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QUANTIZED PRIMITIVE IDEAL SPACES AS QUOTIENTS OF AFFINE ALGEBRAIC VARIETIES

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Abstract. Given an affine algebraic variety V and a quantization $\mathcal{O}_q(V)$ of its coordinate ring, it is conjectured that the primitive ideal space of $\mathcal{O}_q(V)$ can be expressed as a topological quotient of V. Evidence in favor of this conjecture is discussed, and positive solutions for several types of varieties (obtained in joint work with E. S. Letzter) are described. In particular, explicit topological quotient maps are given in the case of quantum toric varieties.

Introduction

A major theme in the subject of quantum groups is the philosophy that in the passage from a classical coordinate ring to a quantized analog, the classical geometry is replaced by structures that should be treated as 'noncommutative geometry'. Indeed, much work has been invested into the development of theories of noncommutative differential geometry and noncommutative algebraic geometry. We would like to pose the question whether these theories are entirely noncommutative, or whether traces of classical geometry are to be found in the noncommutative geometry. This rather vague question can, of course, be focused in any number of different directions. We discuss one particular direction here, which was developed in joint work with E. S. Letzter [**6**]; it concerns situations in which quantized analogs of classical varieties contain certain quotients of these varieties.

To take an ideal-theoretic perspective on the question posed above, recall the way in which an affine algebraic variety V is captured in its coordinate ring $\mathcal{O}(V)$: the space of maximal ideals, $\max \mathcal{O}(V)$, is homeomorphic to V, under their respective Zariski topologies. In the passage from commutative to noncommutative algebras, the natural analog of a maximal ideal space is a primitive ideal space – this point of view is taken partly on practical grounds, because noncommutative rings often have too few maximal ideals for various purposes, but also in order to reflect the ideal theory connected with the study of irreducible representations. (Recall that the primitive ideals of an algebra A are precisely the annihilators of the irreducible A-modules.) Thus, given a quantized version of $\mathcal{O}(V)$, say $\mathcal{O}_q(V)$, the natural analog of V is the primitive ideal space $\operatorname{prim} \mathcal{O}_q(V)$. The aspect of the general problem which we wish to discuss here, then, is that of finding classical geometric structure in $\operatorname{prim} \mathcal{O}_q(V)$, and relating it to the structure of V.

We report below on some success in expressing $\operatorname{prim} \mathcal{O}_q(V)$ as a quotient of V with respect to the respective Zariski topologies. Modulo a minor technical assumption, this is

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done in three situations – when V is an algebraic torus, a full affine space, or (generalizing both of those cases) an affine toric variety. We discuss these cases in Sections 2, 3, and 4, respectively; we sketch some of the methods and indicate relationships among the three cases. In particular, in the last section we establish precise formulas for topological quotient maps in the case of quantum toric varieties, maps were only given an existence proof in [**6**].

Recall that a map $\phi : X \to Y$ between topological spaces is a *topological quotient map* provided ϕ is surjective and the topology on Y coincides with the quotient topology induced by ϕ, that is, a subset $C \subseteq Y$ is closed in Y precisely when $\phi^{-1}(C)$ is closed in X. In that case, Y is completely determined (as a topological space) by the topology on X together with the partition of X into fibres of ϕ.

Throughout the paper, we fix an algebraically closed base field k. All the varieties we discuss will be affine algebraic varieties over k.

1. Quantum semisimple groups

We follow the usual practice in writing 'quantum semisimple groups' as an abbreviation for 'quantized coordinate rings of semisimple groups'. Thus, 'quantum SL_n' refers to any quantization of the coordinate ring $\mathcal{O}(SL_n(k))$. We begin by displaying the primitive ideal space of the most basic example, the standard single parameter quantization of SL_2.

1.1. Example. Consider the algebra $\mathcal{O}_q(SL_2(k))$, where $q \in k^\times$ is a non-root of unity. This is the k-algebra with generators X_{11}, X_{12}, X_{21}, X_{22} satisfying the following relations:

$$\begin{array}{lll} X_{11}X_{12} = qX_{12}X_{11} & X_{11}X_{21} = qX_{21}X_{11} & X_{12}X_{21} = X_{21}X_{12} \\ X_{21}X_{22} = qX_{22}X_{21} & X_{12}X_{22} = qX_{22}X_{12} & \\ X_{11}X_{22} - X_{22}X_{11} = (q-q^{-1})X_{12}X_{21} & & X_{11}X_{22} - qX_{12}X_{21} = 1 \end{array}$$

Since q is not a root of unity, the primitive ideal space of $\mathcal{O}_q(SL_2(k))$ is not large, and can be completely described as follows:

$$\langle X_{11} - \alpha,\ X_{12},\ X_{21},\ X_{22} - \alpha^{-1} \rangle$$
$$(\alpha \in k^\times)$$

$$\langle X_{12} \rangle \qquad\qquad\qquad\qquad \langle X_{21} \rangle$$

$$\langle X_{12} - \beta X_{21} \rangle$$
$$(\beta \in k^\times)$$

With the Zariski topology, this space seems more like a scheme than an affine variety, since it has many non-closed points. When taken apart as pictured, however, it can be viewed as a disjoint union of four classical varieties: two single points, and two punctured affine lines. Taken as a whole, $\operatorname{prim}\mathcal{O}_q(SL_2(k))$ can be related to the variety $SL_2(k)$ via the following map:

$$\begin{aligned}
&\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \mapsto \langle X_{11}-\alpha,\ X_{12},\ X_{21},\ X_{22}-\alpha^{-1}\rangle && (\alpha \in k^\times)\\
&\left(\begin{smallmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{smallmatrix}\right) \mapsto \langle X_{12}\rangle && (\alpha,\gamma \in k^\times)\\
&\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \mapsto \langle X_{21}\rangle && (\alpha,\beta \in k^\times)\\
&\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \mapsto \langle X_{12}-\beta\gamma^{-1}X_{21}\rangle && (\alpha,\delta \in k;\ \beta,\gamma \in k^\times)\\
& && (\alpha\delta-\beta\gamma=1)
\end{aligned}$$

We leave as an exercise for the reader to show that the map $SL_2(k) \twoheadrightarrow \operatorname{prim}\mathcal{O}_q(SL_2(k))$ given above is Zariski continuous; in fact, it is a topological quotient map.

1.2. Let G be a connected semisimple algebraic group over $\mathbb{C}$, and let $q \in \mathbb{C}^\times$ be a non-root of unity. Quantized coordinate rings of G have been defined in both single parameter and multiparameter versions, which we denote $\mathcal{O}_q(G)$ and $\mathcal{O}_{q,p}(G)$ respectively. We shall not recall the definitions here, but refer the reader to [**8**], [**9**], [**10**], [**12**], [**13**], or [**4**]. The first classification results for primitive ideals in this context were proved in the single parameter case:

Theorem. [Hodges-Levasseur, Joseph] *There exists a bijection*

$$\operatorname{prim}\mathcal{O}_q(G) \longleftrightarrow \{\, \textit{symplectic leaves in } G \,\},$$

where the symplectic leaves are computed relative to the associated Poisson structure on G arising from the quantization.

Proof. This was proved for the cases $G = SL_3(\mathbb{C})$ and $G = SL_n(\mathbb{C})$ by Hodges and Levasseur in [**8**, Theorem 4.4.1] and [**9**, Theorem 4.2], and then for arbitrary G by Joseph [**12**, Theorem 9.2], [**13**, Theorems 10.3.7, 10.3.8]. □

Let us rewrite this result to bring in the points of G more directly: There exists a surjection $G \twoheadrightarrow \operatorname{prim}\mathcal{O}_q(G)$ whose fibres are precisely the symplectic leaves in G. In the multiparameter case, the situation is slightly more complicated, as follows.

1.3. Theorem. [Hodges-Levasseur-Toro] *There exists a surjection $G \twoheadrightarrow \operatorname{prim}\mathcal{O}_{q,p}(G)$, but the fibres are symplectic leaves only for certain choices of p.*

Proof. The first statement follows from the results in [**10**, Section 4]; for the second, see [**10**, Theorems 1.8, 4.18]. □

1.4. We can fill in a bit more detail about these surjections by bringing in a group of symmetries. Let H be a maximal torus in G, acting on $\mathcal{O}_{q,p}(G)$ by 'winding automorphisms' (cf. [**13**, (1.3.4)]). For example, if $G = SL_2(\mathbb{C})$ and $H = \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \;\middle|\; \alpha \in \mathbb{C}^\times \right\}$, then H acts on $\mathcal{O}_q(G)$ so that $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha X_{11} & \alpha X_{12} \\ \alpha^{-1} X_{21} & \alpha^{-1} X_{22} \end{smallmatrix}\right)$. In this case, the induced action of H on $\operatorname{prim}\mathcal{O}_q(G)$ has four orbits, which are the sets displayed in Example 1.1.

It follows from the analyses in [**8**, **9**, **12**, **13**, **10**] that

• The number of H-orbits in $\operatorname{prim} \mathcal{O}_{q,p}(G)$ is finite.
• Each of these H-orbits is Zariski homeomorphic to H modulo the relevant stabilizer.
• The preimage of each H-orbit, under the surjection in Theorem 1.3, is a locally closed subset of G.

These properties hint strongly that there is some topological connection between G and $\operatorname{prim} \mathcal{O}_{q,p}(G)$.

1.5. Conjecture. There exists an H-equivariant surjection $G \twoheadrightarrow \operatorname{prim} \mathcal{O}_{q,p}(G)$ which is a topological quotient map (with respect to the Zariski topologies).

If this conjecture were verified, then it together with knowledge of the fibres of the map would completely determine $\operatorname{prim} \mathcal{O}_{q,p}(G)$ as a topological space.

In the case of $\mathcal{O}_q(SL_2(\mathbb{C}))$, the conjecture can be readily verified 'by hand', as already suggested above. There is a further piece of evidence in the SL_3 case: Brown and the author have constructed an H-equivariant topological quotient map $B^+ \twoheadrightarrow \operatorname{prim} \mathcal{O}_q(B^+)$ where B^+ is the upper triangular Borel subgroup of $SL_3(\mathbb{C})$ [Work in progress].

The major difficulty in the way of the conjecture is that no good global description of the Zariski topology on $\operatorname{prim} \mathcal{O}_{q,p}(G)$ is known, only the Zariski topologies on the separate H-orbits.

In order to try to obtain a better feel for the problem, let us widen the focus:

1.6. Conjecture. If $\mathcal{O}_{\mathbf{q}}(V)$ is a quantized coordinate ring of an affine algebraic variety (or group) V, there exists a topological quotient map $V \twoheadrightarrow \operatorname{prim} \mathcal{O}_{\mathbf{q}}(V)$, equivariant with respect to an appropriate group of automorphisms.

Here $\mathcal{O}_{\mathbf{q}}(V)$ is meant to stand for any quantization of $\mathcal{O}(V)$, a phrase which is only slightly less vague than the symbolism, since no definition of a "quantization" of a coordinate ring exists. For descriptions of many commonly studied classes of examples, see [**4**].

In the remainder of the paper, we discuss some cases where the extended conjecture has been verified.

2. Quantum tori

The simplest case in which to examine our conjecture is that of a quantization of the coordinate ring of an algebraic torus. The parameters required can be arranged as a matrix $\mathbf{q} = (q_{ij}) \in M_n(k^\times)$ which is *multiplicatively antisymmetric*, that is, $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j. (These conditions are assumed in order to avoid degeneracies in the resulting algebra.) The *quantum torus* over k with respect to these parameters is the algebra

$$\mathcal{O}_{\mathbf{q}}((k^\times)^n) := k\langle x_1^{\pm 1}, \dots, x_n^{\pm 1} \mid x_i x_j = q_{ij} x_j x_i \text{ for all } i, j \rangle.$$

The structure of this algebra has been analyzed in various cases by many people, among them McConnell and Pettit [**15**], De Concini, Kac, and Procesi [**3**], Hodges [**7**], Vancliff [**16**], Brown and the author [**2**], Letzter and the author [**5**], and Ingalls [**11**]. In particular, it is known that $\operatorname{prim} \mathcal{O}_{\mathbf{q}}((k^\times)^n)$ and $\operatorname{spec} \mathcal{O}_{\mathbf{q}}((k^\times)^n)$ are homeomorphic to the maximal and prime spectra, respectively, of the center $Z(\mathcal{O}_{\mathbf{q}}((k^\times)^n))$, via contraction and extension, and that the center is a Laurent polynomial ring over k (e.g., [**5**, Lemma 1.2, Corollary 1.5]).

2.1. Theorem. [**6**, Theorem 3.11] *For any multiplicatively antisymmetric matrix* $\mathbf{q} \in M_n(k^\times)$*, there exist topological quotient maps*

$$(k^\times)^n \twoheadrightarrow \operatorname{prim} \mathcal{O}_{\mathbf{q}}((k^\times)^n)$$
$$\operatorname{spec} k[y_1^{\pm 1}, \dots, y_n^{\pm 1}] \twoheadrightarrow \operatorname{spec} \mathcal{O}_{\mathbf{q}}((k^\times)^n),$$

equivariant with respect to the natural actions of the torus $(k^\times)^n$*.* $\square$

This case, in many ways, works out too smoothly to really put Conjecture 1.6 to the test. In that regard, the case of quantum affine spaces is much more interesting. Before moving on to that case, however, we describe the form of the maps in Theorem 2.1, since this will preview essential aspects of the quantum affine space case. We proceed by rewriting quantum tori as twisted group algebras, as follows.

2.2. Fix a quantum torus $A = \mathcal{O}_{\mathbf{q}}((k^\times)^n)$. Set $\Gamma = \mathbb{Z}^n$, and define $\sigma : \Gamma \times \Gamma \to k^\times$ by the rule

$$\sigma(\alpha, \beta) = \prod_{i,j=1}^{n} q_{ij}^{\alpha_i \beta_j}.$$

Then σ is an alternating bicharacter on Γ, and it determines the commutation rules in A, since $x^\alpha x^\beta = \sigma(\alpha, \beta) x^\beta x^\alpha$ for all $\alpha, \beta \in \Gamma$, where we have used the standard multi-index notation $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Recall that the *radical* of σ is the subgroup

$$\operatorname{rad}(\sigma) = \{\alpha \in \Gamma \mid \sigma(\alpha, -) \equiv 1\}$$

of Γ. It plays the following role: $Z(A) = k[x^\alpha \mid \alpha \in \operatorname{rad}(\sigma)]$ (cf. [**5**, Lemma 1.2] or [**11**, Proposition 5.2]).

Now choose a 2-cocycle $c : \Gamma \times \Gamma \to k^\times$ such that $c(\alpha, \beta) c(\beta, \alpha)^{-1} = \sigma(\alpha, \beta)$ for $\alpha, \beta \in \Gamma$. (This can be done in many ways – see [**1**, Proposition 1, p. 888] or [**6**, (3.2)], for instance.) We may identify A with the twisted group algebra $k^c\Gamma$, that is, the k-algebra with a basis $\{x_\alpha \mid \alpha \in \Gamma\}$ such that

$$x_\alpha x_\beta = c(\alpha, \beta) x_{\alpha+\beta}$$

for $\alpha, \beta \in \Gamma$.

Let us write the group algebra $k\Gamma$ in terms of a basis $\{y_\alpha \mid \alpha \in \Gamma\}$, where $y_\alpha y_\beta = y_{\alpha+\beta}$ for $\alpha, \beta \in \Gamma$. There is a k-linear (vector space) isomorphism

$$\Phi_c : A \to k\Gamma$$

such that $\Phi_c(x_\alpha) = y_\alpha$ for $\alpha \in \Gamma$. While this map is not multiplicative, it satisfies $\Phi_c(x_\alpha x_\beta) = c(\alpha, \beta) \Phi_c(x_\alpha) \Phi_c(x_\beta)$ for $\alpha, \beta \in \Gamma$ (cf. [**6**, (3.5)]).

The final ingredient needed to describe the topological quotient maps in the current situation is the torus $H = (k^\times)^n$ and its natural actions on A and $k\Gamma$ by k-algebra automorphisms. The induced actions of H on the prime and primitive spectra of A and $k\Gamma$ will also be required. We identify H with $\operatorname{Hom}(\Gamma, k^\times)$, which allows us to write the above actions as

$$h.x_\alpha = h(\alpha) x_\alpha \qquad \text{and} \qquad h.y_\alpha = h(\alpha) y_\alpha$$

for $h \in H$ and $\alpha \in \Gamma$. Observe that the map Φ_c is H-equivariant. The identification $H = \operatorname{Hom}(\Gamma, k^\times)$ provides a pairing $H \times \Gamma \to k^\times$, and we use $^\perp$ to denote orthogonals with respect to this pairing. In particular,

$$S^\perp = \{h \in H \mid \ker h \supseteq S\}$$

for $S \subseteq \Gamma$.

2.3. It is convenient to have a compact notation for the intersection of the orbit of an ideal under a group of automorphisms. If P is an ideal of a ring B, and T is a group acting on B by ring automorphisms, we write

$$(P : T) = \bigcap_{t \in T} t(P).$$

This ideal can also be described as the largest T-invariant ideal of B contained in P. The same notation can also be applied to any subset of B.

2.4. Theorem. [**6**, Theorem 3.11] *Let $A = \mathcal{O}_{\mathbf{q}}((k^\times)^n)$, and fix $\Gamma, \sigma, c, \Phi_c, H$ as above. Assume that $c \equiv 1$ on $\operatorname{rad}(\sigma) \times \Gamma$. (Such a choice for c is always possible; cf.* [**6**, Lemma 3.12]*.) Then the rule $P \mapsto \Phi_c^{-1}(P : \operatorname{rad}(\sigma)^\perp)$, for prime ideals P of $k\Gamma$, defines H-equivariant topological quotient maps*

$$\operatorname{spec} k\Gamma \twoheadrightarrow \operatorname{spec} A \qquad \textit{and} \qquad \max k\Gamma \twoheadrightarrow \operatorname{prim} A.$$

The fibres of the second map are exactly the $\operatorname{rad}(\sigma)^\perp$-orbits in $\max k\Gamma$. $\square$

2.5. As the formula in Theorem 2.4 indicates, the given maps are compositions of two maps, which themselves have nice properties. To express this in more detail, write $S = \operatorname{rad}(\sigma)$, and recall that an *$S^\perp$-prime ideal* of $k\Gamma$ is any proper $S^\perp$-invariant ideal Q such that whenever Q contains a product of $S^\perp$-invariant ideals, it must contain one of the factors. Denote by $S^\perp\text{-}\operatorname{spec} k\Gamma$ the set of $S^\perp$-prime ideals of $k\Gamma$, and observe that this set supports a Zariski topology (defined in the obvious way). The first map in Theorem 2.4 can be factored in the form

$$\operatorname{spec} k\Gamma \twoheadrightarrow S^\perp\text{-}\operatorname{spec} k\Gamma \xrightarrow{\approx} \operatorname{spec} A,$$

where the map $P \mapsto (P : S^\perp)$ from $\operatorname{spec} k\Gamma$ to $S^\perp\text{-}\operatorname{spec} k\Gamma$ is a topological quotient map, and the map $Q \mapsto \Phi_c^{-1}(Q)$ from $S^\perp\text{-}\operatorname{spec} k\Gamma$ to $\operatorname{spec} A$ is a homeomorphism. That $P \mapsto (P : S^\perp)$ is a topological quotient map is an easy exercise (cf. [**6**, Proposition 1.7] for a generalization); the bulk of the work in proving Theorem 2.4 goes into establishing the homeomorphism $S^\perp\text{-}\operatorname{spec} k\Gamma \approx \operatorname{spec} A$ (cf. [**6**, Proposition 3.10]). The key point is that the hypothesis on c ensures that Φ_c is 'central-semilinear', that is, $\Phi_c(za) = \Phi_c(z)\Phi_c(a)$ for $z \in Z(A)$ and $a \in A$ [**6**, Lemma 3.5].

The factorization above respects maximal and primitive ideals in the following way. Let $S^\perp\text{-}\max k\Gamma$ denote the set of maximal proper $S^\perp$-invariant ideals of $k\Gamma$; this is a subset of $S^\perp\text{-}\operatorname{spec} k\Gamma$, which we equip with the relative topology. Then we have the factorization

$$\max k\Gamma \twoheadrightarrow S^\perp\text{-}\max k\Gamma \xrightarrow{\approx} \operatorname{prim} A,$$

where $P \mapsto (P : S^\perp)$ gives a topological quotient map from $\max k\Gamma$ to $S^\perp\text{-}\max k\Gamma$, and $Q \mapsto \Phi_c^{-1}(Q)$ gives a homeomorphism from $S^\perp\text{-}\max k\Gamma$ to $\operatorname{prim} A$ (cf. [**6**, Propositions 3.9, 3.10]).

3. Quantum affine spaces

Let $\mathbf{q} \in M_n(k^\times)$ again be a multiplicatively antisymmetric matrix. The corresponding *quantum affine space* over k is the algebra

$$\mathcal{O}_{\mathbf{q}}(k^\times) := k\langle x_1, \dots, x_n \mid x_i x_j = q_{ij} x_j x_i \text{ for all } i, j \rangle.$$

3.1. Denote the algebra $\mathcal{O}_{\mathbf{q}}(k^\times)$ by A for purposes of discussion. The torus $H = (k^\times)^n$ again acts naturally as k-algebra automorphisms on A. The primitive spectrum of A consists of 2^n H-orbits, which we denote $\operatorname{prim}_w A$, indexed by the subsets $w \subseteq \{1, \dots, n\}$ [**5**, Theorem 2.3]. Here

$$\operatorname{prim}_w A = \{P \in \operatorname{prim} A \mid x_i \in P \Longleftrightarrow i \in w\}.$$

Similarly, $\operatorname{spec} A$ is a disjoint union of 2^n subsets

$$\operatorname{spec}_w A = \{P \in \operatorname{spec} A \mid x_i \in P \Longleftrightarrow i \in w\}.$$

Each $\operatorname{spec}_w A$ is homeomorphic, via localization, to the prime spectrum of the quantum torus

$$A_w = \big(A/\langle x_i \mid i \in w\rangle\big)[x_j^{-1} \mid j \notin w],$$

and likewise $\operatorname{prim}_w A$ is homeomorphic to $\operatorname{prim} A_w$ (cf. [**5**, Theorem 2.3]).

The partition $\operatorname{prim} A = \bigsqcup_w \operatorname{prim}_w A$ is a quantum analog of the standard stratification of affine n-space by its H-orbits. To emphasize the parallel, we label the H-orbits in k^n as

$$(k^n)_w = \{(\alpha_1, \dots, \alpha_n) \in k^n \mid \alpha_i = 0 \Longleftrightarrow i \in w\},$$

for $w \subseteq \{1, \dots, n\}$. It follows from Theorem 2.1 that there are H-equivariant topological quotient maps

$$(k^n)_w \twoheadrightarrow \operatorname{prim} A_w \approx \operatorname{prim}_w A$$

for all w. The problem is then to patch these maps together coherently. Modulo a minor technical assumption, that can be done:

3.2. Theorem. [**6**, Theorem 4.11] *Let $\mathbf{q} = (q_{ij}) \in M_n(k^\times)$ be a multiplicatively antisymmetric matrix. Assume that either $-1 \notin \langle q_{ij}\rangle$ or $\operatorname{char} k = 2$. Then there exist topological quotient maps*

$$k^n \twoheadrightarrow \operatorname{prim} \mathcal{O}_{\mathbf{q}}(k^n)$$
$$\operatorname{spec} k[y_1, \dots, y_n] \twoheadrightarrow \operatorname{spec} \mathcal{O}_{\mathbf{q}}(k^n),$$

equivariant with respect to the natural actions of the torus $(k^\times)^n$. $\square$

Here $\langle q_{ij}\rangle$ denotes the subgroup of $k^\times$ generated by the q_{ij}.

To describe the maps in the above theorem, and identify the fibres of the first, we use a setup analogous to that in the quantum torus case. In particular, we write $\mathcal{O}_{\mathbf{q}}(k^n)$ as a twisted semigroup algebra with respect to a suitable cocycle.

3.3. Fix a quantum affine space $A = \mathcal{O}_{\mathbf{q}}(k^n)$. Set $\Gamma = \mathbb{Z}^n$ and $\Gamma^+ = (\mathbb{Z}^+)^n$, and define $\sigma : \Gamma \times \Gamma \to k^\times$ as in (2.2). As in the previous case, σ determines the commutation rules in A.

Next, choose a 2-cocycle $c : \Gamma\times\Gamma \to k^\times$ such that $c(\alpha, \beta)c(\beta, \alpha)^{-1} = \sigma(\alpha, \beta)$ for $\alpha, \beta \in \Gamma$. We identify A with the twisted semigroup algebra $k^c\Gamma^+$ with a basis $\{x_\alpha \mid \alpha \in \Gamma^+\}$, and we write the corresponding semigroup algebra $R = k\Gamma^+$ in terms of a basis $\{y_\alpha \mid \alpha \in \Gamma^+\}$. There are partitions $\operatorname{spec} R = \bigsqcup_w \operatorname{spec}_w R$ and $\max R = \bigsqcup_w \max_w R$ analogous to those for $\operatorname{spec} A$ and $\operatorname{prim} A$.

Again as before, identify the torus $H = (k^\times)^n$ with $\operatorname{Hom}(\Gamma, k^\times)$, so that the natural actions of H on A and $k\Gamma^+$ are given by

$$h.x_\alpha = h(\alpha)x_\alpha \qquad \text{and} \qquad h.y_\alpha = h(\alpha)y_\alpha.$$

3.4. It is convenient to identify the localizations A_w with subalgebras of the twisted group algebra $k^c\Gamma$, and to identify the corresponding localizations R_w of R with subalgebras of the group algebra $k\Gamma$. This is done as follows:

$$A_w = \sum_{\alpha\in\Gamma_w} kx_\alpha \qquad \text{and} \qquad R_w = \sum_{\alpha\in\Gamma_w} ky_\alpha = k\Gamma_w,$$

where $\Gamma_w = \{\alpha \in \Gamma \mid \alpha_i = 0 \text{ for } i \in w\}$. (Here we have extended the x_α and the y_α to Γ-indexed bases for $k^c\Gamma$ and $k\Gamma$.) As in (2.2), there is an H-equivariant k-linear isomorphism $\Phi = \Phi_c : k^c\Gamma \to k\Gamma$ such that $\Phi(x_\alpha) = y_\alpha$ for $\alpha \in \Gamma$. This map restricts to H-equivariant k-linear isomorphisms from A onto R and from A_w onto R_w for each w.

Next, let σ_w and c_w denote the restrictions of σ and c to Γ_w. Then the identification above can be restated as $A_w = k^{c_w}\Gamma_w$. We shall need the radical of σ_w, and we emphasize that this is a subgroup of Γ_w. There is a natural pairing between Γ_w and a quotient group of H, but it is more useful to take orthogonals with respect to the pairing of H and Γ; thus, we write

$$\operatorname{rad}(\sigma_w)^\perp = \{h \in H \mid \ker h \supseteq \operatorname{rad}(\sigma_w)\}.$$

3.5. Theorem. [**6**, Theorem 4.11] *Let $A = \mathcal{O}_{\mathbf{q}}(k^n)$, and assume that either $-1 \notin \langle q_{ij}\rangle$ or* char $k = 2$. *Fix $\Gamma, \sigma, c, \Phi, H, \sigma_w$ as above. Assume that c is an alternating bicharacter on Γ with $c^2 = \sigma$ and such that $c(\alpha, \beta) = 1$ whenever $\sigma(\alpha, \beta) = 1$. (Such a choice for c is possible by* [**6**, Lemma 4.2]*.) Then there are H-equivariant topological quotient maps*

$$\operatorname{spec} k\Gamma^+ \twoheadrightarrow \operatorname{spec} A \qquad \textit{and} \qquad \max k\Gamma^+ \twoheadrightarrow \operatorname{prim} A$$

such that $P \mapsto \Phi^{-1}(P : \operatorname{rad}(\sigma_w)^\perp)$ for $P \in \operatorname{spec}_w k\Gamma^+$. The fibres of the second map over points in $\operatorname{prim}_w A$ are exactly the $\operatorname{rad}(\sigma_w)^\perp$-orbits within $\max_w k\Gamma^+$. $\square$

While the precise formula given for the maps in Theorem 3.5 is particularly useful for computations, it may round out the picture to give a global formula (independent of w), as follows.

3.6. Lemma. *Let $\phi : \operatorname{spec} k\Gamma^+ \twoheadrightarrow \operatorname{spec} A$ be the topological quotient map given in Theorem 3.5. For all $P \in \operatorname{spec} k\Gamma^+$, the prime ideal $\phi(P) \in \operatorname{spec} A$ equals the largest ideal of A contained in the set $\Phi^{-1}(P)$.*

Proof. We shall need the generators $y_i = y_{\epsilon_i} \in k\Gamma^+$ and $x_i = x_{\epsilon_i} \in A$, where $\epsilon_1, \dots, \epsilon_n$ is the standard basis for Γ.

Let $P \in \operatorname{spec} k\Gamma^+$; then $P \in \operatorname{spec}_w k\Gamma^+$ for some w. First suppose that $P \in \max_w k\Gamma^+$. Then $\phi(P) \in \operatorname{prim}_w A$, and so $\phi(P)$ is a maximal element of $\operatorname{spec}_w A$ [**5**, Theorem 2.3]. Now $\phi(P)$ contains the ideal $J_w := \langle x_i \mid i \in w \rangle$. On the other hand, the multiplicative set generated by $\{y_j \mid j \notin w\}$ in $k\Gamma^+$ is disjoint from P, and Φ^{-1} sends elements of this set to scalar multiples of elements in the multiplicative set X_w generated by $\{x_j \mid j \notin w\}$ in A. Hence, $\Phi^{-1}(P)$ is disjoint from X_w.

Let I be the largest ideal of A which is contained in $\Phi^{-1}(P)$. Then $J_w \subseteq \phi(P) \subseteq I$ and I is disjoint from X_w. Hence, I/J_w induces a proper ideal, call it IA_w, in the localization $A_w = (A/J_w)[X_w^{-1}]$. Let M be a maximal ideal of A_w containing IA_w, and let Q be the inverse image of M under the localization map $A \to A/J_w \to A_w$. Then $Q \in \operatorname{spec}_w A$ and $\phi(P) \subseteq I \subseteq Q$. Since $\phi(P)$ is maximal in $\operatorname{spec}_w A$, we obtain $\phi(P) = Q$ and thus $\phi(P) = I$. Therefore the lemma holds for $P \in \max_w k\Gamma^+$.

Now consider an arbitrary $P \in \operatorname{spec}_w k\Gamma^+$. Since $k\Gamma^+$ is a commutative affine k-algebra, P is an intersection of maximal ideals. It follows easily, as in [**2**, Proposition 1.3(a)], that $P = \bigcap_r P_r$ for some maximal ideals $P_r \in \max_w k\Gamma^+$. (Namely, $P = \bigcap_{v \subseteq \{1,\dots,n\}} Q_v$ where each Q_v is an intersection of maximal ideals from $\max_v k\Gamma^+$. Observe that $Q_v = k\Gamma^+$ when $v \not\supseteq w$, and that $Q_v \supsetneq P$ when $v \supsetneq w$. Consequently, $P = Q_w$.) Since the set functions $(- : \operatorname{rad}(\sigma_w)^\perp)$ and Φ^{-1} preserve intersections, we see that $\phi(P) = \bigcap_r \phi(P_r)$ and $\Phi^{-1}(P) = \bigcap_r \Phi^{-1}(P_r)$. By the previous paragraph, each $\phi(P_r)$ is the largest ideal of A contained in $\Phi^{-1}(P_r)$. Therefore $\phi(P)$ is the largest ideal of A contained in $\Phi^{-1}(P)$. $\square$

3.7. The maps in Theorem 3.5 have analogous factorizations to those in (2.5), which we write

$$\operatorname{spec} k\Gamma^+ \twoheadrightarrow \mathcal{G}\text{-spec}\, k\Gamma^+ \xrightarrow{\approx} \operatorname{spec} A$$

$$\max k\Gamma^+ \twoheadrightarrow \mathcal{G}\text{-max}\, k\Gamma^+ \xrightarrow{\approx} \operatorname{prim} A.$$

Here $\mathcal{G}$ stands for the indexed family of groups $\{S_w^\perp \mid w \subseteq \{1,\dots,n\}\}$ where $S_w = \operatorname{rad}(\sigma_w)$, while

$$\mathcal{G}\text{-spec}\, k\Gamma^+ = \bigsqcup_{w \subseteq \{1,\dots,n\}} \{(P : S_w^\perp) \mid P \in \operatorname{spec}_w k\Gamma^+\}$$

$$\mathcal{G}\text{-max}\, k\Gamma^+ = \bigsqcup_{w \subseteq \{1,\dots,n\}} \{(P : S_w^\perp) \mid P \in \max_w k\Gamma^+\}.$$

Closed sets for Zariski topologies on $\mathcal{G}$-spec $k\Gamma^+$ and $\mathcal{G}$-max $k\Gamma^+$ are defined in the usual way; a compatibility condition on the groups in $\mathcal{G}$ ensures that the result actually is a topology [**6**, (2.4), (4.9)]. A fairly general piece of commutative algebra, developed in [**6**, Section 2], provides the topological quotient maps from $\operatorname{spec} k\Gamma^+$ and $\max k\Gamma^+$ onto $\mathcal{G}$-spec $k\Gamma^+$ and $\mathcal{G}$-max $k\Gamma^+$. The homeomorphisms of $\mathcal{G}$-spec $k\Gamma^+$ and $\mathcal{G}$-max $k\Gamma^+$ onto $\operatorname{spec} A$ and $\operatorname{prim} A$ are constructed by patching together homeomorphisms of $S_w^\perp$- spec $k\Gamma_w$ and $S_w^\perp$- max $k\Gamma_w$ onto $\operatorname{spec} A_w$ and $\operatorname{prim} A_w$ obtained as in (2.5).

3.8. Example. To illustrate the form of these topological quotients, we give the basic example in dimension 3, where various differences between the classical and quantum cases appear. Recall that the single parameter quantum affine spaces, relative to a scalar $q \in k^\times$, are the algebras

$$\mathcal{O}_q(k^n) = k\langle x_1, \dots, x_n \mid x_i x_j = q x_j x_i \text{ for } i < j \rangle.$$

(a) Assume first that q is not a root of unity, and let p be a square root of q in k. In this case, the topological quotient map

$$k^3 \approx \max k(\mathbb{Z}^+)^3 \twoheadrightarrow \operatorname{prim} \mathcal{O}_q(k^3)$$

described in Theorem 3.5 can be computed as follows [**6**, (5.2)], where the scalars λ_i are all assumed to be nonzero:

$$\begin{aligned}
(0,0,0) &\mapsto \langle x_1,\, x_2,\, x_3 \rangle & (\lambda_1,\lambda_2,0) &\mapsto \langle x_3 \rangle \\
(\lambda_1,0,0) &\mapsto \langle x_1-\lambda_1,\, x_2,\, x_3 \rangle & (\lambda_1,0,\lambda_3) &\mapsto \langle x_2 \rangle \\
(0,\lambda_2,0) &\mapsto \langle x_1,\, x_2-\lambda_2,\, x_3 \rangle & (0,\lambda_2,\lambda_3) &\mapsto \langle x_1 \rangle \\
(0,0,\lambda_3) &\mapsto \langle x_1,\, x_2,\, x_3-\lambda_3 \rangle & (\lambda_1,\lambda_2,\lambda_3) &\mapsto \langle \lambda_2 x_1 x_3 - p\lambda_1\lambda_3 x_2 \rangle
\end{aligned}$$

The right column demonstrates both the compression effect of the quotient process and the fact that $\mathcal{O}_q(k^3)$ has 'many fewer' primitive ideals than $\mathcal{O}(k^3)$. Note the factor p in the last term. Without this factor, we obtain only a surjective map, which is not Zariski continuous.

(b) For contrast, consider the case where q is a primitive t-th root of unity for some odd $t > 1$, and take $p = q^{(t+1)/2}$. Then the topological quotient map $k^3 \twoheadrightarrow \operatorname{prim} \mathcal{O}_q(k^3)$ has the form below [**6**, (5.3)]:

$$\begin{aligned}
(0,0,0) &\mapsto \langle x_1,\, x_2,\, x_3 \rangle & (\lambda_1,\lambda_2,0) &\mapsto \langle x_1^t-\lambda_1^t,\, x_2^t-\lambda_2^t,\, x_3 \rangle \\
(\lambda_1,0,0) &\mapsto \langle x_1-\lambda_1,\, x_2,\, x_3 \rangle & (\lambda_1,0,\lambda_3) &\mapsto \langle x_1^t-\lambda_1^t,\, x_2,\, x_3^t-\lambda_3^t \rangle \\
(0,\lambda_2,0) &\mapsto \langle x_1,\, x_2-\lambda_2,\, x_3 \rangle & (0,\lambda_2,\lambda_3) &\mapsto \langle x_1,\, x_2^t-\lambda_2^t,\, x_3^t-\lambda_3^t \rangle \\
(0,0,\lambda_3) &\mapsto \langle x_1,\, x_2,\, x_3-\lambda_3 \rangle
\end{aligned}$$

$$(\lambda_1,\lambda_2,\lambda_3) \mapsto \langle x_1^t-\lambda_1^t,\, x_2^t-\lambda_2^t,\, x_3^t-\lambda_3^t,\, \lambda_2 x_1 x_3 - p\lambda_1\lambda_3 x_2 \rangle$$

In this case, $\operatorname{prim} \mathcal{O}_q(k^3)$ is more 'classical', in that all its points are closed.

4. Quantum toric varieties and cocycle twists

In this section, we discuss quantizations of toric varieties [**11**] and extend Theorem 3.5 to that setting. There are several ways in which (affine) toric varieties may be defined. For our purposes, the simplest definition is a variety on which a torus acts (morphically) with a dense orbit. Some equivalent conditions are given in the following theorem.

4.1. Theorem. [**14**, Satz 5, p. 105] *Let H be an algebraic torus, acting morphically on an irreducible affine variety V. The following conditions are equivalent:*

(a) *There are only finitely many H-orbits in V.*
(b) *Some H-orbit is dense in V.*
(c) *The fixed field $k(V)^H = k$.*
(d) *All H-eigenspaces in $k(V)$ are 1-dimensional.* □

For comparison, we state a ring-theoretic version of this theorem. Since it will not be needed below, however, we leave the proof to the reader.

4.2. Theorem. *Let R be a commutative affine domain over k, and let H be an algebraic torus acting rationally on R by k-algebra automorphisms. Then the following conditions are equivalent:*

(a) *There are only finitely many H-orbits in* $\max R$.
(b) *Some H-orbit is dense in* $\max R$.
(c) $(\operatorname{Fract} R)^H = k$, *where* $\operatorname{Fract} R$ *is the quotient field of R.*
(d) *All H-eigenspaces in R are 1-dimensional.*
(e) *There are only finitely many H-invariant prime ideals in R.* □

In the noncommutative setting, consider a prime, noetherian, affine k-algebra R, let $\operatorname{Fract} R$ be the Goldie quotient ring of R, and replace the maximal ideal space by $\operatorname{prim} R$. The conditions of Theorem 4.2 are no longer equivalent, since $\operatorname{prim} R$ may well have a dense point (if 0 is a primitive ideal). It is thus natural to focus on the stronger conditions, (c) and (d).

Following Ingalls [**11**], we define a *quantum (affine) toric variety* to be an affine domain over k equipped with a rational action of an algebraic torus H by k-algebra automorphisms, such that the H-eigenspaces are 1-dimensional. If we fix H and require the action to be faithful, such algebras can be classified by elements of $\wedge^2 H$ together with finitely generated submonoids that generate the character group $\widehat{H}$ [**11**, Theorem 2.6].

Quantum toric varieties may also be presented as 'cocycle twists' (see below) of coordinate rings of classical toric varieties, analogous to the way quantum tori and quantum affine spaces are twisted versions of (Laurent) polynomial rings. Furthermore, quantum toric varieties can be written as factor algebras of quantum affine spaces, which shows, in particular, that they are noetherian. It also allows us to apply Theorem 3.2 and express their primitive spectra as topological quotients of classical varieties. For this purpose, a presentation as a cocycle twist of a commutative affine algebra suffices; one-dimensionality of eigenspaces is not needed. Hence, we can work with a somewhat larger class of algebras than quantum toric varieties.

4.3. Suppose that R is a k-algebra graded by an abelian group G, and that $c : G \times G \to k^\times$ is a 2-cocycle. The underlying G-graded vector space of R can be equipped with a new multiplication $*$, where $r * s = c(\alpha, \beta) rs$ for all homogeneous elements $r \in R_\alpha$ and $s \in R_\beta$. This new multiplication is associative, and gives R a new structure as G-graded k-algebra, called the *twist of R by c* (see [**1**, Section 3] for details). The following alternative presentation is helpful in keeping computations straight. Let R' be an isomorphic graded vector space copy of R, equipped with a G-graded k-linear isomorphism $R \to R'$ denoted $r \mapsto r'$. Then R' becomes a G-graded k-algebra such that $r's' = c(\alpha, \beta)(rs)'$ for $r \in R_\alpha$

and $s \in R_\beta$. The *twist map* $r \mapsto r'$ then gives an isomorphism of the twist of R by c onto R'.

Now suppose that R is a commutative affine k-algebra, that G is torsionfree, and that either -1 is not in the subgroup of $k^\times$ generated by the image of c or $\operatorname{char} k = 2$. Then there exist topological quotient maps

$$\operatorname{spec} R \twoheadrightarrow \operatorname{spec} R' \qquad \text{and} \qquad \max R \twoheadrightarrow \operatorname{prim} R'$$

[**6**, Theorem 6.3]. We shall show that these maps can be given by formulas analogous to those in Theorem 3.5.

It is convenient to immediately make a reduction to a special choice of c. First, we can replace G by any finitely generated subgroup containing the support of R. Then, as in the proof of [**6**, Theorem 6.3], there exists an alternating bicharacter d on G, satisfying the same hypotheses as c, such that R' is isomorphic to the twist of R by d. Thus, there is no loss of generality in assuming that c is an alternating bicharacter, and we shall describe the topological quotient maps above under this assumption.

The torsionfreeness hypothesis on G is only needed for the reduction just indicated. Hence, we drop this assumption in the discussion to follow.

4.4. To recap, we are now assuming that R is a commutative affine k-algebra graded by an abelian group G, that $c : G \times G \to k^\times$ is an alternating bicharacter, and that either $-1 \notin \langle \operatorname{im} c \rangle$ or $\operatorname{char} k = 2$. Let $A = R'$ be the twist of R by c.

Choose homogeneous k-algebra generators $r_1, \dots, r_n$ for R, and set $a_i = r_i' \in A$ for $i = 1, \dots, n$. We use these elements to define sets $\operatorname{spec}_w R$, $\operatorname{spec}_w A$, $\max_w R$, $\operatorname{prim}_w A$ for $w \subseteq \{1, \dots, n\}$ along the same lines as in (3.1). Thus

$$\begin{aligned} \operatorname{spec}_w R &= \{P \in \operatorname{spec} R \mid r_i \in P \iff i \in w\} \\ \operatorname{spec}_w A &= \{P \in \operatorname{spec} A \mid a_i \in P \iff i \in w\}, \end{aligned}$$

while $\max_w R = (\max R) \cap (\operatorname{spec}_w R)$ and $\operatorname{prim}_w A = (\operatorname{prim} A) \cap (\operatorname{spec}_w A)$. Of course, some of these sets may be empty.

Set $H = \operatorname{Hom}(G, k^\times)$, an abelian group under pointwise multiplication. Because R and A are G-graded k-algebras, H acts on them by k-algebra automorphisms such that $h.r = h(x)r$ and $h.a = h(x)a$ for $h \in H$, $r \in R_x$, $a \in A_x$. Note that the sets $\operatorname{spec}_w R$ and $\operatorname{spec}_w A$ are invariant under the induced H-actions, since the elements r_i and a_i are H-eigenvectors.

For $i = 1, \dots, n$, let $\delta_i \in G$ denote the degree of r_i and a_i. For $w \subseteq \{1, \dots, n\}$, set $G_w = \sum_{i \notin w} \mathbb{Z}\delta_i$ and let c_w denote the restriction of c to G_w. By definition of H, we have a pairing $H \times G \to k^\times$, and we use $^\perp$ to denote orthogonals with respect to this pairing. In particular, for each w we can define $\operatorname{rad}(c_w)^\perp$, which is a subgroup of H.

We can now state the following more precise version of [**6**, Theorem 6.3]:

4.5. Theorem. *Let R be a commutative affine k-algebra, graded by an abelian group G. Let $c : G \times G \to k^\times$ be an alternating bicharacter, and let $A = R'$ be the twist of R by c. Assume that either $-1 \notin \langle \operatorname{im} c \rangle$ or $\operatorname{char} k = 2$.*

With notation as in (4.4), there exist H-equivariant topological quotient maps

$$\phi_s : \operatorname{spec} R \twoheadrightarrow \operatorname{spec} A \qquad \text{and} \qquad \phi_m : \max R \twoheadrightarrow \operatorname{prim} A$$

such that $P \mapsto (P : \operatorname{rad}(c_w)^{\perp})'$ for $P \in \operatorname{spec}_w R$. Alternatively, $\phi_s(P)$ is the largest ideal of A contained in P'. Moreover, the fibres of ϕ_m over points in $\operatorname{prim}_w A$ are precisely the $\operatorname{rad}(c_w)^{\perp}$-orbits in $\max_w R$.

Note that the second description of ϕ_s shows that this map depends only on R, G, and c, that is, it is independent of the choice of homogeneous generators $r_i \in R$ as in (4.4).

The proof of Theorem 4.5 will be distributed over the following three subsections.

4.6. Let $\Phi : A \to R$ be the inverse of the twist map; this is a G-graded k-linear isomorphism. In particular, $X' = \Phi^{-1}(X)$ for subsets $X \subseteq R$.

Set $\Gamma = \mathbb{Z}^n$ and $\Gamma^+ = (\mathbb{Z}^+)^n$, and let $\rho : \Gamma \to G$ be the group homomorphism given by the rule $\rho(\alpha_1, \dots, \alpha_n) = \alpha_1\delta_1 + \cdots + \alpha_n\delta_n$. Set $\tilde{c} = c \circ (\rho \times \rho)$, which is an alternating bicharacter on Γ, as is $\sigma = \tilde{c}^2$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \Gamma^+$, define

$$r^{\alpha} = r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n} \in R_{\rho(\alpha)} \qquad \text{and} \qquad a_{\alpha} = (r^{\alpha})' \in A_{\rho(\alpha)}.$$

Then $a_\alpha a_\beta = \tilde{c}(\alpha, \beta) a_{\alpha+\beta}$ for $\alpha, \beta \in \Gamma^+$.

Next, set $\widetilde{R} = k\Gamma^+$ and $\widetilde{A} = k^{\tilde{c}}\Gamma^+$, expressed with k-bases $\{y_\alpha\}$ and $\{x_\alpha\}$ such that $y_\alpha y_\beta = y_{\alpha+\beta}$ and $x_\alpha x_\beta = \tilde{c}(\alpha, \beta) x_{\alpha+\beta}$ for $\alpha, \beta \in \Gamma^+$. Let $\widetilde{\Phi} : \widetilde{A} \to \widetilde{R}$ be the inverse of the twist map; this is a Γ-graded k-linear isomorphism such that $\widetilde{\Phi}(x_\alpha) = y_\alpha$ for $\alpha \in \Gamma^+$. There exist surjective k-algebra maps $\tau : \widetilde{R} \to R$ and $\pi : \widetilde{A} \to A$ such that $\tau(y_\alpha) = r^\alpha$ and $\pi(x_\alpha) = a_\alpha$ for $\alpha \in \Gamma^+$. Thus, we obtain a commutative diagram

$$\begin{tikzcd} \widetilde{R} \arrow[d, "\tau"'] & \widetilde{A} \arrow[l, "\tilde{\Phi}"'] \arrow[d, "\pi"] \\ R & A \arrow[l, "\Phi"] \end{tikzcd}$$

Let $\widetilde{H} = \operatorname{Hom}(\Gamma, k^\times)$, acting via k-algebra automorphisms on $\widetilde{A}$ and $\widetilde{R}$ in the usual way. For $w \subseteq \{1, \dots, n\}$, define $\operatorname{spec}_w \widetilde{A}$, $\operatorname{spec}_w \widetilde{R}$, Γ_w, $\widetilde{A}_w$, $\widetilde{R}_w$, σ_w, $\tilde{c}_w$ as in (3.1), (3.3), (3.4). (To define $\operatorname{spec}_w \widetilde{A}$ and $\operatorname{spec}_w \widetilde{R}$, for instance, use the elements $x_{\epsilon_i} \in \widetilde{A}$ and $y_{\epsilon_i} \in \widetilde{R}$, where $\epsilon_1, \dots, \epsilon_n$ denotes the standard basis for Γ.) Because of our hypotheses on c, the group $\langle \operatorname{im} \tilde{c} \rangle$ contains no elements of order 2, and hence $\operatorname{rad}(\sigma_w) = \operatorname{rad}(\tilde{c}_w)$ for all w. We set $\widetilde{S}_w = \operatorname{rad}(\tilde{c}_w)$.

Now Theorem 3.5 gives us $\widetilde{H}$-equivariant topological quotient maps

$$\widetilde{\phi}_s : \operatorname{spec} \widetilde{R} \twoheadrightarrow \operatorname{spec} \widetilde{A} \qquad \text{and} \qquad \widetilde{\phi}_m : \max \widetilde{R} \twoheadrightarrow \operatorname{prim} \widetilde{A}$$

such that $P \mapsto \widetilde{\Phi}^{-1}(P : \widetilde{S}_w^{\perp})$ for $P \in \operatorname{spec}_w \widetilde{R}$. Moreover, the fibres of $\widetilde{\phi}_m$ over points in $\operatorname{prim}_w \widetilde{A}$ are the $\widetilde{S}_w^{\perp}$-orbits in $\max_w \widetilde{R}$.

Set $V = \{P \in \operatorname{spec} \widetilde{R} \mid P \supseteq \ker \tau\}$ and $W = \{P \in \operatorname{spec} \widetilde{A} \mid P \supseteq \ker \pi\}$. The proof of [**6**, Theorem 6.3] shows that $\widetilde{\phi}_s$ and $\widetilde{\phi}_m$ restrict to topological quotient maps $V \twoheadrightarrow W$ and $V \cap \max \widetilde{R} \twoheadrightarrow W \cap \operatorname{prim} \widetilde{A}$. Therefore there are topological quotient maps ϕ_s and ϕ_m such that the following diagrams commute:

$$\begin{tikzcd}
\operatorname{spec} \widetilde{R} \arrow[r, two heads, "\tilde{\phi}_s"] & \operatorname{spec} \widetilde{A} \\
V \arrow[u, "\subseteq"] \arrow[r, two heads] & W \arrow[u, "\subseteq"'] \\
\operatorname{spec} R \arrow[u, "\tau^{-1}"] \arrow[r, two heads, "\phi_s"] & \operatorname{spec} A \arrow[u, "\pi^{-1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\max \widetilde{R} \arrow[r, two heads, "\tilde{\phi}_m"] & \operatorname{prim} \widetilde{A} \\
V \cap \max \widetilde{R} \arrow[u, "\subseteq"] \arrow[r, two heads] & W \cap \operatorname{prim} \widetilde{A} \arrow[u, "\subseteq"'] \\
\max R \arrow[u, "\tau^{-1}"] \arrow[r, two heads, "\phi_m"] & \operatorname{prim} A \arrow[u, "\pi^{-1}"']
\end{tikzcd}$$

It remains to show that ϕ_s and ϕ_m have the properties announced in the statement of Theorem 4.5.

Given any $P \in \operatorname{spec} R$, we have $\pi^{-1}(\phi_s(P)) = \widetilde{\phi}_s(\tau^{-1}(P))$, which by Lemma 3.6 equals the largest ideal of $\widetilde{A}$ contained in $\widetilde{\Phi}^{-1}\tau^{-1}(P)$. If I is an ideal of A contained in $P' = \Phi^{-1}(P)$, then $\pi^{-1}(I)$ is an ideal of $\widetilde{A}$ contained in $\pi^{-1}\Phi^{-1}(P) = \widetilde{\Phi}^{-1}\tau^{-1}(P)$. Hence, $\pi^{-1}(I) \subseteq \pi^{-1}(\phi_s(P))$, and so $I \subseteq \phi_s(P)$. This shows that $\phi_s(P)$ is the largest ideal of A contained in P'.

The following lemma records some facts and conditions that we shall need to finish the proof.

4.7. Lemma. *Let $w \subseteq \{1, \dots, n\}$, and set $S_w = \operatorname{rad}(c_w)$.*

(a) $\ker \tau \subseteq (\tau^{-1}(P) : \widetilde{S}_w^{\perp}) \subseteq h(\tau^{-1}(P))$ *for $P \in \operatorname{spec}_w R$ and $h \in \widetilde{S}_w^{\perp}$.*

(b) $(h.(-)) \circ \tau = \tau \circ ((h\rho).(-))$ *for $h \in H$.*

(c) $\rho^{-1}(S_w) \cap \Gamma_w = \widetilde{S}_w$ *and so $\rho(\widetilde{S}_w) = S_w$.*

(d) $(h\rho)(\tau^{-1}(P)) = \tau^{-1}(h(P))$ *for $P \in \operatorname{spec}_w R$ and $h \in S_w^{\perp}$.*

(e) $\widetilde{S}_w^{\perp} = \rho^*(S_w^{\perp})\Gamma_w^{\perp}$, *where $\rho^* : H \to \widetilde{H}$ denotes the homomorphism given by composition with ρ.*

(f) *For $P \in \operatorname{spec}_w R$, the set $\{\tau^{-1}(h(P)) \mid h \in S_w^{\perp}\}$ equals the $\widetilde{S}_w^{\perp}$-orbit of $\tau^{-1}(P)$.*

Proof. (a) Note that $\tau^{-1}(P) \in \operatorname{spec}_w \widetilde{R}$ and $\widetilde{\phi}_s(\tau^{-1}(P)) \in W$. Thus $\widetilde{\Phi}^{-1}(\tau^{-1}(P) : \widetilde{S}_w^{\perp}) \supseteq \ker \pi$, and so $(\tau^{-1}(P) : \widetilde{S}_w^{\perp}) \supseteq \ker \tau$. The remaining inclusion is clear.

(b) This follows from the observation that

$$h.\tau(y_\alpha) = h.r^\alpha = h\rho(\alpha)r^\alpha = \tau(h\rho(\alpha)y_\alpha) = \tau((h\rho).y_\alpha)$$

for $h \in H$ and $\alpha \in \Gamma^+$.

(c) Observe that $\rho(\Gamma_w) = G_w$. Since $S_w \subseteq G_w$ and $\widetilde{S}_w \subseteq \Gamma_w$, the second part of the claim will follow from the first. Now consider an arbitrary element $\gamma \in \Gamma_w$. Then

$\rho(\gamma) \in S_w$ if and only if $c(\rho(\gamma), -) \equiv 1$ on $G_w = \rho(\Gamma_w)$, if and only if $\tilde{c}(\gamma, -) \equiv 1$ on Γ_w, if and only if $\gamma \in \widetilde{S}_w$.

(d) It is clear from part (b) that $\tau(h\rho)(\tau^{-1}(P)) = h(P)$. Since $h \in S_w^\perp$ and $\rho(\widetilde{S}_w) = S_w$ (part (c)), we have $h\rho \in \widetilde{S}_w^\perp$, whence $\ker\tau \subseteq (h\rho)(\tau^{-1}(P))$ (part (a)). Part (d) follows.

(e) Since $\widetilde{S}_w \subseteq \Gamma_w$, it is clear that $\Gamma_w^\perp \subseteq \widetilde{S}_w^\perp$. As in the proof of part (d), it is also clear that $\rho^*(S_w^\perp) \subseteq \widetilde{S}_w^\perp$.

Now consider $h' \in \widetilde{S}_w^\perp$; thus $\widetilde{S}_w \subseteq \ker h'$. Hence, h' induces a homomorphism $h_1 : \Gamma/\widetilde{S}_w \to k^\times$. In view of part (c), the restriction of ρ to Γ_w induces an isomorphism $\rho_w : \Gamma_w/\widetilde{S}_w \to G_w/S_w$. Let $h_2 : G_w \to k^\times$ denote the composition

$$G_w \xrightarrow{\text{quo}} G_w/S_w \xrightarrow{\rho_w^{-1}} \Gamma_w/\widetilde{S}_w \xrightarrow{\subseteq} \Gamma/\widetilde{S}_w \xrightarrow{h_1} k^\times,$$

and observe that the restrictions of $h_2\rho$ and h' to Γ_w coincide. Since k is algebraically closed, $k^\times$ is a divisible group, and hence $k^\times$ is injective in the category of abelian groups. Consequently, h_2 can be extended to a homomorphism $h_3 \in H$. On one hand, $h_3\rho$ and h' agree on Γ_w, whence $h'(h_3\rho)^{-1} \in \Gamma_w^\perp$. On the other hand, $S_w \subseteq \ker h_2 \subseteq \ker h_3$, whence $h_3 \in S_w^\perp$. Therefore $h' \in \rho^*(S_w^\perp)\Gamma_w^\perp$, and part (e) is proved.

(f) If $h \in S_w^\perp$, then $h\rho \in \widetilde{S}_w^\perp$ and we see from part (d) that $\tau^{-1}(h(P))$ lies in the $\widetilde{S}_w^\perp$-orbit of $\tau^{-1}(P)$.

Conversely, consider $h' \in \widetilde{S}_w^\perp$. By part (e), $h' = (h\rho)h_0$ for some $h \in S_w^\perp$ and $h_0 \in \Gamma_w^\perp$. We claim that h_0 leaves $\tau^{-1}(P)$ invariant; it will then follow from part (d) that $h'(\tau^{-1}(P)) = \tau^{-1}(h(P))$.

Since $P \in \operatorname{spec}_w R$, we see that $\tau^{-1}(P)$ contains the ideal $J_w = \langle y_{\epsilon_i} \mid i \in w\rangle$, where $\epsilon_1, \dots, \epsilon_n$ denotes the standard basis for Γ. Note that J_w is invariant under H. Since $\widetilde{R}/J_w$ is spanned by the cosets $y_\alpha + J_w$ for $\alpha \in \Gamma_w \cap \Gamma^+$, we see that the induced action of $\Gamma_w^\perp$ on $\widetilde{R}/J_w$ is trivial. Thus $\tau^{-1}(P)$ is indeed invariant under h_0, which establishes part (f). $\square$

4.8. (Proof of Theorem 4.5) Let $w \subseteq \{1, \dots, n\}$, and consider $P \in \operatorname{spec}_w R$. In view of Lemma 4.7(a) and the commutative diagrams in (4.6), we see that

$$\begin{aligned}\phi_s(P) &= \pi\widetilde{\phi}_s(\tau^{-1}(P)) = \pi\widetilde{\Phi}^{-1}(\tau^{-1}(P) : \widetilde{S}_w^\perp)\\ &= \Phi^{-1}\tau(\tau^{-1}(P) : \widetilde{S}_w^\perp) = \big(\tau(\tau^{-1}(P) : \widetilde{S}_w^\perp)\big)'.\end{aligned}$$

On the other hand, Lemma 4.7(f) implies that

$$(\tau^{-1}(P) : \widetilde{S}_w^\perp) = \bigcap_{h \in S_w^\perp} \tau^{-1}(h(P)) = \tau^{-1}(P : S_w^\perp),$$

whence $\tau(\tau^{-1}(P) : \widetilde{S}_w^\perp) = (P : S_w^\perp)$. Therefore $\phi_s(P) = (P : S_w^\perp)'$.

Since H is abelian, the set functions $(- : S_w^\perp)$ are H-equivariant, as is the twist map $(-)'$. Hence, it follows from the formula just proved that ϕ_s is H-equivariant.

Finally, note that the fibre of ϕ_m over any point of $\operatorname{prim}_w A$ is contained in $\max_w R$. Consider $P, P' \in \max_w R$. Taking account of (4.6), we see that $\phi_m(P) = \phi_m(P')$ if and only if $\widetilde{\phi}_m(\tau^{-1}(P)) = \widetilde{\phi}_m(\tau^{-1}(P'))$, if and only if $\tau^{-1}(P)$ and $\tau^{-1}(P')$ lie in the same

$\widetilde{S}_w^\perp$-orbit. By Lemma 4.7(f), this occurs if and only if $\tau^{-1}(P') = \tau^{-1}(h(P))$ for some $h \in S_w^\perp$, and thus if and only if P and P' lie in the same $S_w^\perp$-orbit. $\square$

Once the precise form of the maps in Theorem 4.5 is given, we can easily construct topological quotient maps between different cocycle twists of R, as follows.

4.9. Corollary. *Let R be a commutative affine k-algebra, graded by an abelian group G. Let $c_1, c_2 : G \times G \to k^\times$ be alternating bicharacters, and let A_i be the twist of R by c_i. Assume that either $-1 \notin \langle \operatorname{im} c_1 \rangle \cup \langle \operatorname{im} c_2 \rangle$ or $\operatorname{char} k = 2$. Set $H = \operatorname{Hom}(G, k^\times)$, and let $\phi_i : \operatorname{spec} R \twoheadrightarrow \operatorname{spec} A_i$ be the H-equivariant topological quotient map given in Theorem 4.5.*

Assume that $c_2(x,y) = 1 \implies c_1(x,y) = 1$, for any $x, y \in G$. Then there exists an H-equivariant topological quotient map $\phi : \operatorname{spec} A_1 \twoheadrightarrow \operatorname{spec} A_2$ such that the following diagram commutes:

$$\begin{tikzcd} & \operatorname{spec} A_1 \arrow[dd, two heads, "\phi"] \\ \operatorname{spec} R \arrow[ur, two heads, "\phi_1"] \arrow[dr, two heads, "\phi_2"'] & \\ & \operatorname{spec} A_2 \end{tikzcd}$$

Proof. Set up notation for A_i and ϕ_i as in (4.4), and abbreviate $(c_i)_w$ by c_{iw}. Let $\Phi_i : A_i \to R$ be the inverse of the twist map. Then ϕ_i is given by the rule

$$\phi_i(P) = \Phi_i^{-1}(P : \operatorname{rad}(c_{iw})^\perp) \qquad \text{for} \quad P \in \operatorname{spec}_w R.$$

Our hypotheses on c_1, c_2 imply that $\operatorname{rad}(c_{2w}) \subseteq \operatorname{rad}(c_{1w})$ for all w, whence $\operatorname{rad}(c_{2w})^\perp \supseteq \operatorname{rad}(c_{1w})^\perp$ and so

$$(P : \operatorname{rad}(c_{2w})^\perp) = \big((P : \operatorname{rad}(c_{1w})^\perp) : \operatorname{rad}(c_{2w})^\perp\big)$$

for all $P \in \operatorname{spec}_w R$. Hence, we can define a map $\phi : \operatorname{spec} A_1 \to \operatorname{spec} A_2$ such that

$$\phi(Q) = \Phi_2^{-1}(\Phi_1(Q) : \operatorname{rad}(c_{2w})^\perp)$$

for $Q \in \operatorname{spec}_w A_1$. It is clear that $\phi\phi_1 = \phi_2$. Since ϕ_1 and ϕ_2 are H-equivariant topological quotient maps, so is ϕ. $\square$

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