces to the left hand side of (5.4).

Similarly, the right hand side of (5.6) reduces to the right hand side of (5.4), and the theorem is proved. \square

5.3. Corollary. *Let $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$. Then*

$$(5.8) \quad q^{|J \cap N|} [I|J][M|N] + q^{|J \cap N|} \sum_{S \in \{<J||N\}} \lambda_S [I|S][M|S^\natural] = \\ q^{|I \cap M|} [M|N][I|J] + q^{|I \cap M|} \sum_{T \in \{>I||M\}} \mu_T [T^\natural|N][T|J],$$

where

$$(5.9) \quad \lambda_S = \widehat{q}^{|J \setminus S|} (-q)^{\mathcal{L}(S, J, N)} \xi_q(J \setminus S; S \setminus J) \\ \mu_T = \widehat{q}^{|T \setminus I|} (-q)^{\mathcal{L}^\natural(T, I, M)} \xi_q(T \setminus I; I \setminus T)$$

for $S \in \{<J||N\}$ and $T \in \{>I||M\}$.

Proof. Interchange the index sets in the statement of Theorem 5.2 as follows: $I \leftrightarrow J$ and $M \leftrightarrow N$. Then apply the automorphism τ to the resulting version of (5.4) to obtain (5.8) (recall (1.13)).

This corollary can also be obtained from Theorem 5.2 by interchanging $I \leftrightarrow M$ and $J \leftrightarrow N$, in which case one should also interchange $S \leftrightarrow T^\natural$ and $T \leftrightarrow S^\natural$. \square

5.4. Further quasicommution. In particular, Theorem 5.2 yields quasicommution relations of the form $q^{|I \cap M|} [I|J][M|N] = q^{|J \cap N|} [M|N][I|J]$ in cases where the index sets $\{<I||M\}$ and $\{>J||N\}$ are empty. This occurs, for instance, if either $[I|J] = [1, \dots, r|n+1-r, \dots, n]$ or $[M|N] = [n+1-r, \dots, n|1, \dots, r]$, recovering the well known fact that the northeasternmost and southwesternmost quantum minors are normal elements of A . Moreover,

$$(5.10) \quad [1, \dots, r|J][M|1, \dots, s] = q^{|J \cap [1, s]| - |[1, r] \cap M|} [M|1, \dots, s][1, \dots, r|J],$$

which is part of [10, Proposition 1.1] (with q^2 replaced by q). Also, (5.10) immediately implies the type A case of [1, Equation (10.3)].

We record the general quasicommution relations of the above type in the corollary below. Part (a) recovers one case of [27, Theorem 2]. It does not seem, however, that the relations (3.13) and (3.14) follow directly from equations such as (5.4) or (5.8).

5.5. Corollary. *Let $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$.*

(a) *If $\max(M \setminus I) < \min(I \setminus M)$ and $\max(J \setminus N) < \min(N \setminus J)$, then*

$$(5.11) \quad [I|J][M|N] = q^{|I \cap M| - |J \cap N|} [M|N][I|J].$$

(b) *If $\max(I \setminus M) < \min(M \setminus I)$ and $\max(N \setminus J) < \min(J \setminus N)$, then*

$$(5.12) \quad [I|J][M|N] = q^{|J \cap N| - |I \cap M|} [M|N][I|J].$$

Proof. (a) If $S \in \{<J||N\}$, then $S \setminus (J \cap N) < J \setminus N$, whence

$$\max(S \setminus (J \cap N)) \leq \max(J \setminus N) < \min(N \setminus J).$$

But then S is disjoint from $N \setminus J$. Since $J \cap N \subseteq S \subseteq J \cup N$ and $|S| = |J|$, this forces $S = J$, which is ruled out by the assumption $S < J$. Thus, $\{<J||N\} = \emptyset$. Similarly, $\{>I||M\} = \emptyset$, and thus (5.11) follows from (5.8).

(b) Interchange $I \leftrightarrow M$ and $J \leftrightarrow N$, and apply part (a). \square

5.6. Example. [$n = 6$] Let $J = N = \{1, 2, 3\}$, and take $I = \{1, 4, 5\}$ and $M = \{2, 3, 6\}$. We first apply Theorem 5.2. Note that $\{>J||N\}$ is empty because $J = N$. For $S \in \{<I||M\}$, we make the following calculations, where commas have been deleted for the sake of abbreviation (for instance, $\{123\}$ stands for the index set $\{1, 2, 3\}$).

S	$\{123\}$	$\{124\}$	$\{125\}$	$\{134\}$	$\{135\}$
S^\natural	$\{456\}$	$\{356\}$	$\{346\}$	$\{256\}$	$\{246\}$
$I \setminus S$	$\{45\}$	$\{5\}$	$\{4\}$	$\{5\}$	$\{4\}$
$S \setminus I$	$\{23\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{3\}$
$(S \setminus S^\natural) \cup (M \setminus I)$	$\{1236\}$	$\{12346\}$	$\{12356\}$	$\{12346\}$	$\{12356\}$
$\ell((S \setminus S^\natural) \cup (M \setminus I); I \setminus S)$	2	1	2	1	2
$\ell((S \setminus S^\natural) \cup (M \setminus I); S \setminus I)$	3	3	3	2	2
$\mathcal{L}(S, I, M)$	-1	-2	-1	-1	0
$\xi_q(I \setminus S; S \setminus I)$	$-q - q^{-1}$	1	1	1	1

Consequently, Theorem 5.2 implies that

$$\begin{aligned}
 q^3[236|J][145|J] &= [145|J][236|J] + \widehat{q}^2(-q)^{-1}(-q - q^{-1})[123|J][456|J] \\
 (5.13) \quad &+ \widehat{q}(-q)^{-2}[124|J][356|J] + \widehat{q}(-q)^{-1}[125|J][346|J] \\
 &+ \widehat{q}(-q)^{-1}[134|J][256|J] + \widehat{q}[135|J][246|J].
 \end{aligned}$$

The relation (5.13) matches the one calculated by Fioresi in [3, Example 2.22] (cf. the first display on page 435, where one must replace q by q^{-1} to account for the difference between (1.6) and the relations used in [3]).

For contrast, we record the relation obtained from Corollary 5.3 for the current choices of I, J, M, N :

$$\begin{aligned}
 q^3[145|J][236|J] &= \\
 (5.14) \quad &[236|J][145|J] + \widehat{q}[235|J][146|J] + \widehat{q}(-q)^{-1}[234|J][156|J] \\
 &+ \widehat{q}[136|J][245|J] + \widehat{q}^2[135|J][246|J] + \widehat{q}^2(-q)^{-1}[134|J][256|J] \\
 &+ \widehat{q}(-q)^{-1}[126|J][345|J] + \widehat{q}^2(-q)^{-1}[125|J][346|J] \\
 &+ \widehat{q}^2(-q)^{-2}[124|J][356|J] + \widehat{q}(-q)^{-4}[123|J][456|J].
 \end{aligned}$$

We derive two further relations from Theorem 5.2 and Corollary 5.3 with the help of the isomorphism β of §1.6, as in §3.5. For use in the upcoming proof, note that since ω_0 reverses inequalities of integers, it also reverses the ordering on index sets: if U and V are subsets of $\{1, \dots, n\}$ with $|U| = |V|$, then $U \leq V$ if and only if $\omega_0 U \geq \omega_0 V$.

5.7. Theorem. *Let $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$. Then*

$$\begin{aligned}
 (5.15) \quad q^{|J \cap N|}[I|J][M|N] + q^{|J \cap N|} \sum_{S \in \{>I||M\}} \widetilde{\mu}_S[S|J][S^\natural|N] &= \\
 q^{|I \cap M|}[M|N][I|J] + q^{|I \cap M|} \sum_{T \in \{<J||N\}} \widetilde{\lambda}_T[M|T^\natural][I|T], &
 \end{aligned}$$

where

$$(5.16) \quad \begin{aligned} \tilde{\mu}_S &= (-\hat{q})^{|S \setminus I|} (-q)^{-\mathcal{L}^\natural(S, I, M)} \xi_q(S \setminus I; I \setminus S) \\ \tilde{\lambda}_T &= (-\hat{q})^{|J \setminus T|} (-q)^{-\mathcal{L}^\natural(T, J, N)} \xi_q(J \setminus T; T \setminus J) \end{aligned}$$

for $S \in \{>I \parallel M\}$ and $T \in \{<J \parallel N\}$.

Proof. Just for this proof, write $\tilde{U} = \omega_0 U$ for index sets U , and observe that

$$\omega_0(\{>I \parallel M\}) = \{<\tilde{I} \parallel \tilde{M}\} \quad \omega_0(\{<J \parallel N\}) = \{>\tilde{J} \parallel \tilde{N}\}.$$

Note also that $\tilde{S}^\natural = \tilde{S}$ for $S \in \{>I \parallel M\}$, and similarly $\tilde{T}^\natural = \tilde{T}$ for $T \in \{<J \parallel N\}$.

Set $A' = \mathcal{O}_{q^{-1}}(M_n(k))$, with generators X'_{ij} and braiding form \mathbf{r}' , and label quantum minors in A' in the form $[I|J]'$. Recall the isomorphism $\beta : A \rightarrow A'$ from §1.6, and equation (1.14). Note that when specializing general results to A' , the scalars q and \hat{q} change to q^{-1} and $-\hat{q}$, respectively.

Now apply Theorem 5.2 to the quantum minors $[\tilde{I}|\tilde{J}]'$ and $[\tilde{M}|\tilde{N}]'$ in A' . We obtain

$$(5.17) \quad \begin{aligned} q^{-|\tilde{I} \cap \tilde{M}|} [\tilde{I}|\tilde{J}]' [\tilde{M}|\tilde{N}]' + q^{-|\tilde{I} \cap \tilde{M}|} \sum_{S \in \{>I \parallel M\}} \lambda'_{\tilde{S}} [\tilde{S}|\tilde{J}]' [\tilde{S}^\natural|\tilde{N}]' = \\ q^{-|\tilde{J} \cap \tilde{N}|} [\tilde{M}|\tilde{N}]' [\tilde{I}|\tilde{J}]' + q^{-|\tilde{J} \cap \tilde{N}|} \sum_{T \in \{<J \parallel N\}} \mu'_{\tilde{T}} [\tilde{M}|\tilde{T}^\natural]' [\tilde{I}|\tilde{T}]', \end{aligned}$$

where

$$\begin{aligned} \lambda'_{\tilde{S}} &= (-\hat{q})^{|\tilde{I} \setminus S|} (-q)^{-\mathcal{L}^\natural(\tilde{S}, \tilde{I}, \tilde{M})} \xi_q(\tilde{I} \setminus \tilde{S}; \tilde{S} \setminus \tilde{I}) \\ \mu'_{\tilde{T}} &= (-\hat{q})^{|\tilde{T} \setminus J|} (-q)^{-\mathcal{L}^\natural(\tilde{T}, \tilde{J}, \tilde{N})} \xi_q(\tilde{T} \setminus \tilde{J}; \tilde{J} \setminus \tilde{T}) \end{aligned}$$

for $S \in \{>I \parallel M\}$ and $T \in \{<J \parallel N\}$. (Here we have simplified the exponents of the $-\hat{q}$ terms and invested the observation that $\xi_{q^{-1}}(U; V) = \xi_q(U; V)$ for any U, V .) Applying the isomorphism β^{-1} to (5.17) yields

$$(5.18) \quad \begin{aligned} q^{-|I \cap M|} [I|J][M|N] + q^{-|I \cap M|} \sum_{S \in \{>I \parallel M\}} \lambda'_S [S|J][S^\natural|N] = \\ q^{-|J \cap N|} [M|N][I|J] + q^{-|J \cap N|} \sum_{T \in \{<J \parallel N\}} \mu'_T [M|T^\natural][I|T] \end{aligned}$$

in A . Equation (5.15) will follow from (5.18) once we see that $\lambda'_S = \tilde{\mu}_S$ and $\mu'_T = \tilde{\lambda}_T$ for all S and T .

Let $S \in \{>I \parallel M\}$, and observe that

$$(5.19) \quad \begin{aligned} S \cap S^\natural &= I \cap M & S \cup S^\natural &= I \cup M \\ S^\natural \setminus M &= I \setminus S & M \setminus S^\natural &= S \setminus I. \end{aligned}$$

It follows from Theorem 4.6 and Lemma 1.7 that

$$q^{-|I \cap M|} \lambda'_S = q^{-|\tilde{I} \cap \tilde{M}|} \lambda'_{\tilde{S}} = \mathbf{r}'([\tilde{I}|\tilde{S}]', [\tilde{M}|\tilde{S}^\natural]') = \mathbf{r}'([M|S^\natural]', [I|S]').$$

With the help of (5.19), a second application of Theorem 4.6 shows that

$$\mathbf{r}'([M|S^\natural]', [I|S]') = q^{-|I \cap M|} \tilde{\mu}_S,$$

and therefore $\lambda'_S = \tilde{\mu}_S$. Similarly, $\mu'_T = \tilde{\lambda}_T$ for all $T \in \{<J||N\}$, and the theorem is proved. \square

The following corollary is obtained from Theorem 5.7 in the same way as Corollary 5.3 from Theorem 5.2.

5.8. Corollary. *Let $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$. Then*

$$(5.20) \quad \begin{aligned} q^{|I \cap M|} [I|J][M|N] + q^{|I \cap M|} \sum_{S \in \{>J||N\}} \tilde{\mu}_S [I|S][M|S^\natural] = \\ q^{|J \cap N|} [M|N][I|J] + q^{|J \cap N|} \sum_{T \in \{<I||M\}} \tilde{\lambda}_T [T^\natural|N][T|J], \end{aligned}$$

where

$$(5.21) \quad \begin{aligned} \tilde{\mu}_S &= (-\hat{q})^{|S \setminus J|} (-q)^{-\mathcal{L}^\natural(S, J, N)} \xi_q(S \setminus J; J \setminus S) \\ \tilde{\lambda}_T &= (-\hat{q})^{|I \setminus T|} (-q)^{-\mathcal{L}(T, I, M)} \xi_q(I \setminus T; T \setminus I) \end{aligned}$$

for $S \in \{>J||N\}$ and $T \in \{<I||M\}$. \square

6. SOME VARIANTS

Consider the general form of a commutation relation for quantum minors $[I|J]$ and $[M|N]$, namely an equation that allows a product $[I|J][M|N]$ to be replaced by a scalar multiple of the reverse product $[M|N][I|J]$, at the cost of some additional terms. In an equation such as (5.4), the additional terms are of two types – scalar multiples of $[S|J][S^\natural|N]$ and of $[M|T^\natural][I|T]$. In some applications, one type may be more useful than the other. For instance, the *preferred bases* constructed in [5] consist of certain products of quantum minors in which quantum minors with larger index sets must occur to the left of those with smaller index sets. Thus, if $|I| < |M|$, then $[M|N][I|J]$ and the terms $[M|T^\natural][I|T]$ are in preferred order, but $[I|J][M|N]$ and the terms $[S|J][S^\natural|N]$ are not. A commutation relation in which all the extra terms are in preferred order can be achieved by iteration – after a first application of (5.4), apply (5.4) to any products $[S|J][S^\natural|N]$ which appear, and continue until all terms have the desired form. This produces a relation in which $q^{|I \cap M|} [I|J][M|N]$ is expressed as $q^{|J \cap N|} [M|N][I|J]$ plus a linear combination of products $[S^\natural|T^\natural][S|T]$ where $S \in \{\leq I||M\}$ and $T \in \{\geq J||N\}$. We begin by illustrating the iteration process in Example 6.1 below.

The aim of this section is to derive closed formulas (i.e., without iterations) for commutation relations of the type just discussed.

6.1. Example. [$n = 4$] Consider $[I|J] = [23|12]$ and $[M|N] = [14|23]$. First, (5.4) leads to the relation

$$(6.1) \quad \begin{aligned} [23|12][14|23] - q[14|23][23|12] = \\ q\hat{q}[14|12][23|23] - \hat{q}(-q)^{-1}[12|12][34|23] - \hat{q}[13|12][24|23]. \end{aligned}$$

The last two terms on the right hand side of (6.1) must now be treated. Applying (5.4) in each case, we obtain

$$(6.2)(i) \quad [12|12][34|23] = q[34|23][12|12] + q\widehat{q}[34|12][12|23]$$

$$(6.2)(ii) \quad [13|12][24|23] = q[24|23][13|12] + q\widehat{q}[24|12][13|23] - \widehat{q}[12|12][34|23].$$

Note that (6.2)(ii) contains a term involving $[12|12][34|23]$. Hence, we first substitute that equation into (6.1), and then combine the two $[12|12][34|23]$ terms, before substituting (6.2)(i) into the result. The final relation is as follows:

$$(6.3) \quad \begin{aligned} [23|12][14|23] - q[14|23][23|12] &= q\widehat{q}[14|12][23|23] - \widehat{q}q[24|23][13|12] \\ &\quad - \widehat{q}^2q[24|12][13|23] + \widehat{q}q^2[34|23][12|12] \\ &\quad + \widehat{q}^2q^2[34|12][12|23]. \end{aligned}$$

In each of the terms on the right hand side of (6.3), the second factor is of the form $[S|T]$ where $S \in \{23, 13, 12\} = \{\leq I \| M\}$ and $T \in \{23, 12\} = \{\geq J \| N\}$.

6.2. Lemma. *Let $s \in \{1, \dots, n-1\}$, and let B and C be the following subalgebras of $A = \mathcal{O}_q(M_n(k))$:*

$$\begin{aligned} B &= k\langle X_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq s \rangle \\ C &= k\langle X_{ij} \mid 1 \leq i \leq n, s+1 \leq j \leq n \rangle. \end{aligned}$$

Then the multiplication map $\mu : B \otimes_k C \rightarrow A$ is a vector space isomorphism.

Proof. Let $X, Y,$ and Z be the standard PBW bases of the respective algebras $B, C,$ and A . Thus,

$$\begin{aligned} X &= \{(X_{11}^{b_{11}} \cdots X_{1s}^{b_{1s}})(X_{21}^{b_{21}} \cdots X_{2s}^{b_{2s}}) \cdots (X_{n1}^{b_{n1}} \cdots X_{ns}^{b_{ns}}) \mid b_{ij} \in \mathbb{Z}^+\} \\ Y &= \{(X_{1,s+1}^{c_{1,s+1}} \cdots X_{1n}^{c_{1n}})(X_{2,s+1}^{c_{2,s+1}} \cdots X_{2n}^{c_{2n}}) \cdots (X_{n,s+1}^{c_{n,s+1}} \cdots X_{nn}^{c_{nn}}) \mid c_{ij} \in \mathbb{Z}^+\} \\ Z &= \{(X_{11}^{a_{11}} \cdots X_{1n}^{a_{1n}})(X_{21}^{a_{21}} \cdots X_{2n}^{a_{2n}}) \cdots (X_{n1}^{a_{n1}} \cdots X_{nn}^{a_{nn}}) \mid a_{ij} \in \mathbb{Z}^+\}, \end{aligned}$$

where the variables occur in each monomial in lexicographic order. Observe that the monomials $X_{i1}^{b_{i1}} \cdots X_{is}^{b_{is}}$ and $X_{l,s+1}^{c_{l,s+1}} \cdots X_{ln}^{c_{ln}}$ commute whenever $i > l$. Hence, any product of a monomial from X with a monomial from Y can be rewritten as follows:

$$\begin{aligned} &[(X_{11}^{b_{11}} \cdots X_{1s}^{b_{1s}})(X_{21}^{b_{21}} \cdots X_{2s}^{b_{2s}}) \cdots (X_{n1}^{b_{n1}} \cdots X_{ns}^{b_{ns}})] [(X_{1,s+1}^{c_{1,s+1}} \cdots X_{1n}^{c_{1n}}) \cdot \\ &\quad (X_{2,s+1}^{c_{2,s+1}} \cdots X_{2n}^{c_{2n}}) \cdots (X_{n,s+1}^{c_{n,s+1}} \cdots X_{nn}^{c_{nn}})] \\ &= (X_{11}^{b_{11}} \cdots X_{1s}^{b_{1s}})(X_{1,s+1}^{c_{1,s+1}} \cdots X_{1n}^{c_{1n}})(X_{21}^{b_{21}} \cdots X_{2s}^{b_{2s}})(X_{2,s+1}^{c_{2,s+1}} \cdots X_{2n}^{c_{2n}}) \cdots \\ &\quad (X_{n1}^{b_{n1}} \cdots X_{ns}^{b_{ns}})(X_{n,s+1}^{c_{n,s+1}} \cdots X_{nn}^{c_{nn}}). \end{aligned}$$

Consequently, μ maps the set $\{x \otimes y \mid x \in X, y \in Y\}$ bijectively onto Z , and the lemma follows. \square

6.3. Theorem. *Let $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$. Then*

$$(6.4) \quad q^{|I \cap M|} [I|J][M|N] = q^{|J \cap N|} [M|N][I|J] + q^{|J \cap N|} \sum_{\substack{S \in \{\leq I || M\} \\ T \in \{\geq J || N\} \\ (S, T) \neq (I, J)}} \tilde{\lambda}_S \mu_T [S^\natural | T^\natural] [S|T],$$

where

$$(6.5) \quad \begin{aligned} \tilde{\lambda}_S &= (-\hat{q})^{|I \setminus S|} (-q)^{-\mathcal{L}(S, I, M)} \xi_q(I \setminus S; S \setminus I) \\ \mu_T &= \hat{q}^{|T \setminus J|} (-q)^{\mathcal{L}^\natural(T, J, N)} \xi_q(T \setminus J; J \setminus T) \end{aligned}$$

for $S \in \{\leq I || M\}$ and $T \in \{\geq J || N\}$.

Remark. We have isolated the term $q^{|J \cap N|} [M|N][I|J]$ on the right hand side of (6.4) to emphasize that this equation is a commutation relation. It may, of course, be incorporated in the given summation as a term where $(S, T) = (I, J)$, since $\tilde{\lambda}_I \mu_J = 1$.

Proof. Note that the coefficients λ_S and μ_T defined in (5.5) also depend on I, J, M, N . For purposes of the present proof, we record that dependence by writing

$$\begin{aligned} \lambda_S^{X, Y} &= \hat{q}^{|X \setminus S|} (-q)^{\mathcal{L}(S, X, Y)} \xi_q(X \setminus S; S \setminus X) \\ \mu_T^{J, N} &= \hat{q}^{|T \setminus J|} (-q)^{\mathcal{L}^\natural(T, J, N)} \xi_q(T \setminus J; J \setminus T) \end{aligned}$$

for $S \in \{\leq X || Y\}$ and $T \in \{\geq J || N\}$. Note that $\lambda_X^{X, Y} = 1$ and $\mu_J^{J, N} = 1$. For $S \in \{\leq I || M\}$, set

$$\alpha_S^{I, M} = \sum_{\substack{S_1 \in \{\leq I || M\} \\ S_2 \in \{\leq S_1 || S_1^\natural\} \\ \dots \\ S \in \{\leq S_{i-1} || S_{i-1}^\natural\}}} (-1)^i \lambda_{S_1}^{I, M} \lambda_{S_2}^{S_1, S_1^\natural} \dots \lambda_S^{S_{i-1}, S_{i-1}^\natural},$$

where we interpret $S_0 = I$ and $S_0^\natural = M$ in terms where $i = 1$. Finally, set $\alpha_I^{I, M} = 1$. We claim that

$$(6.6) \quad q^{|I \cap M|} [I|J][M|N] = q^{|J \cap N|} \sum_{\substack{S \in \{\leq I || M\} \\ T \in \{\geq J || N\}}} \alpha_S^{I, M} \mu_T^{J, N} [S^\natural | T^\natural] [S|T].$$

Let $t = |I|$, and let \mathcal{N}_t denote the collection of t -element subsets of $\{1, \dots, n\}$, partially ordered as in §1.10. In proving (6.6), we proceed by induction on I relative to the ordering in \mathcal{N}_t . To start, suppose that I is minimal in \mathcal{N}_t (that is, $I = \{1, \dots, t\}$). In this case, $\{\leq I || M\}$ is empty, and so Theorem 5.2 implies that

$$q^{|I \cap M|} [I|J][M|N] = q^{|J \cap N|} [M|N][I|J] + q^{|J \cap N|} \sum_{T \in \{\geq J || N\}} \mu_T^{J, N} [M|T^\natural] [I|T],$$

which verifies (6.6).

Now suppose that I is not minimal in \mathcal{N}_t , but that (6.6) holds whenever I is replaced by an index set $I' < I$. Theorem 5.2 implies that

$$(6.7) \quad \begin{aligned} q^{|I \cap M|} [I|J][M|N] &= q^{|J \cap N|} \sum_{T \in \{\geq J|N\}} \mu_T^{J,N} [M|T^\natural][I|T] \\ &\quad - q^{|I \cap M|} \sum_{S_1 \in \{< I|M\}} \lambda_{S_1}^{I,M} [S_1|J][S_1^\natural|N]. \end{aligned}$$

Recall that $S_1 \cap S_1^\natural = I \cap M$ for $S_1 \in \{< I|M\}$, by definition of S_1^\natural . Hence, our induction hypothesis yields

$$(6.8) \quad q^{|I \cap M|} [S_1|J][S_1^\natural|N] = q^{|J \cap N|} \sum_{\substack{S \in \{\leq S_1 \| S_1^\natural\} \\ T \in \{\geq J|N\}}} \alpha_S^{S_1, S_1^\natural} \mu_T^{J,N} [S^\natural|T^\natural][S|T]$$

for all $S_1 \in \{< I|M\}$. Substitute (6.8) in (6.7), which yields

$$(6.9) \quad \begin{aligned} q^{|I \cap M|} [I|J][M|N] &= q^{|J \cap N|} \sum_{T \in \{\geq J|N\}} \mu_T^{J,N} [M|T^\natural][I|T] \\ &\quad - q^{|J \cap N|} \sum_{\substack{S_1 \in \{< I|M\} \\ S \in \{\leq S_1 \| S_1^\natural\} \\ T \in \{\geq J|N\}}} \lambda_{S_1}^{I,M} \alpha_S^{S_1, S_1^\natural} \mu_T^{J,N} [S^\natural|T^\natural][S|T]. \end{aligned}$$

Since $\alpha_I^{I,M} = 1$, the coefficients in the first summation of (6.9) match the corresponding coefficients in (6.6). The second summation of (6.9) may be rewritten in the form

$$q^{|J \cap N|} \sum_{\substack{S \in \{< I|M\} \\ T \in \{\geq J|N\}}} \beta_S \mu_T^{J,N} [S^\natural|T^\natural][S|T],$$

where each

$$\beta_S = - \sum_{\substack{S_1 \in \{< I|M\} \\ S \in \{\leq S_1 \| S_1^\natural\}}} \lambda_{S_1}^{I,M} \alpha_S^{S_1, S_1^\natural} = \alpha_S^{I,M}.$$

Consequently, (6.9) yields (6.6), establishing the induction step. This proves (6.6).

It remains to show that $\alpha_S^{I,M} = \tilde{\lambda}_S$ for $S \in \{\leq I|M\}$.

Observe that all quantities appearing in (6.6) involve index sets contained in the union $I \cup J \cup M \cup N$, and so they remain the same if we work in $\mathcal{O}_q(M_\nu(k))$ for some $\nu > n$. Hence, there is no loss of generality in assuming that $n \geq |I| + |M|$. Thus, if we set

$$J^* = \{n - |I| + 1, \dots, n\} \quad N^* = \{1, \dots, |M|\},$$

we have $\max(N^*) < \min(J^*)$. Note also that J^* is maximal among $|I|$ -element subsets of $\{1, \dots, n\}$. The quantum minors $[U|N^*]$, for $U \subseteq \{1, \dots, n\}$ with $|U| = |M|$, are homogeneous elements of distinct degrees with respect to the grading on A discussed in §1.2. Hence, the $[U|N^*]$ are linearly independent over k . Similarly, the $[V|J^*]$, for $V \subseteq \{1, \dots, n\}$ with $|V| = |I|$, are linearly independent,

and thus it follows from Lemma 6.2 that the products $[U|N^*][V|J^*]$ are linearly independent over k .

Now apply (6.6) to the quantum minors $[I|J^*]$ and $[M|N^*]$. Since $\{>J^*||N^*\}$ is empty, we obtain

$$(6.10) \quad q^{I \cap M} [I|J^*][M|N^*] = \sum_{S \in \{\leq I||M\}} \alpha_S^{I,M} [S^\natural|N^*][S|J^*].$$

However, we also have a relation of this type from Corollary 5.8, which may be written in the form

$$(6.11) \quad q^{I \cap M} [I|J^*][M|N^*] = \sum_{T \in \{\leq I||M\}} \tilde{\lambda}_T [T^\natural|N^*][T|J^*].$$

Since the products $[S^\natural|N^*][S|J^*]$ are linearly independent, it follows from (6.10) and (6.11) that $\alpha_S^{I,M} = \tilde{\lambda}_S$ for all $S \in \{\leq I||M\}$. Therefore (6.6) implies (6.4), as desired. \square

As is easily checked, Theorem 6.3 directly yields equation (6.3).

We next consider the derivation of new relations from Theorem 6.3. Unlike the situation in Section 4, however, the methods used there to prove Corollary 5.3 and Theorem 5.7 yield the same result when applied to Theorem 6.3. Hence, we use the method of Corollary 5.3.

6.4. Corollary. *Let $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$. Then*

$$(6.12) \quad q^{|J \cap N|} [I|J][M|N] = q^{|I \cap M|} [M|N][I|J] + q^{|I \cap M|} \sum_{\substack{S \in \{\geq I||M\} \\ T \in \{\leq J||N\} \\ (S,T) \neq (I,J)}} \mu_S \tilde{\lambda}_T [S^\natural|T^\natural][S|T],$$

where

$$(6.13) \quad \begin{aligned} \mu_S &= \hat{q}^{|S \setminus I|} (-q)^{\mathcal{L}^\natural(S,I,M)} \xi_q(S \setminus I; I \setminus S) \\ \tilde{\lambda}_T &= (-\hat{q})^{|J \setminus T|} (-q)^{-\mathcal{L}(T,J,N)} \xi_q(J \setminus T; T \setminus J) \end{aligned}$$

for $S \in \{\geq I||M\}$ and $T \in \{\leq J||N\}$.

Proof. Interchange $I \leftrightarrow J$ and $M \leftrightarrow N$ in the statement of Theorem 6.3, and also interchange the roles of S and T in the summation. This yields

$$(6.14) \quad q^{|J \cap N|} [J|I][N|M] = q^{|I \cap M|} [N|M][J|I] + q^{|I \cap M|} \sum_{\substack{T \in \{\leq J||N\} \\ S \in \{\geq I||M\} \\ (T,S) \neq (J,I)}} \tilde{\lambda}_T^{J,N} \mu_S^{I,M} [T^\natural|S^\natural][T|S],$$

where we have placed the superscripts on $\tilde{\lambda}_T^{J,N}$ and $\mu_S^{I,M}$ as reminders of the changes required when carrying over (6.5) to the present situation. Thus, observe that $\tilde{\lambda}_T^{J,N}$ and $\mu_S^{I,M}$ are equal to the scalars denoted $\tilde{\lambda}_T$ and μ_S in (6.13). Consequently, an application of the automorphism τ to (6.14) yields (6.12) (recall (1.13)). \square

6.5. Remark. In addition to (6.4) and (6.12), one can derive two commutation relations for quantum minors $[I|J]$ and $[M|N]$ in which the additional terms involve products in the same order as $[I|J][M|N]$, rather than in reverse order. To obtain such results, simply interchange the roles of $[I|J]$ and $[M|N]$ in Theorem 6.3 and Corollary 6.4. One may wish to simplify the coefficients – for instance, with the help of observations such as (5.19), one sees that $\mathcal{L}(S^\natural, M, I) = \mathcal{L}^\natural(S, I, M)$. We leave this to the interested reader.

6.6. Example. [$n = 4$] We close the section by applying Corollary 6.4 to the quantum minors $[I|J] = [23|13]$ and $[M|N] = [14|24]$. In this case, equation (6.12) becomes

$$(6.15) \quad \begin{aligned} [23|13][14|24] &= [14|24][23|13] + \widehat{q}[13|24][24|13] + \widehat{q}(-q)^{-1}[12|24][34|13] \\ &+ (-\widehat{q})[14|34][23|12] + \widehat{q}(-\widehat{q})[13|34][24|12] \\ &+ \widehat{q}(-q)^{-1}(-\widehat{q})[12|34][34|12]. \end{aligned}$$

Equation (6.15) matches the relation calculated by Fioresi in [4, Example 6.2] (after replacing q by q^{-1}).

7. POISSON BRACKETS

In this final section, we use the commutation relations for quantum minors obtained above to derive expressions for the standard Poisson bracket on pairs of classical minors in $\mathcal{O}(M_n(k))$. In particular, we recover, for the case of the standard bracket, a formula calculated by Kupershmidt in [17]. Although the study of Poisson brackets is often restricted to characteristic zero, that restriction is not needed for the results below.

7.1. Standard Poisson bracket on $\mathcal{O}(M_n(k))$. Recall that a *Poisson bracket* on a commutative k -algebra B is a k -bilinear map $\{-, -\} : B \times B \rightarrow B$ such that

$$\begin{aligned} B &\text{ is a Lie algebra with respect to } \{-, -\}, \text{ and} \\ \{b, -\} &\text{ is a derivation for each } b \in B. \end{aligned}$$

Note that a Poisson bracket is uniquely determined by its values on pairs of elements from a k -algebra generating set for B .

Write $\mathcal{O}(M_n(k))$ as a commutative polynomial ring over k in indeterminates x_{ij} for $i, j = 1, \dots, n$. The *standard Poisson bracket* on this algebra is the unique Poisson bracket such that

$$(7.1) \quad \begin{aligned} \{x_{ij}, x_{lj}\} &= x_{ij}x_{lj} && (i < l) \\ \{x_{ij}, x_{im}\} &= x_{ij}x_{im} && (j < m) \\ \{x_{ij}, x_{lm}\} &= 0 && (i < l, j > m) \\ \{x_{ij}, x_{lm}\} &= 2x_{im}x_{lj} && (i < l, j < m). \end{aligned}$$

7.2. $\mathcal{O}_q(M_n)$ as a quantization of $\mathcal{O}(M_n)$. It is well known that $\mathcal{O}_q(M_n(K))$ (for a rational function field $K = k(q)$) is a quantization of the Poisson algebra $(\mathcal{O}(M_n(k)), \{-, -\})$ in the sense that the Poisson bracket on $\mathcal{O}(M_n(k))$ is the

“semiclassical limit” (as $q \rightarrow 1$) of the scaled commutator bracket $\frac{1}{q-1}[-, -]$ on $\mathcal{O}_q(M_n(K))$; we indicate the details below.

For the remainder of this section, replace the scalar q by an indeterminate, and consider the quantum matrix algebra $\mathcal{O}_q(M_n(k(q)))$ defined over the rational function field $k(q)$. The $k[q^{\pm 1}]$ -subalgebra A_0 of $\mathcal{O}_q(M_n(k(q)))$ generated by the X_{ij} can be presented (as a $k[q^{\pm 1}]$ -algebra) by the generators X_{ij} and relations (1.6), from which it follows that there is an isomorphism

$$(7.2) \quad A_0/(q-1)A_0 \xrightarrow{\cong} \mathcal{O}(M_n(k))$$

sending the cosets $X_{ij} + (q-1)A_0 \mapsto x_{ij}$ for all i, j . We identify $A_0/(q-1)A_0$ with $\mathcal{O}(M_n(k))$ via (7.2). Since $\mathcal{O}(M_n(k))$ is commutative, the additive commutator $[-, -]$ on A_0 takes all its values in $(q-1)A_0$, and so $\frac{1}{q-1}[-, -]$ is well-defined on A_0 . It follows that the latter bracket induces a well-defined Poisson bracket on $\mathcal{O}(M_n(k))$, such that

$$(7.3) \quad \{\bar{a}, \bar{b}\} = \overline{(ab - ba)/(q-1)}$$

for $a, b \in A_0$, where overbars denote cosets modulo $(q-1)A_0$. This induced bracket is nothing but the standard Poisson bracket on $\mathcal{O}(M_n(k))$, as one easily sees by computing its values on pairs of generators x_{ij}, x_{lm} .

We shall apply (7.3) when \bar{a} and \bar{b} are minors. In order to reserve the notation $[I|J]$ for classical minors, let us denote quantum minors in $\mathcal{O}_q(M_n(k(q)))$ in the form $[I|J]_q$. Note that $[I|J]_q$ is an element of A_0 , and that the isomorphism (7.2) maps the coset of $[I|J]_q$ to $[I|J]$. Hence, for pairs of minors, (7.3) can be written as

$$(7.4) \quad \{[I|J], [M|N]\} = \overline{([I|J]_q[M|N]_q - [M|N]_q[I|J]_q)/(q-1)}.$$

Combining (7.4) with formulas for additive commutators of quantum minors thus yields formulas for Poisson brackets of classical minors. For instance, from (5.10) we obtain

$$(7.5) \quad \{[1, \dots, r|J], [M|1, \dots, s]\} = (|[1, r] \cap J| - |M \cap [1, s]|)[1, \dots, r|J][M|1, \dots, s],$$

which recovers some cases of [15, Theorem 2.6].

7.3. Theorem. *Let $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$. Then*

$$(7.6) \quad \begin{aligned} \{[I|J], [M|N]\} &= (|J \cap N| - |I \cap M|)[I|J][M|N] \\ &\quad + 2 \sum_{\substack{j \in J \setminus N \\ n \in N \setminus J \\ j < n}} (-1)^{|(J \Delta N) \cap (j, n)|} [I|J \sqcup n \setminus j][M|N \sqcup j \setminus n] \\ &\quad - 2 \sum_{\substack{i \in I \setminus M \\ m \in M \setminus I \\ i > m}} (-1)^{|(I \Delta M) \cap (m, i)|} [I \sqcup m \setminus i|J][M \sqcup i \setminus m|N]. \end{aligned}$$

Proof. Write (5.4) in the form

$$\begin{aligned}
(I|J)_q[M|N]_q - [M|N]_q(I|J)_q &= (q^{|J \cap N| - |I \cap M|} - 1)[M|N]_q(I|J)_q \\
&+ q^{|J \cap N| - |I \cap M|} \sum_{T \in \{>J||N\}} \mu_T [M|T^\natural]_q [I|T]_q \\
(7.7) \qquad \qquad \qquad &- \sum_{S \in \{<I||M\}} \lambda_S [S|J]_q [S^\natural|N]_q.
\end{aligned}$$

Since $\widehat{q}^2/(q-1)$ vanishes modulo $q-1$, we only need to consider the terms in the sums for $T \in \{>J||N\}$ with $|T \setminus J| = 1$ and $S \in \{<I||M\}$ with $|I \setminus S| = 1$. Any such T has the form $T = J \sqcup n \setminus j$ with $j \in J \setminus N$ and $n \in N \setminus J$ such that $j < n$, whence $T^\natural = N \sqcup j \setminus n$ and $(T^\natural \setminus T) \cup (J \setminus N) = (J \Delta N) \setminus n$ and so

$$\begin{aligned}
\mathcal{L}^\natural(T, J, N) &= \ell((J \Delta N) \setminus n; n) - \ell((J \Delta N) \setminus n; j) \\
&= \ell(J \Delta N; n) - \ell(J \Delta N; j) + 1 = -|(J \Delta N) \cap (j, n)|.
\end{aligned}$$

Similarly, the indices S that appear have the form $S = I \sqcup m \setminus i$ with $i \in I \setminus M$ and $m \in M \setminus I$ such that $i > m$, whence $S^\natural = M \sqcup i \setminus m$ and $\mathcal{L}(S, I, M) = -|(I \Delta M) \cap (m, i)|$. Consequently, dividing (7.7) by $q-1$ and then reducing the resulting equation modulo $q-1$ yields (7.6). \square

Similarly, Corollary 5.3 yields the following result.

7.4. Theorem. *Let $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$. Then*

$$\begin{aligned}
\{[I|J], [M|N]\} &= (|I \cap M| - |J \cap N|)[I|J][M|N] \\
&+ 2 \sum_{\substack{i \in I \setminus M \\ m \in M \setminus I \\ i < m}} (-1)^{|(I \Delta M) \cap (i, m)|} [I \sqcup m \setminus i | J] [M \sqcup i \setminus m | N] \\
(7.8) \qquad \qquad &- 2 \sum_{\substack{j \in J \setminus N \\ n \in N \setminus J \\ j > n}} (-1)^{|(J \Delta N) \cap (n, j)|} [I | J \sqcup n \setminus j] [M | N \sqcup j \setminus n]. \quad \square
\end{aligned}$$

Finally, provided k does not have characteristic 2, we can average equations (7.6) and (7.8) to obtain the equation below.

7.5. Corollary. *Let $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$. If $\text{char}(k) \neq 2$, then*

$$\begin{aligned}
 \{[I|J], [M|N]\} &= \sum_{\substack{i \in I \setminus M \\ m \in M \setminus I \\ i < m}} (-1)^{|(I \Delta M) \cap (i, m)|} [I \sqcup m \setminus i | J] [M \sqcup i \setminus m | N] \\
 &\quad - \sum_{\substack{i \in I \setminus M \\ m \in M \setminus I \\ i > m}} (-1)^{|(I \Delta M) \cap (m, i)|} [I \sqcup m \setminus i | J] [M \sqcup i \setminus m | N] \\
 (7.9) \quad &\quad + \sum_{\substack{j \in J \setminus N \\ n \in N \setminus J \\ j < n}} (-1)^{|(J \Delta N) \cap (j, n)|} [I | J \sqcup n \setminus j] [M | N \sqcup j \setminus n] \\
 &\quad - \sum_{\substack{j \in J \setminus N \\ n \in N \setminus J \\ j > n}} (-1)^{|(J \Delta N) \cap (n, j)|} [I | J \sqcup n \setminus j] [M | N \sqcup j \setminus n]. \quad \square
 \end{aligned}$$

Equation (7.9) is the standard case of Kupershmidt's formula [17, Equation (9)]. To obtain the standard Poisson bracket in his setting, make the following choices for the structure constants:

$$r_{lm}^{ij} = \begin{cases} 1 & (i > j, l = j, m = i) \\ -1 & (i < j, l = j, m = i) \\ 0 & (\text{otherwise}). \end{cases}$$

ACKNOWLEDGEMENT

We thank T. H. Lenagan, L. Rigal, A. Zelevinsky, and the referee for their comments and suggestions concerning this project.

REFERENCES

- [1] A. Berenstein and A. Zelevinsky, *Quantum cluster algebras*, Advances in Math. **195** (2005), 405-455.
- [2] K. A. Brown and K. R. Goodearl, *Lectures on Algebraic Quantum Groups*, Advanced Courses in Math. CRM Barcelona, Basel (2002) Birkhäuser.
- [3] R. Fiorese, *Quantum deformation of the Grassmannian manifold*, J. Algebra **214** (1999), 418-447.
- [4] ———, *Commutation relations among quantum minors in $\mathcal{O}_q(M_n(k))$* , J. Algebra **280** (2004), 655-682.
- [5] K. R. Goodearl and T. H. Lenagan, *Quantum determinantal ideals*, Duke Math. J. **103** (2000), 165-190.
- [6] ———, *Prime ideals invariant under winding automorphisms in quantum matrices*, Internat. J. Math. **13** (2002), 497-532.
- [7] ———, *Winding-invariant prime ideals in quantum 3×3 matrices*, J. Algebra **260** (2003), 657-687.
- [8] T. Hayashi, *Quantum groups and quantum determinants*, J. Algebra **152** (1992), 146-165.
- [9] T. J. Hodges and T. Lerasseur, *Primitive ideals of $\mathbf{C}_q[SL(3)]$* , Comm. Math. Phys. **156** (1993), 581-605.

- [10] ———, *Primitive ideals of $C_q[SL(n)]$* , J. Algebra **168** (1994), 455-468.
- [11] T. J. Hodges, T. Levasseur, and M. Toro, *Algebraic structure of multi-parameter quantum groups*, Advances in Math. **126** (1997), 52-92.
- [12] A. Joseph, *Quantum Groups and their Primitive Ideals*, Ergebnisse der Math. (3) 29, Berlin (1995) Springer-Verlag.
- [13] C. Kassel, *Quantum Groups*, Grad. Texts in Math. 155, Berlin (1995) Springer-Verlag.
- [14] A. U. Klimyk and K. Schmüdgen, *Quantum Groups and their Representations*, Berlin (1997) Springer-Verlag.
- [15] M. Kogan and A. Zelevinsky, *On symplectic leaves and integrable systems in standard complex semisimple Poisson-Lie groups*, Internat. Math. Research Notices (2002), no. 32, 1685-1702.
- [16] D. Krob and B. Leclerc, *Minor identities for quasi-determinants and quantum determinants*, Commun. Math. Phys. **169** (1995), 1-23.
- [17] B. A. Kupershmidt, *Poisson relations between minors and their consequences*, J. Physics A: Math. Gen. **27** (1994), L507-L513.
- [18] V. Lakshmibai and N. Reshetikhin, *Quantum deformations of flag and Schubert schemes*, C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), 121-126.
- [19] ———, *Quantum flag and Schubert schemes*, in *Deformation Theory and Quantum Groups with Applications to Mathematical Physics (Amherst 1990)*, pp. 145-181, Providence (1992) Amer. Math. Soc.
- [20] R. G. Larson and D. E. Radford, *Introduction to the Quantum Yang-Baxter Equation and Quantum Groups: An Algebraic Approach*, Dordrecht (1997) Kluwer.
- [21] R. G. Larson and J. Towber, *Two dual classes of bialgebras related to the concepts of "quantum group" and "quantum Lie algebra"*, Communic. in Algebra **19** (1991), 3295-3345.
- [22] B. Leclerc and A. Zelevinsky, *Quasicommuting families of quantum Plücker coordinates*, in *Kirillov's Seminar on Representation Theory*, pp. 85-108, AMS Translations, Series 2, 181, Providence (1998) Amer. Math. Soc.
- [23] T. H. Lenagan and L. Rigal, *Quantum graded algebras with a straightening law and the AS-Cohen-Macaulay property for quantum determinantal rings and quantum grassmannians*, posted at www.arxiv.org/abs/math.QA/0403021.
- [24] M. Noumi, H. Yamada, and K. Mimachi, *Finite dimensional representations of the quantum group $GL_q(n; \mathbb{C})$ and the zonal spherical functions on $U_q(n-1) \backslash U_q(n)$* , Japanese J. Math. **19** (1993), 31-80.
- [25] B. Parshall and J.-P. Wang, *Quantum linear groups*, Memoirs Amer. Math. Soc. **439** (1991).
- [26] N. Yu. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. **1** (1990), 193-225.
- [27] J. Scott, *Quasi-commuting families of quantum minors*, J. Algebra **290** (2005), 204-220.
- [28] Ya. S. Soibelman, *The algebra of functions on a compact quantum group, and its representations*, Leningrad Math. J. **2** (1991), 161-178; *Correction (Russian)*, Algebra i Analiz **2** (1990), 256.
- [29] E. Taft and J. Towber, *Quantum deformation of flag schemes and Grassmann schemes. I. A q -deformation of the shape-algebra for $GL(n)$* , J. Algebra **142** (1991), 1-36.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106, USA

E-mail address: goodearl@math.ucsb.edu